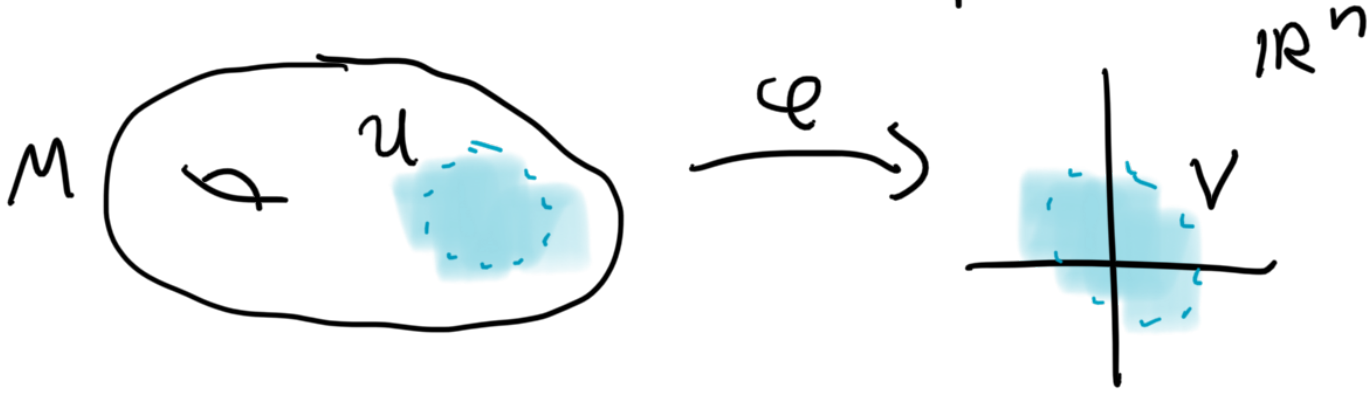
Smooth Manifolds

Def: Topological manifold M of dim n is a paracompact Hausdorff space s.t. each pt has a nbhd homeomorphic to an open set in $\mathbb{R}^n$.

Coord chart: $(\mathcal{U}, \varphi)$ with $\mathcal{U} \subseteq M$ open and $\varphi : \mathcal{U} \to V \subseteq \mathbb{R}^n$ homeomorphism.

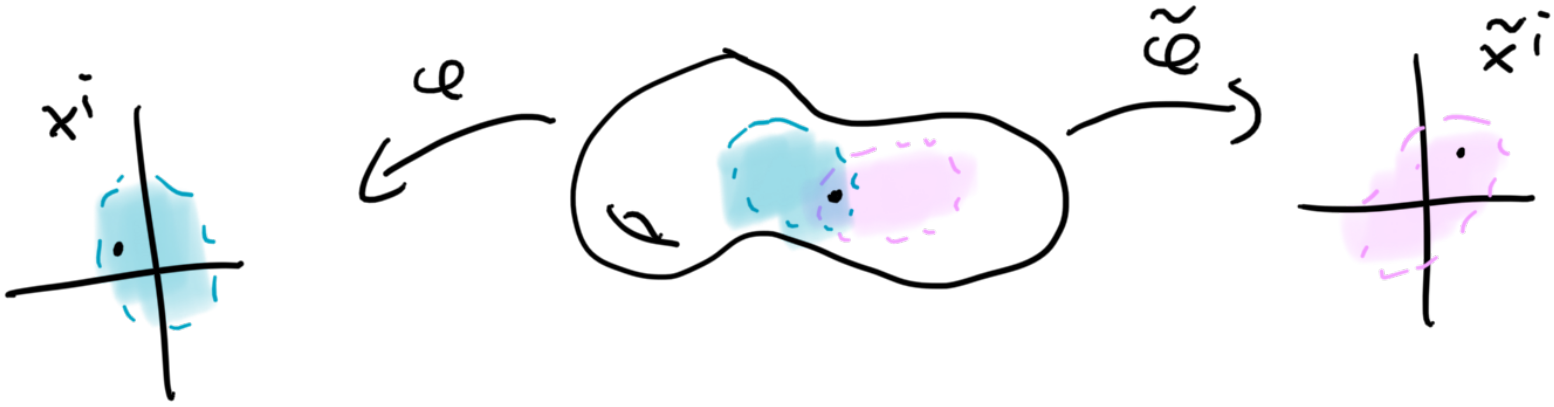


Atlas: Collection $\mathcal{A}$ of coord charts covering M.

