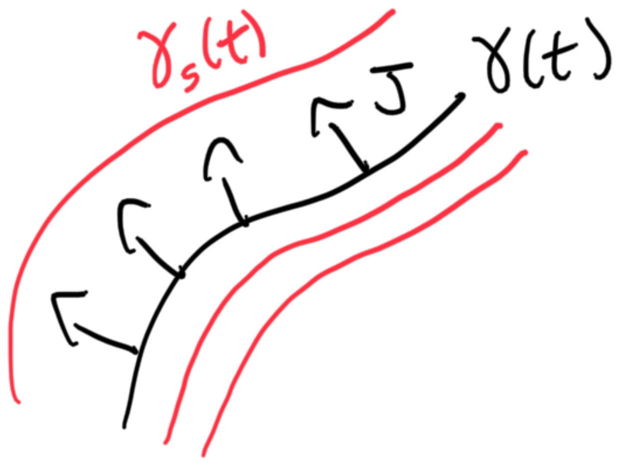
Jacobi Fields : moduli of geodesics

Theme: Understand how objects move in families
(Moduli space)



Given geodesic $\gamma(t)$, how to deform?

Suppose $\gamma_s(t)$ family of geodesics, $s \in (-\varepsilon, \varepsilon)$.
$\gamma_0 =$ given γ

$$\frac{d}{ds}\Big|_{s=0} \gamma_s := J$$

Constraint on deformation field J:

$$\ddot{\gamma}_s^k + \Gamma_{ij}^k(\gamma_s)\dot{\gamma}_s^i\dot{\gamma}_s^j = 0 \quad \text{holds } \forall s \in (-\varepsilon, \varepsilon).$$

Differentiate in s, eval at $s=0$:

$$\ddot{J}^k + 2\dot{\gamma}^i \Gamma_{ij}^k \dot{J}^j + \partial_i \Gamma_{jp}^k J^i \dot{\gamma}^j \dot{\gamma}^p \overset{(1)}{=} 0.$$

Vector fields J along $\gamma(t)$ solving (1) are called Jacobi fields. "infinitesimal deformations"

(1) is 2nd order ODE. Fix geodesic $\gamma(t)$.
$\Rightarrow$ Given $J(0)$ and $\dot{J}(0)$, $\exists$ unique corresponding Jacobi field $J(t)$.

Instead of $\dot{J}(0)$, usually specify pair $(J(0), D_t J(0))$:

Cor: Fix geodesic $\gamma(t)$. Given $J(0)$, $D_t J(0)$, $\exists!$ Jacobi field $J(t)$.

How to write Jacobi eqn using tensors? Γ_{ij}^{k} not a tensor.

Claim: can rewrite Jacobi eqn as

$$D_t D_t J^k + R_{ijp}{}^{k} J^i \dot\gamma^j \dot\gamma^p = 0. \quad (2)$$

without indices:

$$D_t D_t J + R(J, \dot\gamma)\dot\gamma = 0.$$

Calculation for $(1) = (2)$:

Recall notation:

$$D_t V^k = \dot V^k + \dot\gamma^i \Gamma_{ij}^{k}(\gamma(t)) V^j, \quad V \text{ along } \gamma(t)$$

$$R_{ijp}{}^{k} = \partial_i \Gamma_{jp}^{k} + \Gamma_{il}^{k} \Gamma_{jp}^{l} - (i \leftrightarrow j)$$

