Let $\gamma : [a,b] \to \mathbb{R}^n$, $\gamma(a) = p$, $\gamma(b) = q$.

Action S:

$$S(\gamma) = \int_a^b L(\gamma, \dot\gamma)\, dt, \qquad \dot\gamma = \frac{d\gamma}{dt}$$

(Each possible path from p to q) $\rightsquigarrow$ (number)

principle of least action: True path in nature minimizes action.

ex) $S(\gamma) = \int_a^b |\dot\gamma|^2 dt$ $\qquad L(x,\dot x) = |\dot x|^2$

Kinetic energy: $v(t) = \dot x(t)$, $KE = m|v|^2$

ex) g_{ij} metric, $S(\gamma) = \int_a^b g_{ij}(\gamma)\, \dot\gamma^i \dot\gamma^j dt$

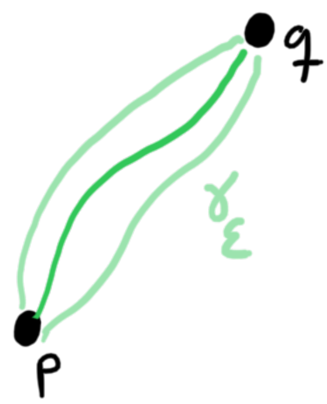
$L(x,\dot x) = g_{ij}(x)\dot x^i \dot x^j$

Euler-Lagrange eqn:

suppose $\gamma(t)$ minimizes action S.

Let $\gamma_\varepsilon(t)$ be family of curves s.t.

$\gamma_0 = \gamma$, $\quad \gamma_\varepsilon(a) = p$, $\gamma_\varepsilon(b) = q$.



As function of ε, $S(\gamma_\varepsilon)$ has min at $\varepsilon = 0$.

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon)$$

$$= \int_a^b \left(\frac{\partial L}{\partial x^\mu}\frac{\partial x^\mu}{\partial \varepsilon} + \frac{\partial L}{\partial \dot x^\mu}\frac{\partial \dot x^\mu}{\partial \varepsilon} \right) dt$$

$L(x,\dot x)$ Lagrangian density

Denote $V = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \gamma_\varepsilon$ Variation field

Note $V(a) = 0$, $V(b) = 0$.

$$0 = \int_a^b \left(\frac{\partial L}{\partial x^\mu} V^\mu + \frac{\partial L}{\partial \dot{x}^\mu} \frac{d}{dt} V^\mu \right) dt$$

$$= \int_a^b \frac{\partial L}{\partial x^\mu} V^\mu - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu} \right) V^\mu + \underbrace{\Big|_a^b \frac{\partial L}{\partial \dot{x}^\mu} V^\mu}_{=0}$$

$$0 = \int_a^b \left(\frac{\partial L}{\partial x^\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu} \right) \right) V^\mu$$

$\forall$ vector fields V^μ with $V(a) = 0$, $V(b) = 0$.

$$\Rightarrow \boxed{\frac{\partial L}{\partial x^\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu} \right) = 0}$$

Euler-Lagrange eqn

ex) $L = |\dot{x}|^2$,

$$0 - \frac{d}{dt}\left(2\dot{x}^\mu \right) = 0 \qquad \forall \mu$$

$$\ddot{x}^\mu = 0 \Rightarrow x(t) = t \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} + \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix}$$

optimal paths are straight lines

Back to Riemannian geometry:

consider energy functional

$$E(\gamma) = \frac{1}{2} \int_a^b |\dot{\gamma}|_g^2 \, dt, \qquad |\dot{\gamma}|_g^2 = g_{ij}(\gamma)\dot{\gamma}^i \dot{\gamma}^j$$

$$L(x, \dot{x}) = \frac{g_{ij}(x)}{2} \dot{x}^i \dot{x}^j$$

$$\frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu i}\dot{x}^i$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \partial_j g_{\mu i}\dot{x}^j\dot{x}^i + g_{\mu i}\ddot{x}^i$$

$$\frac{\partial L}{\partial x^\mu} = \frac{1}{2}\partial_\mu g_{ij}\dot{x}^i\dot{x}^j.$$

Euler-Lagrange eqn:

$$0 = \frac{1}{2}\partial_\mu g_{ij}\dot{x}^i\dot{x}^j - \underbrace{\partial_j g_{\mu i}\dot{x}^i\dot{x}^j}_{\frac{1}{2}(\partial_j g_{\mu i} + \partial_i g_{\mu j})\dot{x}^i\dot{x}^j} - g_{\mu i}\ddot{x}^i$$

$$\Rightarrow g_{\mu i}\ddot{x}^i + \frac{1}{2}(-\partial_\mu g_{ij} + \partial_j g_{\mu i} + \partial_i g_{\mu j})\dot{x}^i\dot{x}^j = 0$$