$$M = \bigcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha .$$

Coord transformation: $(\mathcal{U}, \varphi), (\tilde{\mathcal{U}}, \tilde{\varphi})$

$\tilde{\varphi} \circ \varphi^{-1}$ is coord transformation.



Notation: $\varphi(p) = (x^1, \dots, x^n) \in \mathbb{R}^n$.

Write: $\tilde{x}^i = f^i(x)$, $\quad f = \tilde{\varphi} \circ \varphi^{-1}$

$f : x(\mathcal{U} \cap \tilde{\mathcal{U}}) \to \tilde{x}(\mathcal{U} \cap \tilde{\mathcal{U}})$.

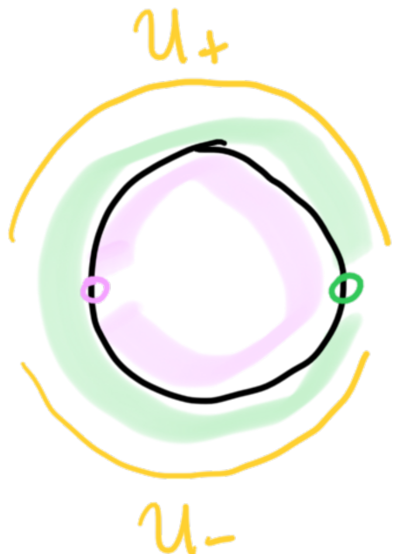
Def: Smooth mfd is topological mfd with a smooth structure (defined below).

Smooth Structure: Equivalence class of atlases s.t. coord transformations $\tilde{x}^i = f^i(x)$ are smooth with smooth inverse.

Equiv relation: $\mathcal{A} \sim \mathcal{A}'$ if $\forall\, (\mathcal{U}, \varphi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{A}'$

$\Rightarrow$ $\varphi \circ \psi^{-1} : \psi(\mathcal{U} \cap V) \to \varphi(\mathcal{U} \cap V)$ is smooth with smooth inverse.

ex) $S^1 = \{ e^{i\theta} : \theta \in [0, 2\pi] \}$



$S^1 = \mathcal{U} \cup V$

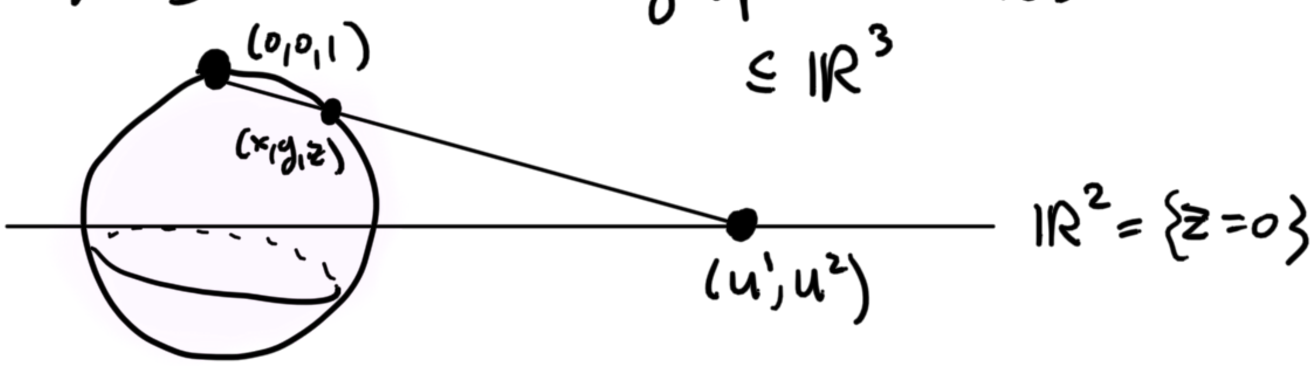
$\mathcal{U} = \{ e^{i\theta} : \theta \in (0, 2\pi) \}$