Compute:

$$D_t D_t J^k = \partial_t D_t J^k + \dot\gamma^i \Gamma_{ij}^{k}(\gamma(t)) D_t J^j$$

$$= \partial_t\left(\partial_t J^k + \dot\gamma^i \Gamma_{ij}^{k}(\gamma(t)) J^j\right) + \dot\gamma^i \Gamma_{ij}^{k}\left(\partial_t J^j + \dot\gamma^r \Gamma_{rs}^{j} J^s\right)$$

$$= \ddot J^k + \underbrace{\ddot\gamma^i}_{= -\Gamma_{rs}^{i}\dot\gamma^r\dot\gamma^s \text{ geodesic eqn}} \Gamma_{ij}^{k} J^j + \dot\gamma^i \partial_r \Gamma_{ij}^{k} \dot\gamma^r J^j + 2\dot\gamma^i \Gamma_{ij}^{k} \dot J^j + \Gamma_{ij}^{k}\Gamma_{rs}^{j}\dot\gamma^i\dot\gamma^r J^s$$

$$= \ddot J^k + 2\dot\gamma^i \Gamma_{ij}^{k} \dot J^j + \left(-\Gamma_{il}^{k}\Gamma_{jp}^{l} + \partial_j \Gamma_{ip}^{k} + \Gamma_{jl}^{k}\Gamma_{ip}^{l}\right) \underbrace{J^i \dot\gamma^j \dot\gamma^p}_{\text{relabel}}$$

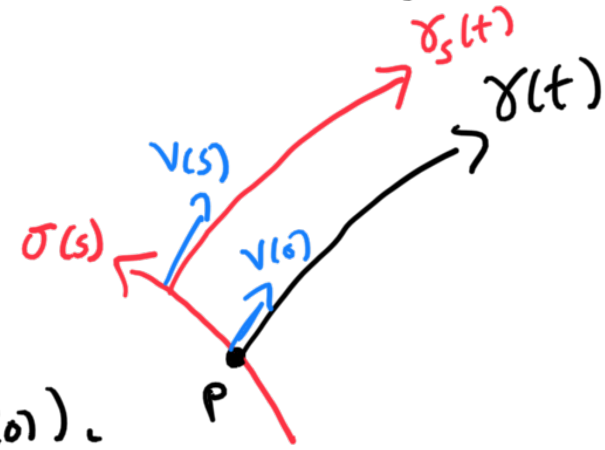
$$\Rightarrow D_t D_t J^k + R_{ijp}{}^{k} J^i \dot\gamma^j \dot\gamma^p$$

$$= \ddot J^k + 2\dot\gamma^i \Gamma_{ij}^{k} \dot J^j + \partial_i \Gamma_{jp}^{k} J^i \dot\gamma^j \dot\gamma^p \quad ✓$$

Prop: Let $\gamma : [0,1] \to (M, g)$ be geodesic.
Every Jacobi field $J(t)$ along $\gamma(t)$ is the first derivative of a family of geodesics $\gamma_s(t)$ with $\gamma_0 = \gamma$.
(infinitesimal deformation $\rightsquigarrow$ genuine deformation)

Pf: Try $\exp_{\sigma(s)}(t V(s)) := \gamma_s(t)$
for suitable $\sigma(s)$, $V(s)$. For fixed s, $\gamma_s(t)$ is geodesic.

Let $\tilde{J}(t) = \frac{d}{ds}\Big|_{s=0} \gamma_s(t)$, solves Jacobi eqn by construction.



By uniqueness, need to choose σ, V to attain initial cond $(J(0), D_t J(0))$.

$$\tilde{J}(0) = \frac{d}{ds}\Big|_{s=0} \gamma_s(0) = \frac{d}{ds}\Big|_{s=0} \sigma(s) = \dot{\sigma}(0)$$
$$= J(0)$$ choose s.t.

$$D_t \tilde{J}^k = \partial_t \tilde{J}^k + \dot{\gamma}^p \Gamma^k_{p\ell} \tilde{J}^\ell$$
$$D_s \dot{\gamma}_s^k = \partial_s \dot{\gamma}_s^k + \dot{\sigma}^p \Gamma^k_{p\ell} \dot{\gamma}_s^\ell$$

$\Rightarrow D_t \tilde{J}(0) = D_s\big|_{s=0} \dot{\gamma}_s(0)$ ($D_t \tilde{J}$ = how tangent vectors $\dot{\gamma}$ vary)

$\Rightarrow D_t \tilde{J}(0) = D_s\big|_0 \left(\frac{d}{dt}\Big|_{t=0} \gamma_s \right)$
$= D_s V(0) = D_t J(0)$ choose V s.t.

