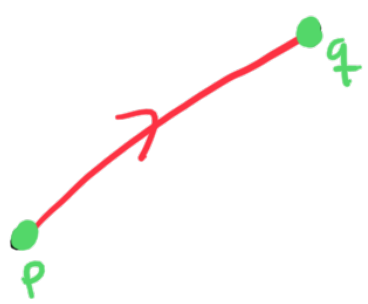
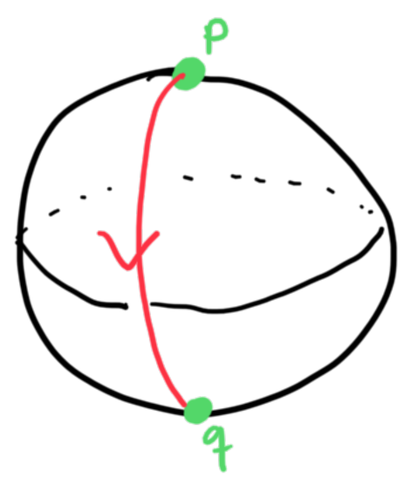
Completeness

<u>Def</u>: (M, g) is complete if geodesics exist for all time.

Summary of facts we will use: (Hopf-Rinow)

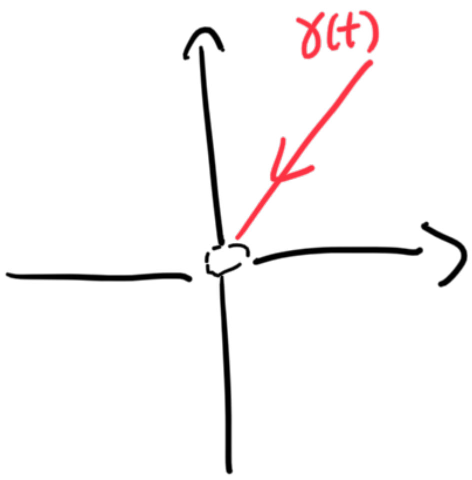
- compact manifolds are complete
- (M, g) connected + complete implies
  $\forall p, q \in M$, $\exists$ length minimizing geodesic from p to q.



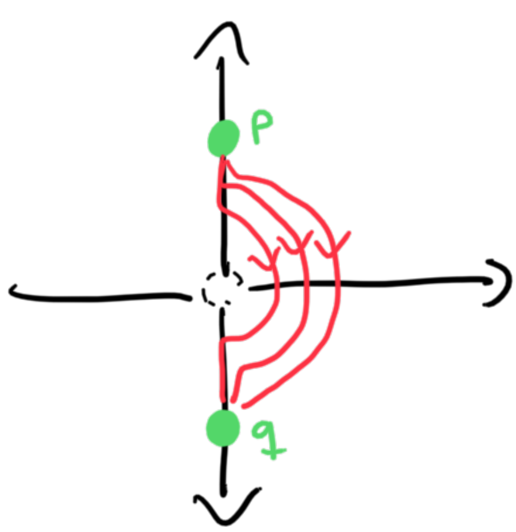


- (M, g) connected, if $\exists p$ s.t. $\exp_p : T_pM \to M$ defined on all T_pM, then (M, g) complete.

ex) $(\mathbb{R}^2 \setminus \{0\}, g_{Euc})$ incomplete



does not exist $\forall t$



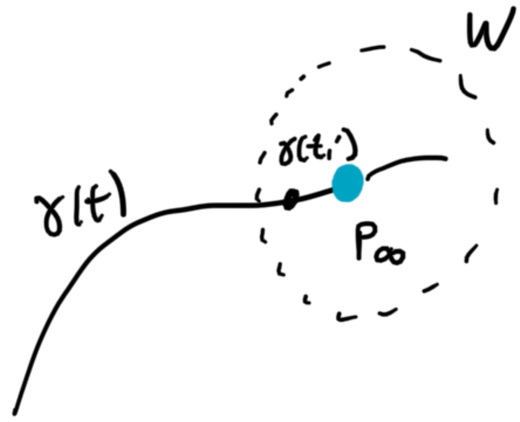
cannot attain minimum distance

Prop: (M, g) compact.
Then geodesics exist for all time.

