

Problem Set 6: Metrics and Integration

Problem 1. Let (M, g) be a manifold with metric. The gradient of a function ∇f is a vector field defined by $(df)^\sharp$, or in local coordinates

$$\nabla f \stackrel{\text{loc}}{=} \left[g^{ik} \partial_k f \right] \frac{\partial}{\partial x^i}.$$

Compute ∇f in polar coordinates over $\mathbb{R}^2 \setminus \{0\}$ with the Euclidean metric.

Solution. The Euclidean metric

$$g = dx dx + dy dy,$$

in polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, becomes

$$g = (\cos \theta dr - r \sin \theta d\theta)^2 + (\sin \theta dr + r \cos \theta d\theta)^2,$$

which simplifies to

$$g = dr dr + r^2 d\theta d\theta.$$

Its inverse is

$$g^{rr} = 1, \quad g^{r\theta} = 0, \quad g^{\theta\theta} = r^{-2}.$$

Therefore

$$\nabla f = \left[g^{rr} \partial_r f \right] \frac{\partial}{\partial r} + \left[g^{\theta\theta} \partial_\theta f \right] \frac{\partial}{\partial \theta}$$

which becomes

$$\nabla f = \partial_r f \frac{\partial}{\partial r} + \frac{1}{r^2} \partial_\theta f \frac{\partial}{\partial \theta}.$$

□

Problem 2. In this problem, we rederive formulas from calculus on arclength and surface area.

(a) Let $f : (a, b) \rightarrow \mathbb{R}$ and consider the submanifold

$$S = \{(x, f(x)) : x \in (a, b)\} \subset \mathbb{R}^2.$$

In other words, S is the graph of $y = f(x)$. Equip S with coordinates $\varphi : S \rightarrow U$, $U = (a, b)$ given by

$$\varphi(x, f(x)) = x.$$

Denote by g the Euclidean metric on $\mathbb{R}^2$ restricted to S . Find the expression for g in the coordinate chart (U, x) . To be clear, this means $g = (\varphi^{-1})^* g_{\text{Euc}}$. Derive the formula for the arclength by simplifying the general formula

$$\int_U \sqrt{\det g_{ij}} dx^1 \cdots dx^n$$

in this context.

(b) Next, let $f : U \rightarrow \mathbb{R}$ be a function of two variables $f(x, y)$, where $U \subset \mathbb{R}^2$ is an open set. Consider the submanifold

$$S = \{(x, y, f(x, y)) : (x, y) \in U\} \subset \mathbb{R}^3.$$

Equip S with coordinates $\varphi : S \rightarrow U$. Compute the metric g in these coordinates, where g is the pullback of the Euclidean metric. Derive the formula for the area of a surface by simplifying the general formula

$$\int_U \sqrt{\det g_{ij}} dx^1 \cdots dx^n$$

in this context.

(c) Suppose the level set of the function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ defines a submanifold:

$$S = \{F(x, y, z) = 0\} \subseteq \mathbb{R}^3.$$

Let $(a, b, c) \in S$ and suppose $\partial_z F(a, b, c) \neq 0$. By the implicit function theorem, there exists a neighborhood $U \subset \mathbb{R}^2$ of (a, b) , $V \subset \mathbb{R}^3$ of (a, b, c) and a function $f : U \rightarrow \mathbb{R}$ such that

$$S \cap V = \{(x, y, f(x, y)) : (x, y) \in U\}.$$

Use the results from part (b) to derive the formula for area of $S \cap V$:

$$\iint_U \frac{|\nabla F|}{|\partial_z F|} dx dy.$$

Verify this formula by computing the area of the hemisphere by taking $F = x^2 + y^2 + z^2 - 1$ and $U = \{x^2 + y^2 < 1\}$.

Solution. For (a), we compute the pullback

$$(\varphi^{-1})^*(dx \otimes dx + dy \otimes dy) = (1 + (f')^2)dx \otimes dx,$$

and so $g = (1 + f'^2)dx^2$. This is a 1×1 matrix, so $\sqrt{\det g} = \sqrt{1 + f'^2}$ and

$$\int_a^b \sqrt{1 + f'(x)^2} dx$$

is the formula for the arclength.