$\Rightarrow \tilde{J}(t)$ solves Jacobi eqn with same IC as $J(t)$
$\Rightarrow \tilde{J}(t) = J(t)$. $\square$

comes from variation of geodesics

Prop: Let $\gamma(t) = \exp_p(tv)$ geodesic,
$J(t)$ Jacobi field with: $J(0)=0$, $D_t J(0) = w$.

Then: $(\exp_p)_*\big|_{tv}(tw) = J(t)$.

In particular $(\exp_p)_*\big|_v w = J(1)$.

Recall pushforward
$f: M \to N$
$f_*\big|_p V = \frac{d}{dt}\big|_{t=0} f\circ\gamma$
$\gamma(0)=p$
$\dot\gamma(0) = V$

Pf: Redo prev proof in special case. Let:

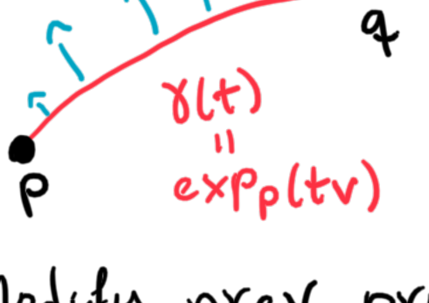
$$\left[\gamma_s(t) = \exp_p(t(v+sw)).\right]$$

Then
$$J(t) = \frac{d}{ds}\Big|_{s=0}\gamma_s$$
$$= (\exp_p)_*\big|_{tv}(tw),$$

and $J(0)=0$,
$$D_t J(0) = D_s\big|_0 \dot\gamma_s(0) = D_s\big|_0 \overbrace{(\exp_p)_*\big|_0}^{id}(v+sw)$$
$$= w.$$
$\square$

Def: Let $p,q \in M$ points connected by geodesic $\gamma(t)$. Say p,q are conjugate points along γ if $\exists$ Jacobi field vanishing at both p and q. (But $J(t) \not\equiv 0$)

Cor:



$p,\ q = \exp_p(v)$
conj pts $\Leftrightarrow$ $(\exp_p)_*\big|_v$ has kernel
$\|$
q is critical value of $\exp_p$

Pf: Modify prev prop. $\square$

Topological consequences: (local curvature cond $\Leftrightarrow$ global topology)

Cartan-Hadamard Thm:

Let (M,g) complete, connected, $\sec_g \leq 0$.

$\Rightarrow \exp_p : T_pM \to M$ covering map.

$\Rightarrow$ Universal cover of M is diffeo to $\mathbb{R}^n$.

(If M simply-connected, then M diffeo to $\mathbb{R}^n$).

Pf: Let $p \in M$. complete $\Rightarrow \exp_p$ defined on all T_pM.
Let $V \in T_pM$.

① Prove $(\exp_p)_*|_V$ is injective.

Let $\gamma(t) = \exp_p(tV)$ geodesic, $J(t)$ Jacobi field with $J(0)=0$, $D_tJ(0) = w$.

$$\frac{d}{dt}|J(t)|^2 = 2\langle D_tJ, J\rangle$$

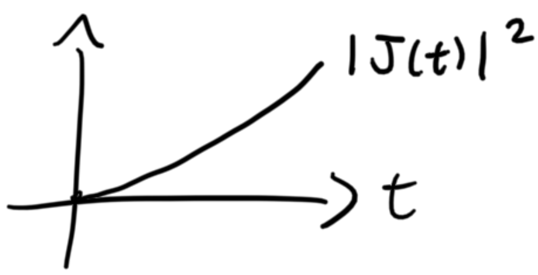
$$\frac{d^2}{dt^2}|J(t)|^2 = \underbrace{2|D_tJ|^2}_{\geq 0} + 2\underbrace{\langle D_tD_tJ, J\rangle}$$

$$= -\langle R(J,\dot\gamma)\dot\gamma, J\rangle$$

$$= -\sec(J,\dot\gamma)|J\wedge\dot\gamma|^2 \geq 0$$

$\Rightarrow \frac{d^2}{dt^2}|J(t)|^2 \geq 0 \Rightarrow |J(t)|^2$ convex + vanishes at $t=0$.
minimum

Since $|J(t)| \not\equiv 0$, then
$|J(1)| \neq 0$.



