

Problem Set 7: Cohomology

Problem 1. Let M be a compact closed connected oriented manifold. Show that if $\alpha \in \Omega^{\dim M}(M)$ satisfies $\int_M \alpha = 0$, then $\alpha = d\beta$ for some $\beta \in \Omega^{\dim M-1}(M)$.

Solution. Recall that $H^0(M)$ consists of constant functions c . Poincaré duality on top forms is the statement that

$$\text{PD} : H^{\dim M}(M) \rightarrow (H^0(M))^*, \quad \text{PD}([\alpha])(c) = c \int_M \alpha$$

is an isomorphism. If $\int_M \alpha = 0$, then $\text{PD}([\alpha])$ is the zero linear functional. Since PD is injective,

$$[\alpha] = 0.$$

This means $\alpha = d\beta$ for some $\beta \in \Omega^{\dim M-1}(M)$. □

Problem 2. Compute $H^*(S^3)$.

Solution. The relevant vector spaces are:

$$H^0(S^3), \quad H^1(S^3), \quad H^2(S^3), \quad H^3(S^3).$$

We know $H^0(S^3) \cong \mathbb{R}$, and by Poincaré duality

$$\dim H^0(S^3) = 1, \quad \dim H^1(S^3) = \dim H^2(S^3), \quad \dim H^3(S^3) = 1.$$

So we only need to compute $H^1(S^3)$. Decompose S^3 using stereographic charts:

$$S^3 = U \cup \tilde{U},$$

where

$$U \cong \mathbb{R}^3, \quad \tilde{U} \cong \mathbb{R}^3, \quad U \cap \tilde{U} \cong \mathbb{R}^3 \setminus \{0\}.$$

Hence

$$U \simeq \{*\}, \quad \tilde{U} \simeq \{*\}, \quad U \cap \tilde{U} \simeq S^2.$$

The Mayer-Vietoris sequence starts with

$$0 \rightarrow H^0(S^3) \rightarrow H^0(U) \oplus H^0(\tilde{U}) \rightarrow H^0(U \cap \tilde{U}) \rightarrow H^1(S^3) \rightarrow 0.$$

Given what we know, this is

$$0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R} \rightarrow H^1(S^3) \rightarrow 0.$$

For an exact sequence of vector spaces starting and ending with zero, the alternating sum of the dimensions is zero.

$$0 = 1 - 2 + 1 - \dim H^1(S^3).$$

Therefore $H^1(S^3) = 0$. Altogether

$$\begin{aligned} H^0(S^3) &= \mathbb{R} \\ H^1(S^3) &= 0 \\ H^2(S^3) &= 0 \\ H^3(S^3) &= \mathbb{R}. \end{aligned}$$

□

Problem 3. Compute $H^*(S^4)$.

Solution. The relevant vector spaces are:

$$H^0(S^4), \quad H^1(S^4), \quad H^2(S^4), \quad H^3(S^4), \quad H^4(S^4).$$

We know $H^0(S^4) \cong \mathbb{R}$, and by Poincaré duality

$$\dim H^0(S^4) = 1, \quad \dim H^1(S^4) = \dim H^3(S^4), \quad \dim H^4(S^4) = 1.$$

So we only need to compute $H^1(S^4)$ and $H^2(S^4)$. Decompose S^4 using stereographic charts:

$$S^4 = U \cup \tilde{U},$$

where

$$U \cong \mathbb{R}^4, \quad \tilde{U} \cong \mathbb{R}^4, \quad U \cap \tilde{U} \cong \mathbb{R}^4 \setminus \{0\}.$$

Hence

$$U \simeq \{*\}, \quad \tilde{U} \simeq \{*\}, \quad U \cap \tilde{U} \simeq S^3.$$

The Mayer-Vietoris sequence starts with

$$0 \rightarrow H^0(S^4) \rightarrow H^0(U) \oplus H^0(\tilde{U}) \rightarrow H^0(U \cap \tilde{U}) \rightarrow H^1(S^4) \rightarrow 0.$$

Given what we know, this is

$$0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R} \rightarrow H^1(S^4) \rightarrow 0.$$

For an exact sequence of vector spaces starting and ending with zero, the alternating sum of the dimensions is zero.

$$0 = 1 - 2 + 1 - \dim H^1(S^4).$$

Therefore $H^1(S^4) = 0$. It remains to calculate $H^2(S^4)$. Part of the Mayer-Vietoris sequence is

$$H^1(U \cap \tilde{U}) \rightarrow H^2(S^4) \rightarrow H^2(U) \oplus H^2(\tilde{U}).$$