For (b), we again compute the pullback

$$(\varphi^{-1})^*dz = f_x dx + f_y dy,$$

and so

$$(\varphi^{-1})^*g_{\text{Euc}} = (1 + (f_x)^2)dx \otimes dx + (1 + (f_y)^2)dy \otimes dy + f_x f_y dx \otimes dy + f_x f_y dy \otimes dx.$$

where $g_{\text{Euc}} = dx^2 + dy^2 + dz^2$. Written in another way:

$$g_{ij} = \begin{bmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{bmatrix}.$$

The determinant is

$$\det g_{ij} = 1 + f_x^2 + f_y^2$$

and so the area formula is

$$\int_U \sqrt{1 + f_x^2 + f_y^2} dx dy.$$

For (c), we differentiate $F(x, y, f(x, y)) = 0$ to derive

$$F_1 + F_3 \partial_x f = 0$$

$$F_2 + F_3 \partial_y f = 0.$$

Therefore we substitute $f_x = -F_1/F_3$ and $f_y = -F_2/F_3$ into the area formula above to obtain

$$\int_U \sqrt{1 + f_x^2 + f_y^2} dx dy = \int_U \left(1 + \frac{F_1^2}{F_3^2} + \frac{F_2^2}{F_3^2} \right)^{1/2} dx dy$$

which simplifies to

$$\int_U \frac{(F_1^2 + F_2^2 + F_3^2)^{1/2}}{|F_3|} dx dy,$$

as required. If $F = x^2 + y^2 + z^2 - 1$, then

$$\frac{|\nabla F|}{\partial_z F} = \frac{1}{z}.$$

The area is then

$$\int_U \frac{|\nabla F|}{\partial_z F} = \int_U \frac{1}{z} dx dy = \int_0^1 \int_0^1 \frac{dx dy}{\sqrt{1 - x^2 - y^2}}.$$

In polar coordinates we have

$$\int_U \frac{|\nabla F|}{\partial_z F} = \int_0^{2\pi} \int_0^1 \frac{r dr}{\sqrt{1 - r^2}} = 2\pi.$$

□

Problem 3. Let $0 < b < a$. Let $f : [0, 2\pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ be given by

$$f(u, v) = ((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u).$$

This is a parametrization of the torus of revolution T^2 .

(a) Compute the metric g_{ij} on T^2 in (u, v) -coordinates, where the metric is induced by the Euclidean metric on $\mathbb{R}^3$. Give your answer as a 2×2 matrix function of (u, v) .

(b) Compute the length of the curves $c_1 : [0, 2\pi] \rightarrow T^2$ given by

$$c_1(t) = f(\pi, t),$$

and $c_2 : [0, 2\pi] \rightarrow T^2$ given by $c_2(t) = f(t, \pi)$.

(c) Compute the area of the torus.

Solution. For (a), we compute

$$\frac{\partial f}{\partial u} = (-b \sin u \cos v, -b \sin u \sin v, b \cos u),$$

and

$$\frac{\partial f}{\partial v} = (-(a + b \cos u) \sin v, (a + b \cos u) \cos v, 0).$$

Thus

$$g_{uu} = b^2(\sin u)^2(\cos v)^2 + b^2(\sin u)^2(\sin v)^2 + b^2(\cos u)^2 = b^2,$$

$$g_{uv} = (b \sin u \cos v)(a + b \cos u) \sin v - (b \sin u \sin v)(a + b \cos u) \cos v = 0,$$

$$g_{vv} = (a + b \cos u)^2(\sin v)^2 + (a + b \cos u)^2(\cos v)^2 = (a + b \cos u)^2.$$

