Vector Bundles

ex) Vector field V.

$M = \bigcup \mathcal{U}_\alpha$, $\quad V_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}^n$

$$\left[V_{\tilde{\mathcal{U}}}\right] = \left[\frac{\partial \tilde{x}}{\partial x}\right]\left[V_{\mathcal{U}}\right] \quad \text{on } \tilde{\mathcal{U}} \cap \mathcal{U}$$

Generalization: section s.

$$s_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}^k, \quad \left[s_{\tilde{\mathcal{U}}}\right] = \underbrace{\left[C_{\tilde{\mathcal{U}}\mathcal{U}}\right]}_{k \times k \text{ matrix}}\left[s_{\mathcal{U}}\right]$$

Defn A: $E \to M$ vector bundle of rank k.

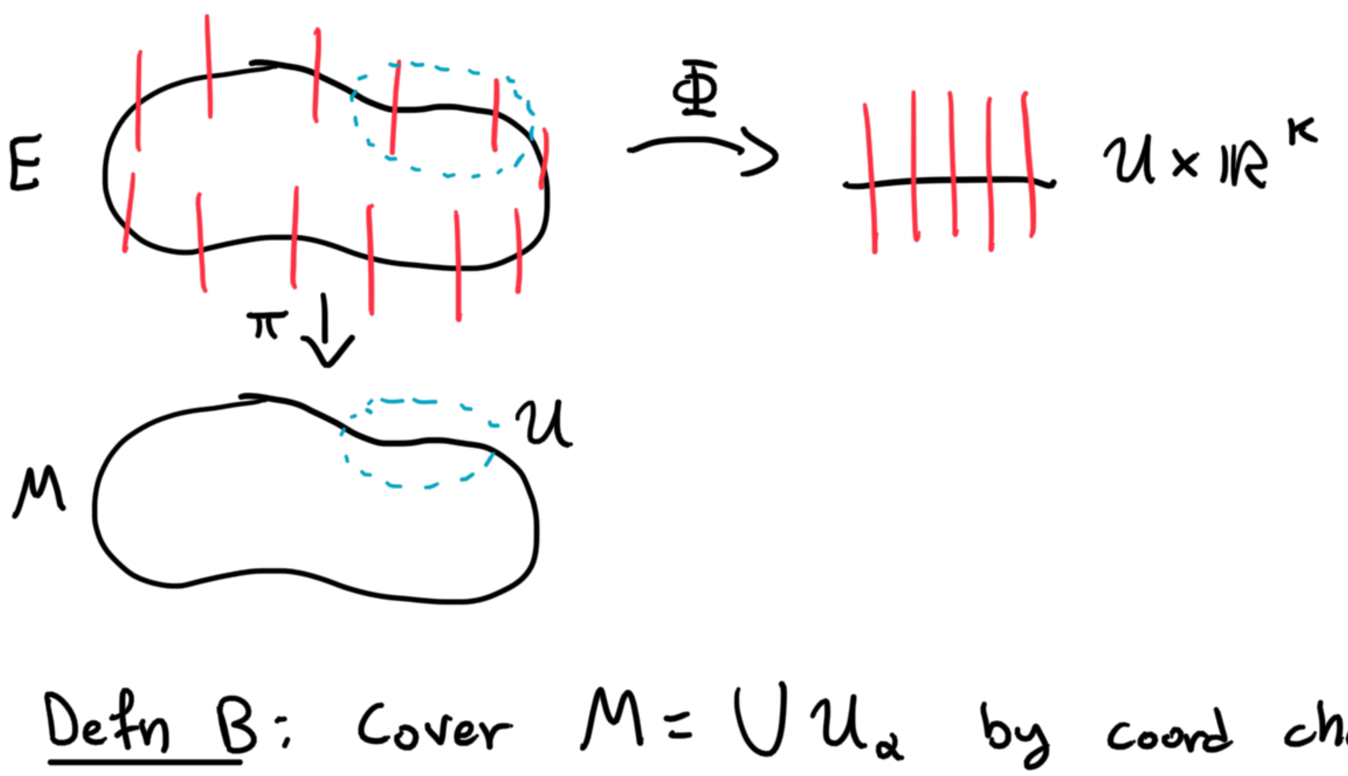
E, M mfd. $\pi : E \to M$ smooth surjection.

$\forall p \in M$, then: (1) $\pi^{-1}(p)$ is k-dim vector space

(2) $\exists$ nbhd $\mathcal{U} \ni p$ and $\Phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{R}^k$ diffeo

s.t. $\forall q \in \mathcal{U}$, then

- $\pi \circ \Phi^{-1}(q, v) = q$
- $\Phi : E_q \to \{q\} \times \mathbb{R}^k$ vector space isomorphism.