$g^{\kappa\mu}$

$$\Rightarrow \ddot{x}^\kappa + \underbrace{\frac{g^{\kappa\mu}}{2}(-\partial_\mu g_{ij} + \partial_i g_{\mu j} + \partial_j g_{\mu i})}_{\Gamma_{ij}^{\kappa}}\dot{x}^i\dot{x}^j = 0$$

$$\left[\ddot{\gamma}^\kappa + \Gamma_{ij}^{\kappa}(\gamma)\dot{\gamma}^i\dot{\gamma}^j = 0\right]$$ geodesic eqn

Can be rewritten using notation:

$$D_t V^\kappa = \partial_t V^\kappa + \dot{\gamma}^i\Gamma_{ij}^{\kappa}(\gamma)V^j,$$ V vector field along $\gamma(t)$.

Geo eqn: $D_t\dot{\gamma} = 0$.

<u>Def</u>: A geodesic is a map $\gamma : [a,b] \to M$ satisfying the geodesic eqn.

Note: Geodesics have constant speed.

Recall $\frac{d}{dt}\langle V, W\rangle_g = \langle D_t V, W\rangle_g + \langle V, D_t W\rangle_g$

$$\Rightarrow \frac{d}{dt}|\dot\gamma|_g^2 = 2\langle D_t\dot\gamma, \dot\gamma\rangle_g = 0.$$

$$\Rightarrow |\dot\gamma|_g \equiv \text{const}.$$

Note: Let $f: (M,g) \to (\tilde M, \tilde g)$ be an isometry.

Then if $\gamma(t)$ geodesic on M, then $\tilde\gamma(t) = f\circ\gamma(t)$ geodesic on $\tilde M$.

Isometries take geodesics to geodesics.

Proof: Exercise. Can use transformation law for Γ_{ij}^k.

Prop: (M,g), $p\in M$, $v\in T_pM$.

$\exists\, \varepsilon>0$ and unique $\gamma: (-\varepsilon,\varepsilon)\to M$ satisfying geodesic eqn and $\gamma(0)=p$, $\dot\gamma(0)=v$.

Pf: In chart around p,

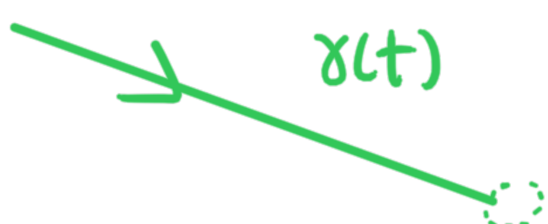
$\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$ and want to solve

$$\ddot\gamma^k + \Gamma_{ij}^k(\gamma(t))\,\dot\gamma^i\dot\gamma^j = 0.$$

Non-linear ODE for $\gamma(t)$.

Picard-Lindelöf $\Rightarrow \exists$ short-time soln. $\square$

ex) Straight line in $\mathbb{R}^n\setminus\{0\}$, $g = g_{Euc}$

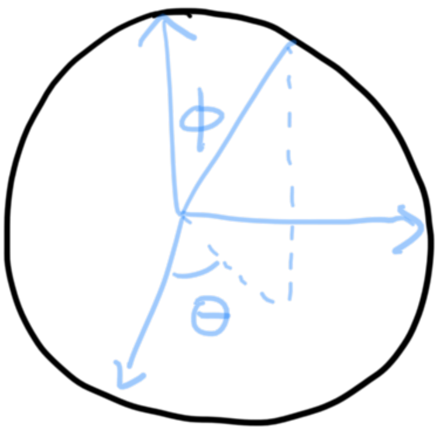


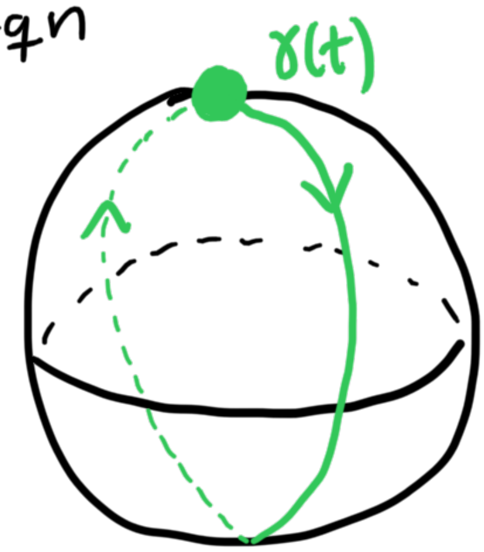
only short-time existence.