Altogether, we have

$$g_{ij} = \begin{pmatrix} b^2 & 0 \\ 0 & (a + b \cos u)^2 \end{pmatrix}.$$

For (b), in the parametrization U , the curve c_1 is given by $\gamma_1(t) = (\pi, t)$ with constant speed $\dot{\gamma}_1 = (0, 1)$. Its length is

$$|\dot{\gamma}_1|_g^2 = g_{ij}(\gamma)(\dot{\gamma}_1)^i(\dot{\gamma}_1)^j = g_{vv}.$$

Evaluated at the point (π, t) , we obtain

$$|\dot{\gamma}_1|_g^2(t) = (a + b \cos \pi)^2 = (a - b)^2.$$

Therefore

$$\begin{aligned} L_1 &= \int_0^{2\pi} |\dot{\gamma}_1|_g dt \\ &= \int_0^{2\pi} |a - b| dt \\ &= (2\pi)(a - b). \end{aligned}$$

In the parametrization U , the curve c_2 is given by $\gamma_2(t) = (t, \pi)$ with constant speed $\dot{\gamma}_2 = (1, 0)$. Its length is

$$|\dot{\gamma}_2|_g^2 = g_{ij}(\gamma)(\dot{\gamma}_2)^i(\dot{\gamma}_2)^j = g_{uu} = b^2.$$

Therefore the length of c_2 is given by

$$\begin{aligned} L_2 &= \int_0^{2\pi} |\dot{\gamma}_2|_g dt \\ &= 2\pi b. \end{aligned}$$

For (c), the area is given by

$$A = \int_0^{2\pi} \int_0^{2\pi} \sqrt{\det g_{ij}} du dv.$$

Since $0 < b < a$, we have $|a + b \cos u| = (a + b \cos u)$ and so

$$A = (2\pi b) \int_0^{2\pi} (a + b \cos u) du.$$

Therefore

$$A = 4\pi^2 ab.$$

□

Problem 4. Show that the manifold TM is always orientable, even if M is not.

Solution. Let $\{(U, x)\}$ be an atlas on M . Given a chart (U, x) , denote the coordinates on the corresponding chart on TM by

$$w = (x^i, v^i).$$

If $\tilde{x}^i = f^i(x)$ are a change of coordinates on M , then

$$\begin{aligned} \tilde{x}^i &= f^i(x) \\ \tilde{v}^i &= \frac{\partial \tilde{x}^i}{\partial x^p} v^p. \end{aligned}$$

We need to compute

$$\frac{\partial \tilde{w}}{\partial w} = \begin{bmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial x} & \frac{\partial \tilde{v}}{\partial v} \end{bmatrix}$$

which is

$$\frac{\partial \tilde{w}}{\partial w} = \begin{bmatrix} \frac{\partial \tilde{x}}{\partial x} & 0 \\ * & \frac{\partial \tilde{x}}{\partial x} \end{bmatrix}.$$

Therefore

$$\det \frac{\partial \tilde{w}}{\partial w} = \left(\det \frac{\partial \tilde{x}}{\partial x} \right)^2 > 0.$$

Hence every change of coordinates on this atlas on TM has positive Jacobian determinant. Therefore TM is orientable. □

Problem 5. Let $L \rightarrow \mathbb{RP}^1$ be the tautological line bundle

$$L = \left\{ ([x_0, x_1], w) \in \mathbb{RP}^1 \times \mathbb{R}^2 : w \in [x_0, x_1] \right\}.$$

Recall that the transition function is $\tau_{01} = x_0/x_1$. Show that the manifold L is not orientable.