$V = \{ e^{i\varphi} : \varphi \in (-\pi, \pi) \}$

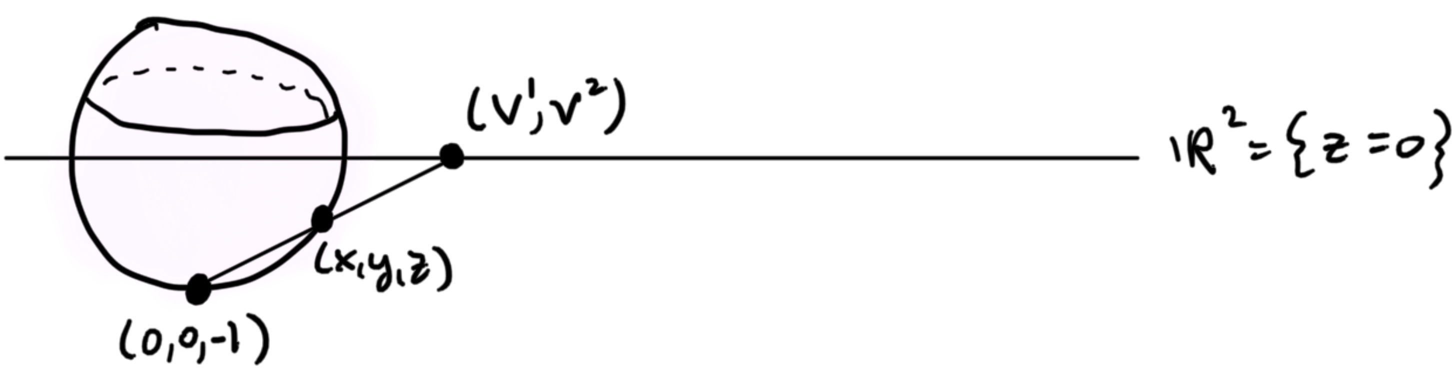
$\mathcal{U} \cap V = \mathcal{U}_+ \sqcup \mathcal{U}_-$

Coord trans: $\begin{cases} \varphi = \theta & \text{on } \mathcal{U}_+ \\ \varphi = \theta - 2\pi & \text{on } \mathcal{U}_- \end{cases}$

ex) S^2 in stereographic coords $\subseteq \mathbb{R}^3$



$$(u^1, u^2) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right).$$



$$(v^1, v^2) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right)$$

<u>Exercise</u>: Coord trans is

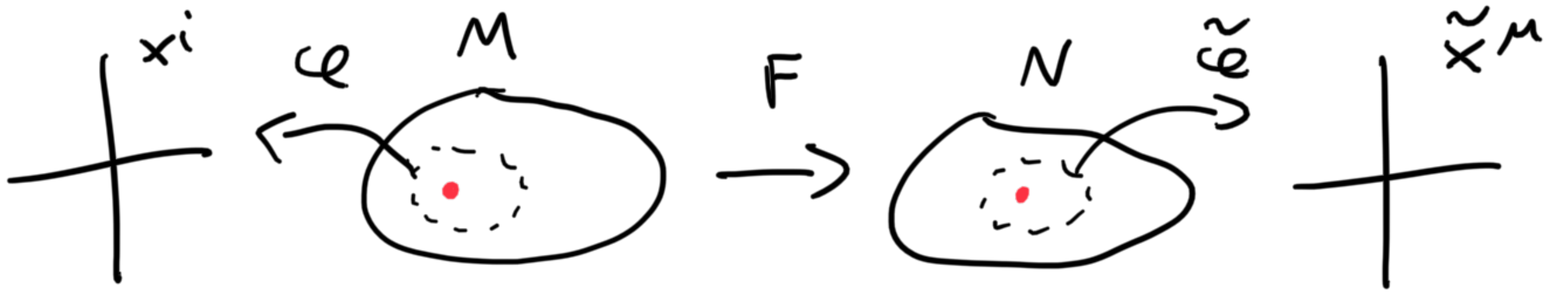
$$(v^1, v^2) = \left(\frac{u^1}{(u^1)^2 + (u^2)^2}, \frac{u^2}{(u^1)^2+(u^2)^2}\right)$$