Since $U \cap \tilde{U} \simeq S^3$, and we calculated in an earlier problem that $H^1(S^3) = 0$, this implies

$$0 \rightarrow H^2(S^4) \rightarrow 0.$$

Therefore $H^2(S^4) = 0$. Altogether

$$\begin{aligned} H^0(S^4) &= \mathbb{R} \\ H^1(S^4) &= 0 \\ H^2(S^4) &= 0 \\ H^3(S^4) &= 0 \\ H^4(S^4) &= \mathbb{R}. \end{aligned}$$

□

Problem 4. Let $M = \mathbb{R}^3 \setminus \{p_1, p_2, p_3\}$, where the p_i are distinct points. Compute $H^*(M)$. Conclude that any vector field $\vec{F}$ on $\mathbb{R}^3 \setminus \{p_1, p_2, p_3\}$ satisfying $\nabla \times \vec{F} = \vec{0}$ admits a scalar potential: $\vec{F} = \nabla \phi$.

Solution. Decompose $\mathbb{R}^3 = M \cup V$, where $V = \sqcup_{i=1}^3 D_i$ and each D_i is a small open ball centered at the point p_i . The overlap is

$$M \cap V = \sqcup_{i=1}^3 (D_i \setminus \{p_i\}).$$

These opens are homotopic to

$$V \simeq \sqcup_{i=1}^3 \{*\}, \quad M \cap V \simeq \sqcup_{i=1}^3 S^2.$$

Since M is connected, we know $H^0(M) = \mathbb{R}$. We look for H^1 in the Mayer-Vietoris sequence and find

$$H^1(\mathbb{R}^3) \rightarrow H^1(M) \oplus H^1(V) \rightarrow H^1(M \cap V).$$

We know $H^1(S^2) = 0$, so this sequence is

$$0 \rightarrow H^1(M) \rightarrow 0,$$

and $H^1(M) = 0$. For H^2 , we continue along the Mayer-Vietoris sequence and find the segment

$$H^2(\mathbb{R}^3) \rightarrow H^2(M) \oplus H^2(V) \rightarrow H^2(M \cap V) \rightarrow H^3(\mathbb{R}^3).$$

This sequence is

$$0 \rightarrow H^2(M) \rightarrow \mathbb{R}^3 \rightarrow 0.$$

Therefore $H^2(M) \cong \mathbb{R}^3$. Finally, we find H^3 in the sequence.

$$H^3(\mathbb{R}^3) \rightarrow H^3(M) \oplus H^3(V) \rightarrow H^3(M \cap V).$$

This sequence is

$$0 \rightarrow H^3(M) \rightarrow 0,$$

and so $H^3(M) = 0$. Altogether,

$$H^0(M) = \mathbb{R}$$

$$H^1(M) = 0$$

$$H^2(M) = \mathbb{R}^3$$

$$H^3(M) = 0.$$

Given a vector field

$$\vec{F} = (F_1, F_2, F_3),$$

the condition $\nabla \times \vec{F} = 0$ is

$$\partial_1 F_2 = \partial_2 F_1, \quad \partial_1 F_3 = \partial_3 F_1, \quad \partial_2 F_3 = \partial_3 F_2.$$

Therefore the 1-form

$$\alpha = F_1 dx + F_2 dy + F_3 dz$$

satisfies

$$\begin{aligned} d\alpha &= (\partial_1 F_2 - \partial_2 F_1) dx \wedge dy + (\partial_2 F_3 - \partial_3 F_2) dy \wedge dz + (\partial_3 F_1 - \partial_1 F_3) dz \wedge dx \\ &= 0. \end{aligned}$$

Since $H^1 = 0$, we conclude that there exists a function ϕ such that $\alpha = d\phi$. In components, this means

$$F_1 dx + F_2 dy + F_3 dz = \partial_1 \phi dx + \partial_2 \phi dy + \partial_3 \phi dz$$

and hence $\vec{F} = \nabla \phi$. □

Problem 5. Let $M = \mathbb{R}^3 \setminus \{z\text{-axis}\}$. Compute $H^*(M)$.

Solution. Since

$$M = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R},$$

this manifold deformation retracts onto a circle.

$$M \simeq S^1.$$

This is because $\mathbb{R} \rightarrow \{*\}$ and $\mathbb{R}^2 \setminus \{0\} \rightarrow S^1$. Therefore

$$\begin{aligned} H^0(M) &= \mathbb{R} \\ H^1(M) &= \mathbb{R} \\ H^k(M) &= 0, \quad k \geq 2. \end{aligned}$$

□

Problem 6. Let $M = T^2 \setminus \{\text{pt}\}$. Compute $H^*(M)$.