Solution. We denote a point $[X_0, X_1] \in \mathbb{RP}^1$ with capitals to distinguish from local coordinates with lower case x . The two coordinate charts covering $\mathbb{RP}^1$ are

$$U = \{[X_0, X_1] : X_0 \neq 0\}, \quad \tilde{U} = \{[X_0, X_1] : X_1 \neq 0\}$$

with coordinates

$$x = \frac{X_1}{X_0} \quad \text{over } U \cong \mathbb{R}, \quad \tilde{x} = \frac{X_0}{X_1} \quad \text{over } \tilde{U} \cong \mathbb{R}.$$

Using the relation $v = \tau_{01}\tilde{v}$ and transition function $\tau_{01} = X_0/X_1$, we obtain two charts on L

$$\left(\pi^{-1}(U), (x, v) \right), \quad \left(\pi^{-1}(\tilde{U}), (\tilde{x}, \tilde{v}) \right)$$

with change of coordinates

$$\begin{aligned} \tilde{x} &= \frac{1}{x} \\ \tilde{v} &= xv. \end{aligned}$$

Let $\omega \in \Omega^2(L)$ be a top form. Then

$$\omega = f_0(x, v) dx \wedge dv \quad \text{over } U$$

and

$$\omega = f_1(\tilde{x}, \tilde{v}) d\tilde{x} \wedge d\tilde{v} \quad \text{over } \tilde{U}.$$

Since on the overlap,

$$d\tilde{x} = -\frac{1}{x^2} dx, \quad d\tilde{v} = v dx + x dv$$

we have

$$d\tilde{x} \wedge d\tilde{v} = -\frac{1}{x} dx \wedge dv.$$

Hence $\omega \in \Omega^2(L)$ creates a pair of functions $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$f_0(x, v) = -\frac{1}{x} f_1(x^{-1}, xv) \quad \text{on } \{x \neq 0\}.$$

We can now prove that every top form ω must vanish somewhere. Suppose $f_1(0, 0) > 0$; if $f_1(0, 0) = 0$ then we are done and if $f_1(0, 0) < 0$ the following argument can be

repeated verbatim with all inequalities flipped. By continuity, there exists $\epsilon > 0$ such that $f_1(s, 0) > 0$ for all $|s| < \epsilon$. Then

$$f_0(R, 0) < 0, \quad R > \epsilon^{-1},$$

while on the other hand,

$$f_0(-R, 0) > 0, \quad R > \epsilon^{-1}.$$

It follows that the continuous function $x \mapsto f_0(x, 0)$ must attain zero at some point $x_* \in (-R, R)$. Hence ω vanishes somewhere. An orientable manifold admits a nowhere vanishing top form, thus L is not orientable. □

Problem 6. Let $T^2 = S^1 \times S^1 \subseteq \mathbb{R}^4$ be given by

$$T^2 = \left\{ (w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 = y^2 + z^2 = 1 \right\}.$$

Let

$$\omega = xz \, dw \wedge dy, \quad \omega \in \Omega^2(\mathbb{R}^4).$$

Compute the integral

$$\int_{T^2} \omega.$$

You should declare an orientation on T^2 before evaluating this integral.

Solution. Introduce a local θ -coordinate

$$\begin{aligned} w &= \cos \theta \\ x &= \sin \theta \end{aligned}$$

for the first S^1 and a local φ -coordinate

$$\begin{aligned} y &= \cos \varphi \\ z &= \sin \varphi \end{aligned}$$

for the second S^1 . Orient T^2 by the nowhere vanishing top form

$$d\theta \wedge d\varphi \in \Omega^2(T^2).$$

Note that $d\theta$ and $d\varphi$ are globally well-defined as they are invariant under local coordinate changes $\theta \mapsto \theta + 2\pi$. We write ω in these angle coordinates.

$$\omega = (\sin \theta)(\sin \varphi) d(\cos \theta) \wedge d(\cos \varphi).$$

Simplifying gives

$$\omega = (\sin \theta)^2 (\sin \varphi)^2 d\theta \wedge d\varphi.$$

The integral is then

$$\int_{T^2} \omega = \int_0^{2\pi} (\sin \theta)^2 d\theta \int_0^{2\pi} (\sin \varphi)^2 d\varphi = \pi^2.$$

If the opposite orientation is chosen, the integral will be $-\pi^2$. □

Problem 7. Let $\tilde{\omega} \in \Omega^2(\mathbb{R}^3)$ be defined by

$$\tilde{\omega} = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy.$$

Let $\iota : S^2 \rightarrow \mathbb{R}^3$ and $\omega = \iota^* \tilde{\omega} \in \Omega^2(S^2)$. Give S^2 the metric $g = \iota^* g_{\text{Euc}}$ and the outward-pointing orientation via $S^2 = \partial B^3$.

