

Problem Set 7: Cohomology

Problem 1. Let M be a compact closed connected oriented manifold. Show that if $\alpha \in \Omega^{\dim M}(M)$ satisfies $\int_M \alpha = 0$, then $\alpha = d\beta$ for some $\beta \in \Omega^{\dim M - 1}(M)$.

Problem 2. Compute $H^*(S^3)$.

Problem 3. Compute $H^*(S^4)$.

Problem 4. Let $M = \mathbb{R}^3 \setminus \{p_1, p_2, p_3\}$, where the p_i are distinct points. Compute $H^*(M)$. Conclude that any vector field $\vec{F}$ on $\mathbb{R}^3 \setminus \{p_1, p_2, p_3\}$ satisfying $\nabla \times \vec{F} = \vec{0}$ admits a scalar potential: $\vec{F} = \nabla \phi$.

Problem 5. Let $M = \mathbb{R}^3 \setminus \{z\text{-axis}\}$. Compute $H^*(M)$.

Problem 6. Let $M = T^2 \setminus \{\text{pt}\}$. Compute $H^*(M)$.

Problem 7. This problem is not about cohomology, but results here will be used in the next problems.

(a) Consider the top form

$$dV = dx_1 \wedge \cdots \wedge dx_{n+1}, \quad dV \in \Omega^{n+1}(\mathbb{R}^{n+1}),$$

the radial vector field

$$\xi = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}, \quad \xi \in \Gamma(T\mathbb{R}^{n+1}),$$

and the differential form

$$\tilde{\omega} = (\xi \lrcorner dV), \quad \omega \in \Omega^n(\mathbb{R}^{n+1}).$$

Let $\iota : S^n \rightarrow \mathbb{R}^{n+1}$ be inclusion and let

$$\omega_{S^n} = \iota^* \tilde{\omega} \tag{1}$$

be a top form on S^n . Compute $\int_{S^n} \omega_{S^n}$.

(b) From

$$\tilde{\tau} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}, \quad x \mapsto -x,$$

define the antipodal map

$$\tau : S^n \rightarrow S^n, \quad \tau = \tilde{\tau}|_{S^n}.$$

Compute $\tau^* \omega_{S^n}$.

Problem 8. Compute

$$\begin{aligned} H^0(\mathbb{RP}^2) &= \mathbb{R} \\ H^1(\mathbb{RP}^2) &= 0 \\ H^2(\mathbb{RP}^2) &= 0. \end{aligned}$$

To do this, view $\mathbb{RP}^2$ as the quotient of S^2 by the antipodal map $\tau : S^2 \rightarrow S^2$.

$$\mathbb{RP}^2 = S^2 / \langle \tau \rangle.$$

It comes with a quotient map

$$\pi : S^2 \rightarrow \mathbb{RP}^2, \quad x \mapsto [x].$$

Start from $\chi \in \Omega^k(\mathbb{RP}^2)$, with $k = 1$ or $k = 2$, satisfying $d\chi = 0$. Lift χ to the double cover and show

$$[\pi^* \chi] = 0 \in H^k(S^2).$$

Then descend back down to $\Omega^k(\mathbb{RP}^2)$ to prove $[\chi] = 0$.

Problem 9. In this problem, we show that every vector field on an even dimensional sphere must have a zero.

(a) Suppose S^n admits a nowhere vanishing vector field $V \in \Gamma(TS^n)$. Then

$$F : S^n \times [0, \pi] \rightarrow S^n, \quad F(x, t) = (\cos t)x + (\sin t)\frac{V}{|V|}$$

is a homotopy between the antipodal map τ and id. Use the induced maps on cohomology to prove that an even dimensional sphere cannot admit a nowhere vanishing vector field.

(b) Construct a nowhere vanishing vector field on an odd dimensional sphere.