$S^2 = \mathcal{U} \cup V$, $\quad (\mathcal{U}, u^i) \quad \mathcal{U} = S^2 \setminus (0,0,1)$
$\qquad\qquad\qquad (V, v^i) \quad V = S^2 \setminus (0,0,-1)$

<u>Def</u>: Let $F: M \to N$ be map between mfd.
Let $p \in M$, $(\mathcal{U}, x^i)$ coords near p
$(\tilde{\mathcal{U}}, \tilde{x}^\mu)$ coords near $F(p)$.

Near p, F appears as: $F^\mu(x^i) = \tilde{\varphi} \circ F \circ \varphi^{-1}$

F is smooth if all such $F^\mu(x^i)$ are smooth.



ex) Notion of smooth path $\gamma: (-\varepsilon, \varepsilon) \to M$.

<u>Defn</u>: Vector field: cover M^n by coord charts:

$$M = \bigcup \mathcal{U}_i$$

A VF is defined by collection of smooth vector functions

$$V_{\mathcal{U}}: \mathcal{U} \to \mathbb{R}^n, \quad V_{\mathcal{U}} = (V^1_{\mathcal{U}}, \dots, V^n_{\mathcal{U}}).$$

s.t. on overlaps $V: \mathcal{U} \to \mathbb{R}^n$, $\tilde{V}: \tilde{\mathcal{U}} \to \mathbb{R}^n$, $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \phi$

there holds: $\tilde{V} = \dfrac{\partial \tilde{x}}{\partial x} V$ on $\mathcal{U} \cap \tilde{\mathcal{U}}$ matrix notation

$\tilde{V}^i = \dfrac{\partial \tilde{x}^i}{\partial x^p} V^p$ on $\mathcal{U} \cap \tilde{\mathcal{U}}$ index notation

e.g. $$\begin{pmatrix} \tilde{V}^1 \\ \tilde{V}^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{x}^1}{\partial x^1} & \frac{\partial \tilde{x}^1}{\partial x^2} \\ \frac{\partial \tilde{x}^2}{\partial x^1} & \frac{\partial \tilde{x}^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$

Notation: $V = \underset{\text{components}}{V^i} \underset{\text{basis}}{\frac{\partial}{\partial x^i}} \quad \leftrightsquigarrow \quad V = \begin{pmatrix} V^1 \\ \vdots \\ V^n \end{pmatrix}$

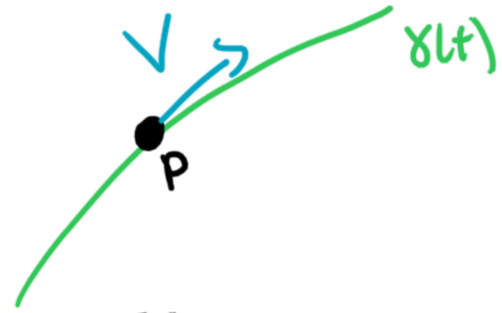
components

basis

Chain rule: $V^p \frac{\partial}{\partial x^p} = \underbrace{V^p \frac{\partial \tilde{x}^i}{\partial x^p}} \frac{\partial}{\partial \tilde{x}^i} = \underbrace{\tilde{V}^i} \frac{\partial}{\partial \tilde{x}^i}$.

If V is vector field, $V(p) \in T_pM$ is tangent vector.
$p \in M$,

Motivation for defn: Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be smooth path, $\gamma(0) = p$.



$(\mathcal{U}, x), (\tilde{\mathcal{U}}, \tilde{x})$ coords near p.

Over $\mathcal{U}$, $\gamma = x^i(t)$.