(a) Show that $\omega = d\text{vol}_g$.

(b) Compute the integral $\int_{S^2} \omega$.

(c) Show that there does not exist $\alpha \in \Omega^1(S^2)$ such that $\omega = d\alpha$.

Solution. For (a), let $S^2 \cap \{z > 0\}$ be given coordinates

$$\varphi(x, y, \sqrt{1 - x^2 - y^2}) = (x, y).$$

We computed in a previous problem set that

$$\iota^* \tilde{\omega} = \frac{1}{z} dx \wedge dy$$

in these coordinates. Similarly, we computed earlier in this problem set that

$$d\text{vol}_g = \sqrt{\det g_{ij}} dx \wedge dy = \frac{|\nabla F|}{|\partial_z F|} dx \wedge dy = \frac{1}{z} dx \wedge dy.$$

Repeating this calculation over $S^2 \cap \{z < 0\}$ with coordinates (x, y) also gives

$$\iota^* \tilde{\omega} = \frac{1}{z} dx \wedge dy = d\text{vol}_g.$$

Therefore

$$\iota^* \tilde{\omega} = d\text{vol}_g \text{ on } S^2 \cap \{z \neq 0\}.$$

By continuity of both sides of the equation, this identity holds on all of S^2 . Therefore $\omega = d\text{vol}_g$.

For (b), we use Stokes's theorem

$$\int_{S^2} \omega = \int_{B^3} d\omega = 3 \int_{B^3} dx dy dz = 3\text{Vol}(B^3) = (3) \frac{4}{3} \pi = 4\pi.$$

For (c), we note that if $\omega = d\alpha$ then

$$\int_{S^2} \omega = \int_{S^2} d\alpha = 0$$

which is a contradiction. □

Problem 8. Let $M = S^2 \cap \{z \geq 0\}$ be the upper hemisphere with outward orientation and inclusion $\iota : M \hookrightarrow \mathbb{R}^3$. Let

$$\tilde{\alpha} = 2dx \wedge dy - dx \wedge dz, \quad \tilde{\alpha} \in \Omega^2(\mathbb{R}^3),$$

and define $\alpha = \iota^* \tilde{\alpha}$. Compute the surface integral $\int_M \alpha$.

Solution. We can find an anti-derivative of $\tilde{\alpha}$ by inspection. For example,

$$\tilde{\alpha} = d \left[2x \, dy - x \, dz \right],$$

so we let

$$\tilde{\eta} = 2x \, dy - x \, dz, \quad \eta = \iota^* \tilde{\eta}.$$

By Stokes's theorem,

$$\int_M \alpha = \int_{\partial M} \eta.$$

The boundary circle is the equator

$$\partial M = \{(x, y, 0) : x^2 + y^2 = 1\},$$

with counterclockwise orientation when viewed from above. It is parametrized via the orientation-preserving map $\gamma : S^1 \rightarrow \partial M$ given by

$$\gamma(e^{i\theta}) = (\cos \theta, \sin \theta, 0).$$

We pullback

$$\gamma^* \eta = 2(\cos \theta) d(\sin \theta) - 0 = 2 \cos^2 \theta \, d\theta.$$

The integral is then

$$\int_{\partial M} \eta = \int_0^{2\pi} (2 \cos^2 \theta) \, d\theta = 2\pi.$$

Therefore $\int_M \alpha = 2\pi$. □

Problem 9. Cover S^1 by the two charts (U, θ) and $(\tilde{U}, \tilde{\theta})$ given by

$$U = \{e^{i\theta} : \theta \in (0, 2\pi)\}, \quad \tilde{U} = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}.$$

Let $c \in \mathbb{R}$ and consider the 1-forms

$$\alpha = \sin^2 \theta \, d\theta, \quad \beta = c \, d\theta.$$

Find c and $f \in C^\infty(S^1)$ such that $\alpha - \beta = df$.