Pf: If maximal time $T < \infty$, take a limit

$p_\infty = \lim \gamma(t_i)$.

Let W be δ-normal nbhd of p_∞.
Let $\gamma(t_i) \in W$.
$\Rightarrow$ any unit speed geo starting at $\gamma(t_i)$ exists for time δ.
If $t_i + \delta > T$, extend past T.
$\Rightarrow \Leftarrow$



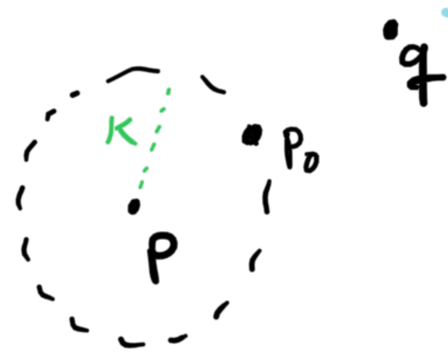
$\square$

Prop: Suppose
(M, g) admits $p \in M$ s.t. $\exp_p : T_pM \to M$ defined on all T_pM. (prev prop $\Rightarrow$ true if M compact)

Then for any $q \in M$, there exists a geodesic curve from p to q whose length is $d(p, q)$.

Pf: Let $r = d(p, q)$
Let $B_{\kappa_0}(p)$ be a geodesic coord ball with $\kappa_0 < r$.
Let $p_0 \in \partial B_{\kappa_0}(p)$ attain $\min\limits_{\partial B_{\kappa_0}(p)} d(\cdot, q)$.

Write $p_0 = \exp_p(tV)$,
$V \in T_pM$, $|V|_g = 1$.



Let $\boxed{\gamma(t) = \exp_p(tv).}$

objective: Show this attains minimum length from p to q.

Need to show: $[\gamma(r) = q]$ $\quad L(\gamma) = \int_0^r \underbrace{|\dot{\gamma}|_g}_{=1} = r$

Let $I = \{ t \in [0,r] : d(\gamma(t), q) = r - t \}$

① $I \neq \emptyset$ since $0 \in I$.

② I is closed since if $t_i \in I$ with $t_i \to t_\infty$, then $t_\infty \in I$ by continuity of $d(\cdot, q)$.

$|d(x,q) - d(y,q)| \leq d(x,y) \leftarrow$ small in normal coords when x, y close

③ Let $T = \sup I$. By ①,②, $T > 0$ + $T \in I$.

Suppose $T < r$. Will derive contradiction so that $r \in I$ and so $d(\gamma(r), q) = 0$
$\Rightarrow \gamma(r) = q$.

let $0 < K < \mathrm{inj}_{\gamma(T)}$, $K \leq r - T$.

Want: $[d(\gamma(T+K), q) = r - (T+K)]$

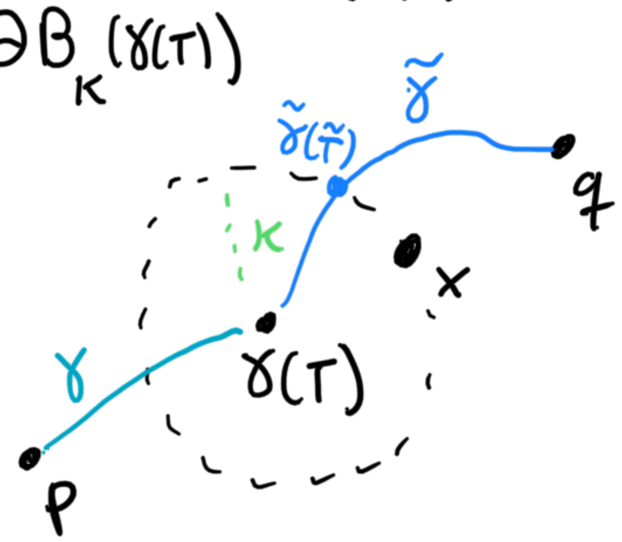
$\Rightarrow T + K \in I$. Contradicts $T = \sup I$.