ex) Param S^2 by $f(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$

Metric $g_{S^2} = f^* g_{EUC}$

Exercise: compute geodesic eqn





Exponential Map:

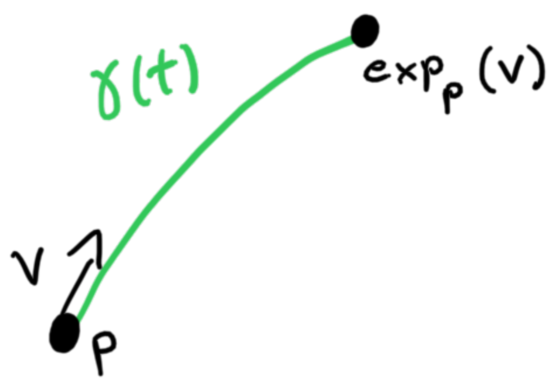
Let $p \in M$. Let $v \in T_pM$ s.t. unique geodesic $\gamma(t)$ with $\gamma(0)=p$, $\dot\gamma(0)=v$ exists on $[0,1]$.

Def: $\exp_p(v) = \gamma(1)$.

Note: $\exists\, \mathcal{O}_p \subseteq T_pM$ open set s.t.

$$\exp_p : \mathcal{O}_p \to M$$

is a smooth map.



Here is why: by the ODE thm, $\exists\, \varepsilon > 0$ s.t. for all initial velocities $v \in T_pM$ with $|v|_g \le 1$, then geodesics γ with $\gamma(0)=p$, $\dot\gamma(0)=v$ exist on $(-2\varepsilon, 2\varepsilon)$.

Claim: $\exp_p(\cdot)$ defined $\forall v \in T_pM$ with $|v|_g < \varepsilon$.

To see this, let $\tilde v = \varepsilon^{-1} v$. Then $\exists$ geo $\tilde\gamma$ on $(-2\varepsilon, 2\varepsilon)$ with $\tilde\gamma(0)=p$, $\dot{\tilde\gamma}(0) = \tilde v$.

Consider $\gamma(t) := \tilde\gamma(\varepsilon t)$. Then: γ defined on $(-2, 2)$, $\gamma(0)=p$, $\dot\gamma(0) = \varepsilon\varepsilon^{-1} v = v$.

Check: $\gamma(t)$ solves $\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$ since $\tilde{\gamma}$ solves it.

$\Rightarrow \exp_p(v) = \gamma(1)$ is well-defined.

(And $\exp_p : \mathcal{O}_p \to M$ is smooth by smooth dependence on initial cond in the ODE thm).

<u>Note</u>: The geodesic with $\gamma(0)=p$, $\dot{\gamma}(0)=v$ is
$\gamma(t) = \exp_p(tv)$.

Indeed, $\exp_p(\varepsilon v) = \tilde{\gamma}(1)$, $\quad \tilde{\gamma}(0)=p$, $\dot{\tilde{\gamma}}(0) = \varepsilon v$
$\tilde{\gamma}(t)$ geo.

In fact $\tilde{\gamma}(t) = \gamma(\varepsilon t)$ by uniqueness, since this $\tilde{\gamma}(t)$ solves geo eqn and satisfies $\tilde{\gamma}(0)=p$,
$\dot{\tilde{\gamma}}(0) = \varepsilon v$.

$\Rightarrow \exp_p(\varepsilon v) = \tilde{\gamma}(1) = \gamma(\varepsilon)$ ✓

<u>Cor</u>: $D\exp_p\big|_0 = \mathrm{id}$.

<u>Pf</u>:
$$D\exp_p\big|_0 v = \frac{d}{dt}\Big|_{t=0} \exp_p(tv)$$
$$= \frac{d}{dt}\Big|_{t=0} \gamma(t) = v. \qquad \square$$

geo with $\gamma(0)=p$, $\dot{\gamma}(0)=v$

<u>Cor</u>: Let $p \in M$.
$\exists$ nbhd of origin $V \subseteq T_pM$, nbhd of p $\mathcal{U} \subseteq M$

s.t. $\exp_p : V \to \mathcal{U}$ is a diffeomorphism.

<u>Pf</u>: $D\exp_p\big|_0 = \mathrm{id}$ + inverse function Thm. $\square$

Normal Coordinates:

Let $p \in (M, g)$.