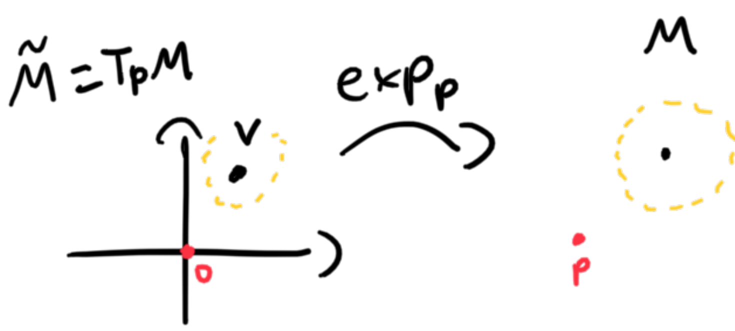
$\Rightarrow (\exp_p)_*|_V\, w = J(1) \neq 0$.

$\Rightarrow [(\exp_p)_*|_V]$ invertible linearized operator

$\Rightarrow \exp_p$ local diffeo near V. IFT

Let $\tilde{M} = T_pM \approx \mathbb{R}^n$

$\tilde{g} = \exp_p^* g$ non-deg



Then $(\tilde{M}, \tilde{g}) \xrightarrow{\exp_p} (M, g)$ local isometry.

② Global properties of $\tilde{M} \xrightarrow{\exp_p} M$:

Completeness $\Rightarrow \forall q \in M$, can write $q = \exp_p(X)$.
(see proof of Hopf–Rinow)

$\Rightarrow \exp_p : \tilde{M} \to M$ is surjective.

Also, $(\tilde{M}, \tilde{g})$ complete since geodesics starting at 0 are $t(V^1, \dots, V^n)$ (normal coords) and exist $\forall t$ since (M, g) complete.

$\Rightarrow \underset{\mathbb{R}^n \approx}{(\tilde{M}, \tilde{g})} \xrightarrow{\exp_p} (M, g)$ surjective local isometry of connected complete Riemannian mfd.

For our purposes, we call this a "covering map".

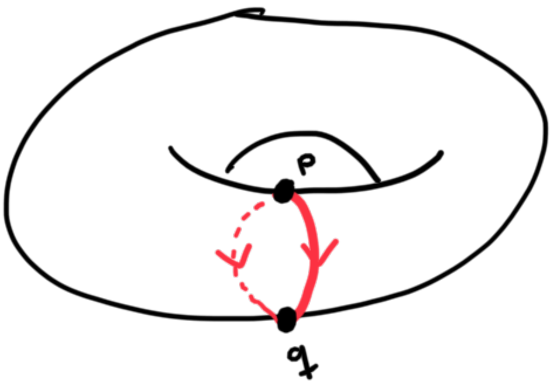
For connection to notion of covering maps in topology, see [Lee] Thm 6.23.

$\square$

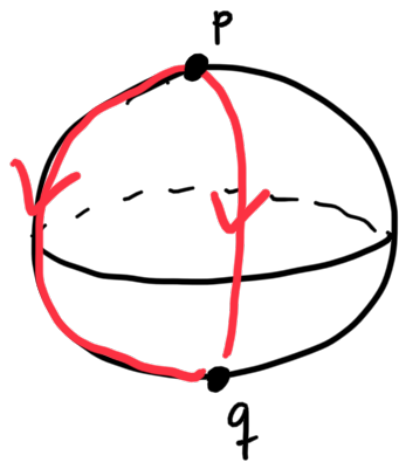
<u>Cor</u>: (M, g) complete, connected, $\sec_g \leq 0$.