Defn B: Cover $M = \bigcup \mathcal{U}_\alpha$ by coord charts.

Give matrix-valued smooth functions on non-zero overlaps

$$c_{\mathcal{U}V} : \mathcal{U} \cap V \to GL(k, \mathbb{R})$$

s.t. $c_{UV} = c_{UW} c_{WV}$ on $U \cap V \cap W$.

(call the c_{UV} transition functions. Note: $c_{UU} = I_{K \times K}$)
$c_{UV}^{-1} = c_{VU}$

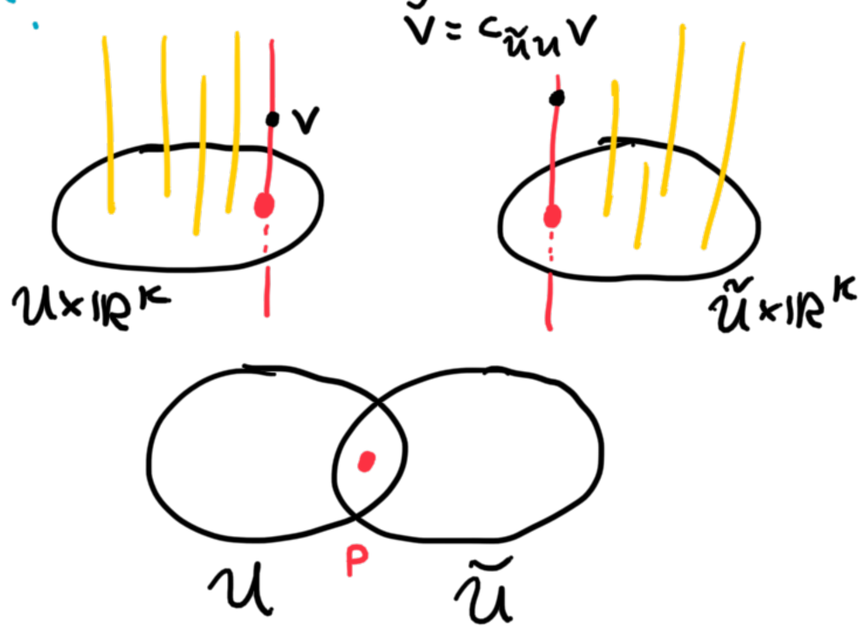
Define $E = \left(\bigcup_\alpha U_\alpha \times \mathbb{R}^K \right) / \sim$

where: for $(p, v) \in U \times \mathbb{R}^K$
$(p, \tilde{v}) \in \tilde{U} \times \mathbb{R}^K$

$$(p, v) \sim (p, \tilde{v}) \iff \tilde{v} = c_{\tilde{U}U} v.$$

Index notation: $\tilde{v}^k = [c_{\tilde{U}U}(p)]^k{}_\ell \, v^\ell$

$v = \begin{pmatrix} v^1 \\ \vdots \\ v^k \end{pmatrix} \in \mathbb{R}^k$.



Going between defn A ⇝ B

(A) ⇝ (B): Have cover $M = \bigcup U_p$ with
$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^K$ diffeo.

On $U \cap \tilde{U}$: have $\Phi, \tilde{\Phi}$.

iso of $\mathbb{R}^K$ = $K \times K$ matrix

$$\tilde{\Phi} \circ \Phi^{-1}(p, v) = (p, c_{\tilde{U}U}(p) v)$$

Obtain transition functions $c_{\tilde{U}U}$.

Check: $c_{UV} = c_{UW} c_{WV}$.

$$\Phi_U \circ \Phi_V^{-1} = (\Phi_U \circ \Phi_W^{-1}) \circ (\Phi_W \circ \Phi_V^{-1})$$

Ⓑ $\rightsquigarrow$ Ⓐ : $E = \bigcup (\mathcal{U}_\alpha \times \mathbb{R}^k)/\sim$. Denote elements by $[(p, v, \alpha)]$.

$p \in \mathcal{U}_\alpha$ $\uparrow$ keep track

projection is $\pi : E \to M$
$\pi([p, v, \alpha]) = p$

$E_p = \pi^{-1}(p)$ is vector space: $[p, u, \alpha]$, $[p, v, \alpha]$ $\in \mathcal{U}_\alpha \times \mathbb{R}^k$

$$a[p, u, \alpha] + b[p, v, \alpha] = [p, au + bv, \alpha].$$

$$\Phi_{\mathcal{U}_\alpha}([p, u, \alpha]) = (p, u).$$

$$\Phi_{\mathcal{U}_\alpha} : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^k.$$

<u>Def:</u> $E \to M$ rank k. Smooth section $s \in \Gamma(M, E)$ is given by local smooth vector-valued functions $\{\mathcal{U}_i, s_{\mathcal{U}_i}\}$ with $s_\mathcal{U} : \mathcal{U} \to \mathbb{R}^k$ s.t.

$$s_\mathcal{U} = [c_{\mathcal{U}V}]\, s_V .$$

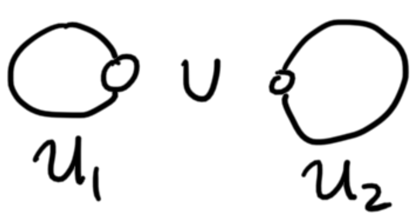
In index notation: $s_\mathcal{U}{}^i = c_{\mathcal{U}V}{}^i{}_k \, s_V{}^k$.