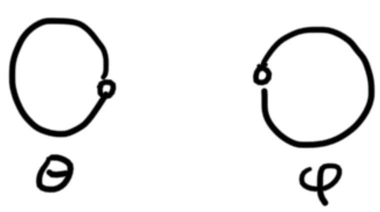
$V^i = \frac{dx^i}{dt}(0)$. defines tangent vector

Over $\tilde{\mathcal{U}}$:

$$\tilde{V}^i = \frac{d\tilde{x}^i}{dt}(0) = \frac{\partial \tilde{x}^i}{\partial x^p} \frac{dx^p}{dt}(0) = \frac{\partial \tilde{x}^i}{\partial x^p} V^p$$

Conversely, every $V \in T_pM$ comes from $V = \dot{\gamma}(0)$.
e.g. in local coords $\gamma(t) = \begin{pmatrix} p^1 \\ \vdots \\ p^n \end{pmatrix} + t \begin{pmatrix} V^1 \\ \vdots \\ V^n \end{pmatrix}$.

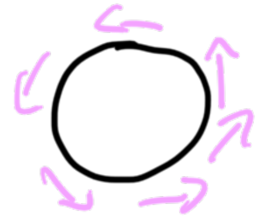
ex) $S^1 = \mathcal{U} \cup \tilde{\mathcal{U}}$



$\varphi = \begin{cases} \theta \\ \theta - 2\pi \end{cases}$

Can define V vector field: $\{(\mathcal{U}, V = 1), (\tilde{\mathcal{U}}, \tilde{V} = 1)\}$

$1 = \overset{1}{\frac{\partial \varphi}{\partial \theta}} \cdot 1$ ✓ other notation:
$V = \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \phi}$.



ex) $S^2 = \mathcal{U} \cup \tilde{\mathcal{U}}$

(v_1, v_2) (u_1, u_2)

$$(v_1, v_2) = \frac{1}{u_1^2 + u_2^2}(u_1, u_2)$$

Can define X vector field on $\tilde{\mathcal{U}}$: $\tilde{X} = \frac{\partial}{\partial u_1}$.

i.e. $\tilde{X} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

How to extend to $\mathcal{U}$?

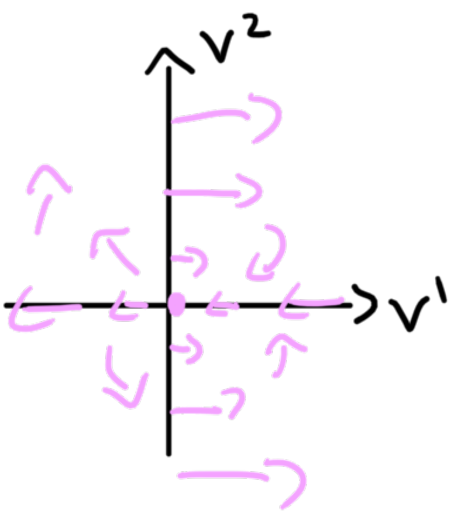
Exercise:

$$\frac{\partial v}{\partial u} = \begin{pmatrix} v_2^2 - v_1^2 & -2v_1v_2 \\ -2v_1v_2 & v_1^2 - v_2^2 \end{pmatrix}$$

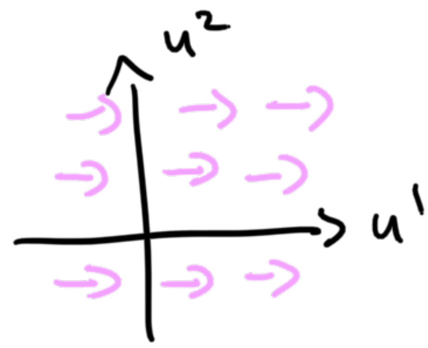
$$\left[\frac{\partial v}{\partial u}\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} v_2^2 - v_1^2 \\ -2v_1v_2 \end{pmatrix} = X.$$

$X = \frac{\partial v}{\partial u}\tilde{X}$

X is defined by: $\left\{ \left(\mathcal{U}, \begin{pmatrix} v_2^2 - v_1^2 \\ -2v_1v_2 \end{pmatrix}\right), \left(\tilde{\mathcal{U}}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \right\}$



only zero at North pole



project through North pole