Given $p, q \in M$, each homotopy class contains a unique geodesic from p to q.

e.g.



e.g.

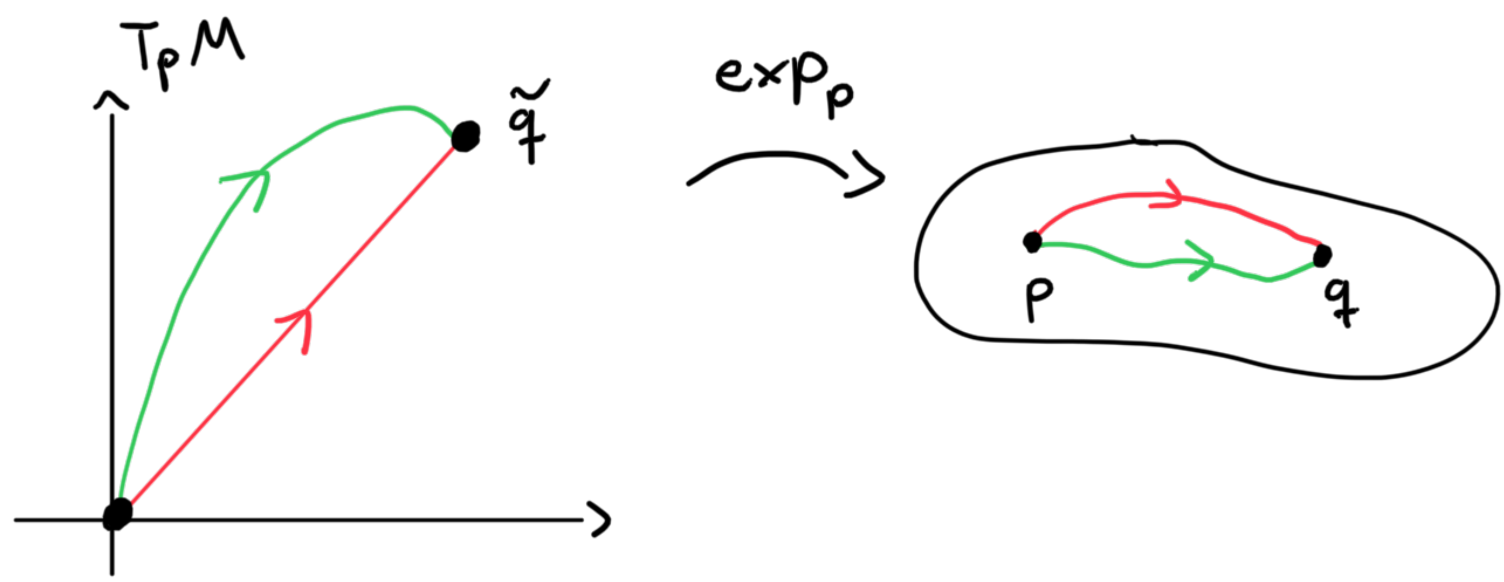


<u>Pf</u>: Recall topology ([Munkres] p.344)

$\pi : \tilde{M} \to M$ covering map, $p \in M$, $\pi(\tilde{p}) = p$.

- Any path on M starting at p has a unique lift starting at $\tilde{p}$.
- Let γ_1, γ_2 be two paths on M from p to q. Let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be their lifts. If γ_1, γ_2 path homotopic $\Rightarrow$ $\tilde{\gamma}_1, \tilde{\gamma}_2$ end at same point $\tilde{q}$.

Apply to $\exp_p : \mathbb{R}^n \to (M, g)$.



- All paths in homotopy class lift to paths in T_pM from 0 to $\tilde{q}$, $\exp_p(\tilde{q}) = q$.
- Geodesics lift to radial lines emanating from origin
- $\exists!$ radial line from 0 to $\tilde{q}$.