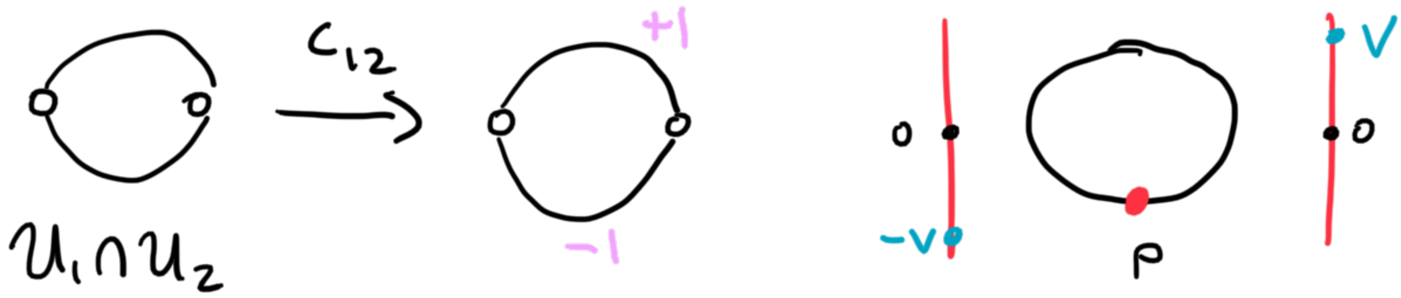
<u>Note:</u> $s \in \Gamma(M, E)$ defines map $s : M \to E$ s.t. $s(p) \in \pi^{-1}(p)$.

Indeed: $s(p) = [p, s_{\mathcal{U}_\alpha}(p), \alpha]$
if $p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ then $(p, s_{\mathcal{U}_\alpha}(p), \alpha) \sim (p, s_{\mathcal{U}_\beta}(p), \beta)$
since $s_{\mathcal{U}_\alpha} = c_{\mathcal{U}_\alpha \mathcal{U}_\beta} s_{\mathcal{U}_\beta}$.

ex) Möbius bundle $E \to S^1$ (rank 1) (line bundle)