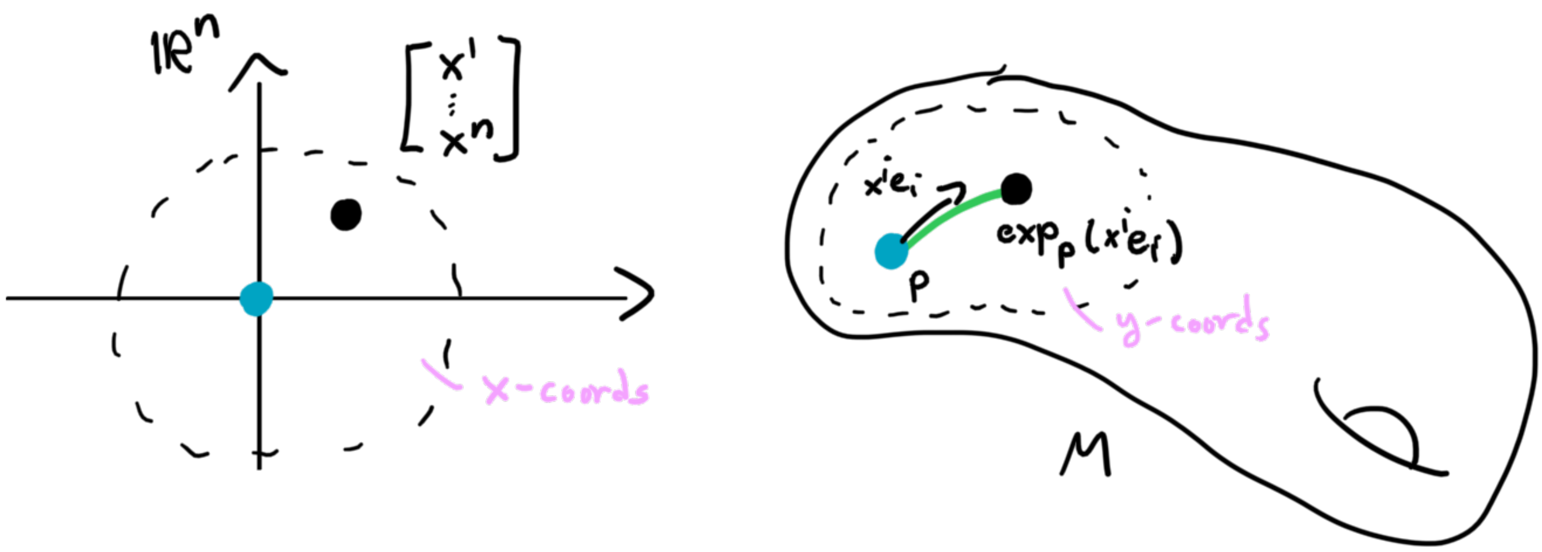
choose $\{e_1, \cdots, e_n\}$ orthonormal basis of T_pM.

$g(e_i, e_j) = \delta_{ij}$.

$\{y^k\}$ initial coords.

Define new coords $\{x^i\}$ by: $y^k = \exp_p^k(x^i e_i)$.

These are called normal coordinates based at p.



Lemma: In normal coords $(\mathcal{U}, x^i)$

$$g_{ij}(0) = \delta_{ij}$$

$$\Gamma_{ij}^k(0) = 0.$$

Pf: $g_{ij} = \frac{\partial y^p}{\partial x^i} \, g^{\text{y-coords}}_{pq} \, \frac{\partial y^q}{\partial x^j}$

components in y-coords

$$\left.\frac{\partial y^k}{\partial x^i}\right|_0 = \left.\frac{\partial \exp_p^k}{\partial y^l}\right|_0 e^l{}_i = e^k{}_i,$$

$e_i = e^k{}_i \frac{\partial}{\partial y^k}$

$D\exp_p|_0 = id$

$$\Rightarrow g_{ij}(0) = e^p{}_i \, g^{\text{y-coord}}_{pq} \, e^q{}_j = g(e_i, e_j) = \delta_{ij}.$$

Next, for $\Gamma_{ij}^k(0)$, note $\gamma(t) = t\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$ is geodesic in these coords.

Geo eqn: $\Gamma_{ij}^k(\gamma(t))\, v^i v^j = 0$

$$\xrightarrow{t=0} \Gamma_{ij}^k(0)\, v^i v^j = 0 \quad \forall \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \in \mathbb{R}^n.$$

$\Rightarrow \Gamma_{ij}^k(0) = 0.$ $\square$

let $v = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$