$\square$

Note: Curvature condition $\sec_g \le 0$ enters for uniqueness: always have existence.

Prop: (M,g) complete, connected.
Given $p,q \in M$, each homotopy class contains a geodesic (possibly many, e.g. sphere).

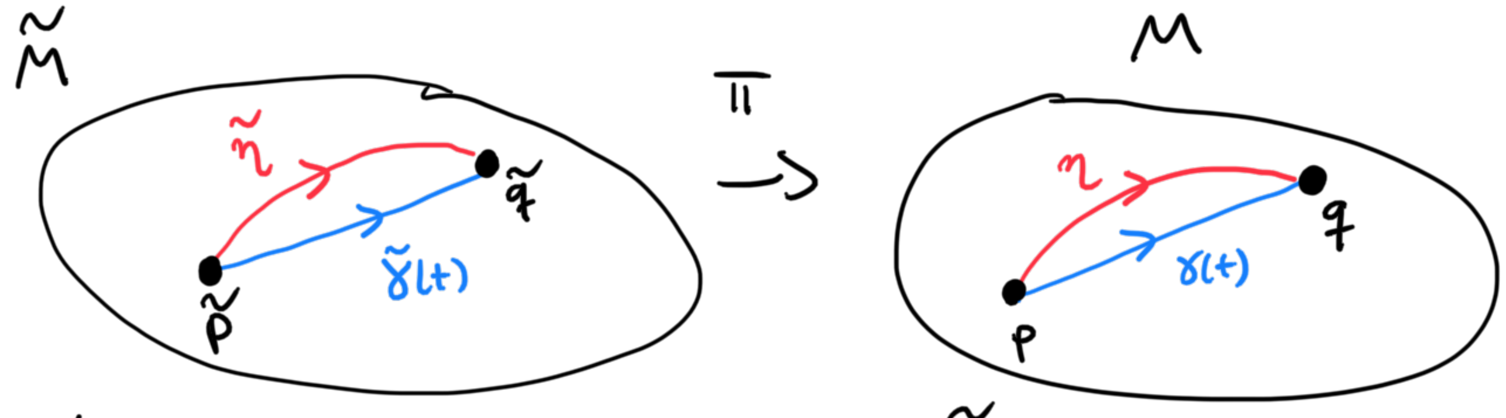
Pf: Let $\pi : \tilde{M} \to M$ be the universal cover of M.

(M,g) complete $\Rightarrow$ $(\tilde{M}, \pi^* g)$ complete.

$\tilde{p} \in \tilde{M}$, $\tilde{v} \in T_{\tilde{p}}\tilde{M}$ $\rightsquigarrow$ $p = \pi(p)$, $v = \pi_* \tilde{v}$ $\rightsquigarrow$ geodesic $\gamma(t)$ on (M,g) with $\gamma(0) = p$, $\dot{\gamma}(0) = v$ exists $\forall t$

$\Rightarrow$ Lift $\tilde{\gamma}(t)$ is geodesic with $\tilde{\gamma}(0) = \tilde{p}$, $\dot{\tilde{\gamma}}(0) = \tilde{v}$, exists $\forall t$.

Fix $p,q \in M$, $\tilde{p} \in \tilde{M}$ s.t. $\pi(\tilde{p}) = p$. $\exists \tilde{q} \in \tilde{M}$ s.t.:
All paths in homotopy class lift to path from $\tilde{p}$ to $\tilde{q}$.



$(\tilde{M}, \pi^* g)$ complete $\Rightarrow$ $\exists$ geodesic $\tilde{\gamma}$ from $\tilde{p}$ to $\tilde{q}$.

$\Rightarrow$ $\gamma(t) = \pi \circ \tilde{\gamma}(t)$ is geodesic from p to q.

Let $\eta(t)$ be a path in the homotopy class.

$\tilde{M}$ simply connected $\Rightarrow$ lift $\tilde{\eta}$ homotopic to $\tilde{\gamma}$
$\Rightarrow$ η homotopic to γ. $\square$