$S^1 =$

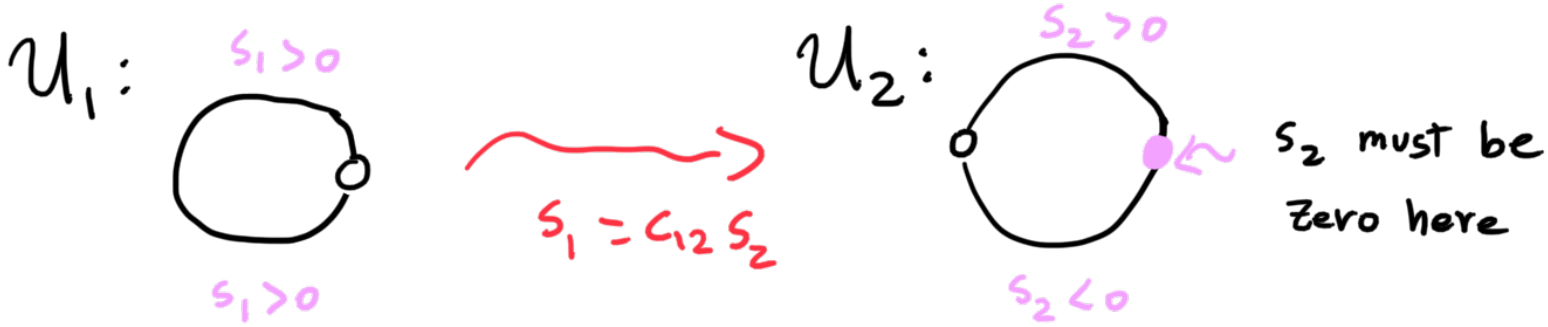




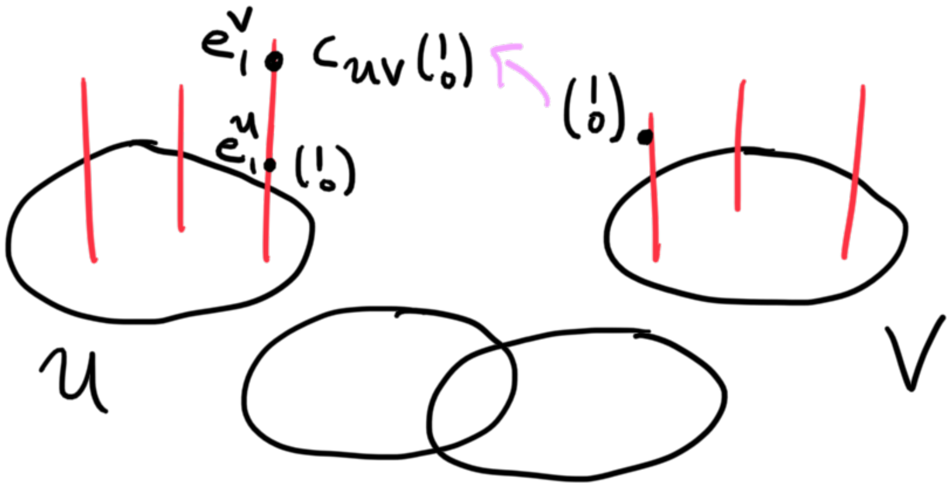
$c_{12}(x) = \pm 1$

Any section must pass through zero. $S = \begin{Bmatrix} (U_1, s_1) \\ (U_2, s_2) \end{Bmatrix}$

If say $s_1 : U_1 \to \mathbb{R}$ is s.t. $s_1 > 0$, then:



<u>Local frames</u>: Take 2 triv $\begin{matrix} U \times \mathbb{R}^k \\ V \times \mathbb{R}^k \end{matrix}$



$U \times \mathbb{R}^k$ has moving basis

$$e_1^U = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, \ e_k^U = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Denote by $\{e_a^V\}$ the basis in $U \times \mathbb{R}^k$ which is coming from $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ in $V \times \mathbb{R}^k$.

$$e_1^V = [c_{UV}] \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, \ e_k^V = [c_{UV}] \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then: $e_a^V = {c_{UV}}^b{}_a \, e_b^U$

Conventions: $c_{UV} = \begin{pmatrix} {c_{UV}}^1{}_1 & {c_{UV}}^1{}_2 \\ {c_{UV}}^2{}_1 & {c_{UV}}^2{}_2 \end{pmatrix}$

<u>Note</u>: Given $s \in \Gamma(M, E)$, over triv $U \times \mathbb{R}^k$

write $s = {s_U}^i e^U_i$.

$s = \begin{pmatrix} s^1 \\ s^2 \end{pmatrix} = s^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Since ${s_U}^i e^U_i \underset{(*)}{=} {s_V}^i e^V_i$, usually just write:

$s \overset{\text{loc}}{=} s^i e_i$.

Check $(*)$: ${s_U}^i e^U_i = ({c_{UV}}^i{}_k s^k_V)({c_{VU}}^\ell{}_i e^V_\ell)$

$= \delta^\ell{}_k s^k_V e^V_\ell$ ✓

$c_{UV} c_{VU} = \text{id} \underset{\substack{\text{index} \\ \text{notation}}}{\rightsquigarrow} {c_{UV}}^i{}_k {c_{VU}}^k{}_j = \delta^i{}_j$

<u>Summary</u>: $s \in \Gamma(M, E)$, $s = s^i e_i$

On $U \cap V$: ${s_U}^i = {c_{UV}}^i{}_k {s_V}^k$ transform components

: $e^U_a = e^V_b {c_{VU}}^b{}_a$ transform frame

ex) $E = TM$ tangent bundle.

$M = \bigcup_\alpha U_\alpha$, (U_α, x_{U_α}) coords

$c_{UV} = \dfrac{\partial x_U}{\partial x_V}$ transition functions

Sections = vector fields: $V = \underbrace{V^i}_{\text{components}} \underbrace{\dfrac{\partial}{\partial x^i}}_{\text{frame}}$.

On $\mathcal{U} \cap \tilde{\mathcal{U}}$:

$$\tilde{V}^i = \frac{\partial \tilde{x}^i}{\partial x^k} V^k$$

transform components

$$\left[\frac{\partial \tilde{x}}{\partial x}\right]\left[\frac{\partial x}{\partial \tilde{x}}\right] = \mathrm{id}$$

$$\frac{\partial}{\partial \tilde{x}^a} = \frac{\partial x^b}{\partial \tilde{x}^a} \frac{\partial}{\partial x^b}$$

transform frame

$$\frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial x^p}{\partial \tilde{x}^j} = \delta^i{}_j$$

$$V = V^i \frac{\partial}{\partial x^i} = \tilde{V}^i \frac{\partial}{\partial \tilde{x}^i} .$$