Let $x \in \partial B_K(\gamma(T))$ attain $\min\limits_{\partial B_K(\gamma(T))} d(\cdot, q)$

To complete the proof, we will show:

Ⓐ $d(x, q) = r - (T+K)$.

Ⓑ $\gamma(T+K) = x$.



Ⓐ First, we notice:

$$d(\gamma(T), q) = \kappa + d(x, q). \quad (*)$$

Reason: $d(\gamma(T), q) \le \underbrace{d(\gamma(T), x)}_{= \kappa} + d(x, q)$ Triangle

For $\ge$, let $\tilde{\gamma}$ arbitrary curve from $\gamma(T)$ to q

$\tilde{\gamma}(\tilde{T}) \in \partial B_\kappa(\gamma(T))$, $\tilde{\gamma}: [0, a] \to M$

$$L(\tilde{\gamma}) = L_{[0, \tilde{T})} + L_{[\tilde{T}, a]}$$
$$\ge \kappa + d(x, q). \quad \text{choice of } x$$

Using $(*)$ + $T \in I$:

$$d(x, q) = d(\gamma(T), q) - \kappa$$
$$d(x, q) = r - T - \kappa \quad \checkmark$$

Remains to prove Ⓑ : $\gamma(T + \kappa) = x$.

Note: $d(p, x) \ge d(p, q) - d(q, x)$

$$= r - (r - T - \kappa) \quad (*)$$

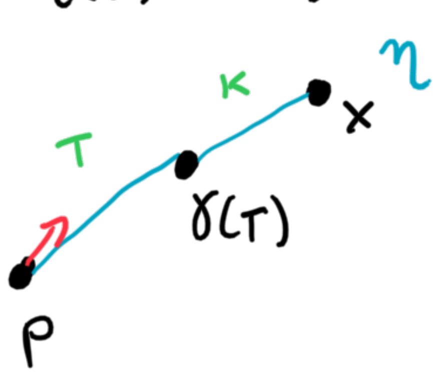
$\Rightarrow$ $d(p, x) \ge \kappa + T$, and bound attained by path η that goes along γ from p to $\gamma(T)$ then along radial geo from $\gamma(T)$ to x.

This path η is length minimizing from p to x.

$\Rightarrow$ η is geodesic

$\Rightarrow$ by uniqueness of initial cond, $\eta(t) = \gamma(t)$

$\Rightarrow$ $\gamma(T + \kappa) = \eta(T + \kappa) = x$. $\square$



Prop: Suppose:

(M, g) admits $p \in M$ s.t.

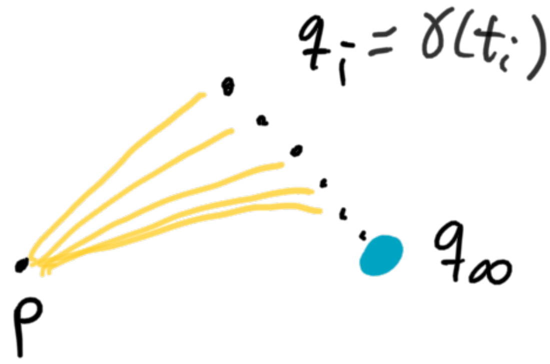
$\exp_p : T_pM \to M$ defined on all T_pM,

Then: Any geodesic exists for all time.

Pf: Let $\gamma : [0, T) \to M$ be unit speed geo that does not extend past T.

$t_i \to T$. Let $q_i = \gamma(t_i)$.

$q_i \to ?$



Prev Thm: write $q_i = \exp_p(d_i v_i)$,

$d_i = d(p, q_i)$, $|v_i|_g = 1$. Note: $d(q_i, q_j) \leq |t_i - t_j|$

$L(\gamma|_{[t_i, t_j]}) = |t_i - t_j|$

$\{d_i\}, \{v_i\}$ bounded seq. After subseq:

$d_i \to d_\infty$, $v_i \to v_\infty$ $\implies$ $q_i \to \exp_p(d_\infty v_\infty) := q_\infty$

Take unif δ-normal nbhd of q_∞.

Let $\gamma(t_i) \in W$. $\implies$ can extend geodesic to $t_i + \delta$.

If $t_i + \delta > T$, contradiction. $\square$