Solution. We write the union

$$T^2 = M \cup D$$

where $D \subseteq T^2$ is a small open disk centered at the removed point. Then $M \cap D$ is a punctured disk and

$$D \simeq \{*\}, \quad M \cap D \simeq S^1.$$

We know $H^0(M) = \mathbb{R}$, and look for H^1 and H^2 in Mayer-Vietoris sequence.

$$H^0(M \cap D) \rightarrow H^1(T^2) \rightarrow H^1(M) \oplus H^1(D) \rightarrow H^1(M \cap D) \xrightarrow{\delta} H^2(T^2) \rightarrow H^2(M) \rightarrow 0.$$

We found a zero on the right, but we also need to find a zero on the left. We extend the sequence one term further to the left.

$$H^0(M) \oplus H^0(D) \xrightarrow{(a,b) \mapsto -a+b} H^0(M \cap D) \xrightarrow{\delta} H^1(T^2).$$

The map

$$H^0(M) \oplus H^0(D) \xrightarrow{(a,b) \mapsto -a+b} H^0(M \cap D)$$

is surjective since $H^0(M) \cong H^0(D) \cong H^0(M \cap D) \cong \mathbb{R}$. The sequence is exact, so

$$\dim H^0(M \cap D) = \dim \ker \delta = 1.$$

Therefore

$$1 = \dim \ker \delta + \dim \text{im } \delta \implies \dim \text{im } \delta = 0.$$

So we can write

$$0 \rightarrow H^1(T^2) \rightarrow H^1(M) \oplus H^1(D) \rightarrow H^1(M \cap D) \xrightarrow{\delta} H^2(T^2) \rightarrow H^2(M) \rightarrow 0, \quad (1)$$

but we're still stuck since we have two unknowns $H^1(M)$ and $H^2(M)$. To move forward, we use information about the connecting homomorphism. Recall that the general formula for this is: if $X = U \cup V$ with Mayer-Vietoris sequence

$$H^k(X) \rightarrow H^k(U) \oplus H^k(V) \xrightarrow{(a,b) \mapsto b-a} H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(X)$$

then

$$\delta[\eta] = [d\rho_U \wedge \eta], \quad \rho_U + \rho_V = 1, \quad \text{supp } \rho_U \subset U, \quad \text{supp } \rho_V \subset V.$$

In our setup (1), this means

$$H^1(M \cap D) \xrightarrow{\delta} H^2(T^2), \quad \delta[\eta] = [d\rho_M \wedge \eta]$$

where ρ_M is a bump function supported in M . We will show $\ker \delta = \{0\}$. Now

$$\dim H^1(M \cap D) = \dim H^2(T^2)$$

so to show $\ker \delta = \{0\}$, we just need to find one nonzero element $\delta[\eta] \neq 0$. Choose polar coordinates (r, θ) on D such that the puncture is at the origin, and bump function $\rho_M(r)$ such that $\rho_M \equiv 0$ on $\{r \leq \epsilon\}$ and $\rho_M \equiv 1$ on $\{r \geq 2\epsilon\}$. Take

$$\eta = d\theta, \quad [\eta] \in H^1(M \cap D),$$

To test whether $\delta[\eta] \in H^2(T^2)$ is zero, we integrate.

$$\int_{T^2} \delta[\eta] = \int_{T^2} d\rho_M \wedge \eta = \int_D \rho'(r) dr \wedge d\theta.$$

Therefore

$$\int_{T^2} \delta[\eta] = \int_0^{2\pi} \int_\epsilon^{2\epsilon} \rho'(r) dr d\theta = 2\pi.$$

It follows that

$$\delta[\eta] \neq 0.$$

This proves $\ker \delta = \{0\}$. We can now add this information to the exact sequence (1).

$$0 \rightarrow H^1(T^2) \rightarrow H^1(M) \oplus H^1(D) \rightarrow 0.$$

Since $H^1(T^2) \cong \mathbb{R}^2$ and $H^1(D) \cong 0$, we conclude

$$H^1(M) \cong \mathbb{R}^2.$$

The exact sequence (1) with $\ker \delta = \{0\}$ also gives

$$0 \rightarrow H^1(M \cap D) \rightarrow H^2(T^2) \rightarrow H^2(M) \rightarrow 0$$

which becomes

$$0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow H^2(M) \rightarrow 0$$

and implies $H^2(M) = 0$. Altogether,

$$\begin{aligned} H^0(M) &= \mathbb{R} \\ H^1(M) &= \mathbb{R}^2 \\ H^2(M) &= 0. \end{aligned}$$

□

Problem 7. This problem is not about cohomology, but results here will be used in the next problems.