Solution. First, we find what the constant c would need to be. Integrating both sides of $\alpha - \beta = df$ gives

$$\int_{S^1} \alpha = \int_{S^1} \beta$$

by Stokes's theorem. This condition is

$$\int_0^{2\pi} \sin^2 \theta \, d\theta = c \int_0^{2\pi} d\theta \implies \pi = (c)(2\pi)$$

and so $c = 1/2$. We now look for $f(\theta)$ solving $\alpha - \beta = df$, which in coordinates is

$$\sin^2 \theta - \frac{1}{2} = \partial_\theta f \implies -\frac{1}{2} \cos(2\theta) = \frac{df}{d\theta}.$$

Taking the antiderivative gives

$$f(\theta) = -\frac{1}{4} \sin(2\theta) + C.$$

Since f is 2π -periodic, the local expressions satisfy $f(\theta) = f(\tilde{\theta})$, and so f is a well-defined function on S^1 . $\square$

Problem 10. Consider the 1-form

$$\alpha = -\frac{y \, dx}{x^2 + y^2} + \frac{x \, dy}{x^2 + y^2}.$$

defined on $\mathbb{R}^2 \setminus \{0\}$.

(a) Show that $d\alpha = 0$.

(b) Show that there does not exist $f : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ such that $\alpha = df$.

(c) Find an explicit $f : \mathbb{R}^2 \setminus \{x = 0\} \rightarrow \mathbb{R}$ such that $\alpha = df$. Hint: consider $\arctan u$.

Solution. For (a), we compute

$$d\alpha = \partial_y \left(-\frac{y}{x^2 + y^2} \right) dy \wedge dx + \partial_x \left(\frac{x}{x^2 + y^2} \right) dx \wedge dy$$

which leads to

$$d\alpha = \left(\frac{-x^2 + y^2}{(x^2 + y^2)^2} \right) dy \wedge dx + \left(\frac{-x^2 + y^2}{(x^2 + y^2)^2} \right) dx \wedge dy.$$

Since $dy \wedge dx = -dx \wedge dy$, these two terms cancel.

$$d\alpha = 0.$$

For (b), suppose $\alpha = df$. Let $S^1 = \{x^2 + y^2 = 1\}$. Then by Stokes's theorem

$$\int_{S^1} \alpha = \int_{S^1} df = 0.$$

On the other hand, using the coordinates $\iota : (0, 2\pi) \rightarrow S^1$, $\theta \mapsto (\cos \theta, \sin \theta)$, we can compute

$$\iota^* \alpha = -(\sin \theta) d(\cos \theta) + (\cos \theta) d(\sin \theta)$$

which simplifies to

$$\iota^* \alpha = d\theta.$$

Thus

$$\int_{S^1 \setminus \{0\}} \alpha = \int_0^{2\pi} d\theta = 2\pi.$$

Adding back the point $0 \in S^1$ does not change the integral, hence

$$\int_{S^1} \alpha = 2\pi \neq 0,$$

which is a contradiction.

For (c), we follow the hint and note that

$$\frac{d}{du} \arctan u = \frac{1}{1+u^2}.$$

We can then guess the combination

$$\frac{\partial}{\partial y} \arctan \frac{y}{x} = \frac{1}{1+(y/x)^2} \frac{1}{x} = \frac{x}{x^2+y^2}$$

and

$$\frac{\partial}{\partial x} \arctan \frac{y}{x} = -\frac{1}{1+(y/x)^2} \frac{y}{x^2} = -\frac{y}{x^2+y^2}.$$

We see that

$$d \arctan \frac{y}{x} = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy,$$

and so $f = \arctan(y/x)$. This function is not defined on all of $\mathbb{R}^2 \setminus \{0\}$, which is consistent with (b).

□