(a) Consider the top form

$$dV = dx_1 \wedge \cdots \wedge dx_{n+1}, \quad dV \in \Omega^{n+1}(\mathbb{R}^{n+1}),$$

the radial vector field

$$\xi = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}, \quad \xi \in \Gamma(T\mathbb{R}^{n+1}),$$

and the differential form

$$\tilde{\omega} = (\xi \lrcorner dV), \quad \omega \in \Omega^n(\mathbb{R}^{n+1}).$$

Let $\iota : S^n \rightarrow \mathbb{R}^{n+1}$ be inclusion and let

$$\omega_{S^n} = \iota^* \tilde{\omega} \tag{2}$$

be a top form on S^n . Compute $\int_{S^n} \omega_{S^n}$.

(b) From

$$\tilde{\tau} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}, \quad x \mapsto -x,$$

define the antipodal map

$$\tau : S^n \rightarrow S^n, \quad \tau = \tilde{\tau}|_{S^n}.$$

Compute $\tau^* \omega_{S^n}$.

Solution. For (a), we expand

$$\tilde{\omega} = \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}$$

and compute

$$d\tilde{\omega} = (n+1) dx_1 \wedge \cdots \wedge dx_{n+1}.$$

By Stokes's theorem,

$$\int_{S^n} \omega_{S^n} = \int_{\partial B^{n+1}} \tilde{\omega} = \int_{B^{n+1}} d\tilde{\omega} = (n+1) \text{Vol}(B^{n+1}).$$

For (b), we compute $\tau^* \omega_{S^n}$.

$$\begin{aligned} \tau^* \omega_{S^n} &= \iota^* \tilde{\tau}^* \left[\sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} \right] \\ &= \iota^* \left[(-1)^{n+1} \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} \right] \\ &= (-1)^{n+1} \omega_{S^n}. \end{aligned} \tag{3}$$

□

Problem 8. Compute

$$\begin{aligned} H^0(\mathbb{RP}^2) &= \mathbb{R} \\ H^1(\mathbb{RP}^2) &= 0 \\ H^2(\mathbb{RP}^2) &= 0. \end{aligned}$$

To do this, view $\mathbb{RP}^2$ as the quotient of S^2 by the antipodal map $\tau : S^2 \rightarrow S^2$.

$$\mathbb{RP}^2 = S^2 / \langle \tau \rangle.$$

It comes with a quotient map

$$\pi : S^2 \rightarrow \mathbb{RP}^2, \quad x \mapsto [x].$$

Start from $\chi \in \Omega^k(\mathbb{RP}^2)$, with $k = 1$ or $k = 2$, satisfying $d\chi = 0$. Lift χ to the double cover and show

$$[\pi^*\chi] = 0 \in H^k(S^2).$$

Then descend back down to $\Omega^k(\mathbb{RP}^2)$ to prove $[\chi] = 0$.

Solution. We proved in a previous problem set that in this setup of a quotient by an involution $S^2 / \langle \tau \rangle$, differential forms can be understood by an isomorphism

$$\pi^* : \Omega^k(\mathbb{RP}^2) \xrightarrow{\sim} \left\{ \alpha \in \Omega^k(S^2) : \tau^*\alpha = \alpha \right\}. \quad (4)$$

Degree 1: Let $\chi \in \Omega^1(\mathbb{RP}^2)$ satisfy $d\chi = 0$. We want to show that $\chi = df$. To do this, we lift to the double cover and consider

$$\alpha = \pi^*\chi \in \Omega^1(S^2).$$

Since $H^1(S^2) = 0$, then $[\alpha] = 0$ and there exists $f \in C^\infty(S^2)$ such that

$$\pi^*\chi = df.$$

We claim that actually f is τ -invariant and so descends to the quotient. To see this, note

$$d(f - \tau^*f) = 0$$

since $\tau^*\pi^* = \pi^*$. Therefore

$$f = \tau^*f + c$$

for some constant c . Applying τ again gives

$$\tau^*f = f + c$$

hence $c = 0$. So $\tau^*f = f$. By (4), f descends to the base: $f = \pi^*f_{P^2}$ where $f_{P^2} \in C^\infty(\mathbb{RP}^2)$. We have shown

$$\pi^*\chi = \pi^*df_{P^2}.$$

Applying the inverse of the map π^* in (4) proves

$$\chi = df_{\mathbb{R}P^2} \implies [\chi] = 0 \in H^1(\mathbb{R}P^2).$$

Degree 2: We move on to show $H^2 = 0$. Given $\chi \in \Omega^2(\mathbb{R}P^2)$ with $d\chi = 0$, we must show $\chi = d\beta$. Lift to the sphere.

$$\alpha = \pi^*\chi \in \Omega^2(S^2).$$

On one hand, since $\tau^*\pi^* = \pi^*$, we know that

$$[\tau^*\alpha] = [\alpha] \in H^2(S^2).$$

On the other hand, $H^2(S^2) \cong \mathbb{R}$ implies $[\alpha] = c[\omega_{S^2}]$ for some constant c , where ω_{S^2} was defined in the previous problem, where (3) shows $\tau^*\omega_{S^2} = -\omega_{S^2}$. Thus

$$[\tau^*\alpha] = -[\alpha] \in H^2(S^2).$$

Therefore $[\alpha] = 0$ on S^2 , which means there exists $\beta_{S^2} \in \Omega^1(S^2)$ such that

$$\pi^*\chi = d\beta_{S^2}.$$

We now need to construct a 1-form β down on $\mathbb{R}P^2$ such that $d\beta = \chi$. For this purpose, we let

$$\bar{\beta} = \frac{1}{2}(\beta_{S^2} + \tau^*\beta_{S^2}), \quad \bar{\beta} \in \Omega^1(S^2).$$

Then

$$d\bar{\beta} = \frac{1}{2}(\pi^*\chi + \tau^*\pi^*\chi) = \pi^*\chi,$$

and by (4), since $\tau^*\bar{\beta} = \bar{\beta}$, there exists $\beta \in \Omega^1(\mathbb{R}P^2)$ such that

$$\bar{\beta} = \pi^*\beta.$$

Taking the exterior derivative gives

$$\pi^*(d\beta) = \pi^*\chi.$$

Applying the inverse of the map π^* in (4) proves

$$d\beta = \chi$$

as required. □

Problem 9. In this problem, we show that every vector field on an even dimensional sphere must have a zero.

(a) Suppose S^n admits a nowhere vanishing vector field $V \in \Gamma(TS^n)$. Then

$$F : S^n \times [0, \pi] \rightarrow S^n, \quad F(x, t) = (\cos t)x + (\sin t)\frac{V}{|V|}$$

is a homotopy between the antipodal map τ and id. Use the induced maps on cohomology to prove that an even dimensional sphere cannot admit a nowhere vanishing vector field.

(b) Construct a nowhere vanishing vector field on an odd dimensional sphere.

Solution. For (a), we first verify that $F(\cdot, t)$ maps to the sphere for all t .

$$\left| (\cos t)x + (\sin t)\frac{V}{|V|} \right|^2 = \cos^2 t + \sin^2 t = 1,$$

using $V \cdot x = 0$. Since $F(x, 0) = x$ and $F(x, \pi) = -x$, we see that F is a homotopy between τ and id . Therefore τ^* and id^* have the same action on cohomology classes. In particular τ^* and id^* are equal on $[\omega_{S^n}]$, where ω_{S^n} was defined in (2).

$$[\tau^* \omega_{S^n}] = [\omega_{S^n}].$$

This implies

$$\int_{S^n} \tau^* \omega_{S^n} = \int_{S^n} \omega_{S^n} = (n+1)\text{Vol}(B^{n+1}) > 0.$$

On the other hand, suppose S^n is even dimensional. Then (3), with even n , implies $\tau^* \omega_{S^n} = -\omega_{S^n}$, so

$$\int_{S^n} \tau^* \omega_{S^n} = - \int_{S^n} \omega_{S^n} = -(n+1)\text{Vol}(B^{n+1}) < 0.$$

This is a contradiction.

For (b), on an odd dimensional sphere we can define

$$V = (-x_2, -x_1, -x_4, x_3, \dots, -x_{n+1}, x_n).$$

Since

$$T_x S^n = \left\{ v \in \mathbb{R}^{n+1} : v \cdot x = 0 \right\}$$

we see that V defines a nowhere zero vector field on an odd dimensional sphere S^n . $\square$