Jacobi Fields II

Def:
$$S_c(t) = \begin{cases} \frac{1}{\sqrt{c}} \sin \sqrt{c}\, t & c > 0 \\ t & c = 0 \\ \frac{1}{\sqrt{c}} \sinh \sqrt{c}\, t & c < 0 \end{cases}$$

Lem:
- (M, g) with constant sectional curv c
- γ unit speed geodesic
- $J(t)$ Jacobi field with $\langle J(t), \dot\gamma(t) \rangle_g \equiv 0$ and $J(0) = 0$.

$$\Rightarrow |J(t)| = |S_c(t)|\, |D_t J(0)|.$$

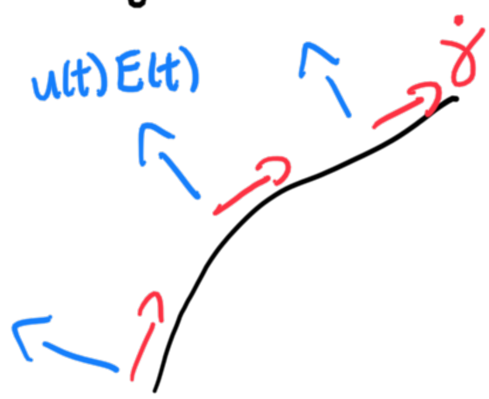
Pf: const $\sec_g$ means:
$$R(V, W)X = c\left(\langle W, X \rangle V - \langle V, X \rangle W \right)$$

Jacobi eqn: $D_t^2 J + R(J, \dot\gamma)\dot\gamma = 0$ becomes

$$D_t^2 J + cJ = 0.$$

Let $E(t)$ be parallel transport of $E(0) = D_t J(0)$
$D_t E \equiv 0.$

Try $J(t) = u(t)\, E(t)$.



Get ODE: $\ddot u(t) + c\,u(t) = 0.$
$u(0) = 0.$
$$D_t J(0) = \underbrace{\dot u(0)}_{=1}\, E(0)$$

$u(t) = S_c(t)$ solves constraints.

$$\Rightarrow |J(t)| = |S_c(t)|\, |E(t)|$$
$$= |S_c(t)|\, |D_t J(0)| \quad \square$$

parallel E preserves length
$\partial_t |E|^2 = 2\langle D_t E, E \rangle = 0.$

Thm: • (M, g) with $\sec_g = c$.

• $p \in M$, $(\mathcal{U}, x^i)$ normal coords at p, $r = (\Sigma (x^i)^2)^{1/2}$

Then: On $\mathcal{U} \setminus \{p\}$, metric appears as

$$g = dr^2 + s_c(r)^2 \hat{g}$$

with $\hat{g} = \pi^* g_{S^{n-1}}$, $\pi(x) = \frac{x}{|x|}$, $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$.

Cor: Constant curvature metrics look the same.
(local uniqueness) (geometries with $\sec_g = c$ are locally isometric)

Recall model geometries:

(1) (S^n, g_{S^n}) : $\sec_{g_{S^n}} = c > 0$

(2) $(\mathbb{R}^n, g_{Euc})$: $\sec_{g_{Euc}} = 0$

(3) (H^n, g_{Hyp}) : $\sec_{g_{Hyp}} = c < 0$

Thm: (Killing - Hopf)
(M, g) complete, simply connected, $\sec_g = \text{const}$, $n \geq 2$.
$\Rightarrow (M, g)$ isometric to either S^n, $\mathbb{R}^n$, H^n.

Will not prove: see Lee chap 12.

Proof of local uniqueness $g = dr^2 + s_c^2 \hat{g}$:

Note: Euclidean metric in polar coords

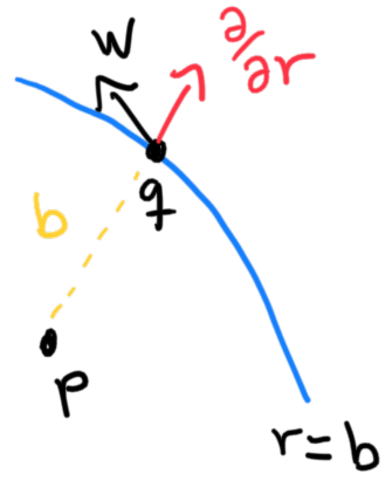
$$g_{Euc} = dr^2 + r^2 \hat{g}$$ ← metric on sphere

Proved earlier (using Gauss lemma) that in normal coords:

$g = dr^2 + g^T$ ← metric on $\{r = \text{const}\}$

Let $w \in T_q M$, $\delta_{ij} w^i q^j = 0$, $b = r(q)$.

Need to show: $\langle w, \frac{\partial}{\partial r} \rangle_{g_{Euc}} = 0$



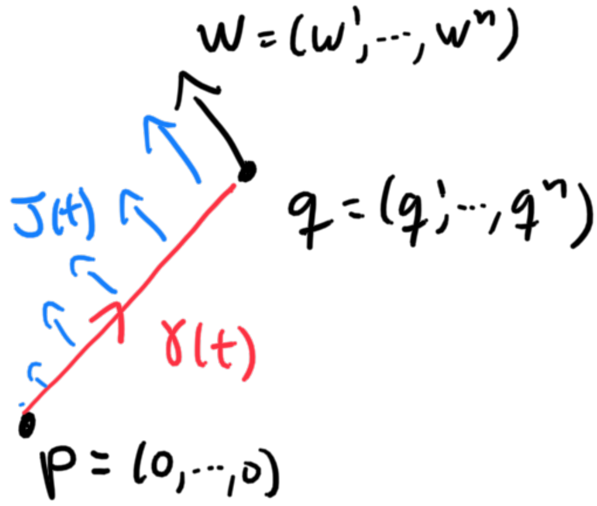
$$g(w,w) = s_c(b)^2 \hat{g}(w,w).$$

$g(w,w) = |w|_g^2$

To compute $|w|_g^2$, reach it by Jacobi field.

$$\gamma(t) = \left(\frac{q^1}{b}, \cdots, \frac{q^n}{b}\right) t$$

unit geo normal coords



$$\gamma_s(t) = \left(t\left(\frac{q^1 + sw^1}{b}\right), \cdots, t\left(\frac{q^n + sw^n}{b}\right)\right)$$

variation of geos

$$J(t) = \frac{d}{ds}\Big|_{s=0} \gamma_s$$

$$= \left(\frac{t}{b} w^1, \cdots, \frac{t}{b} w^n\right)$$

Jacobi field with $J(b) = w$.

Note: $\langle \dot\gamma, J \rangle(t) \equiv 0$ since $g_{ij}\left(\frac{q}{b}t\right) \underbrace{\frac{q^i}{b}}_{\dot\gamma} \underbrace{t\frac{w^j}{b}}_{J}$

Gauss lemma $= \delta_{ij} \frac{q^i t}{b} \frac{w^j}{b} = 0$ ← $\delta_{ij} q^i w^j = 0$.

prev lemma

$\Rightarrow |J(t)| = |s_c(t)|\, |D_t J(0)|$.

$$\Rightarrow |w|_g^2 = |J(b)|_g^2$$

$$= (s_c(b))^2 |D_t J(0)|_g^2$$

$$= s_c(b)^2 \left|\left(\frac{w^1}{b}, \cdots, \frac{w^n}{b}\right)\right|_{g_{Euc}}^2$$

← $g(0) = \delta_{ij}$, $\Gamma(0) = 0$ normal coords

$$= s_c(b)^2 |w|_{\hat{g}}^2. \quad \square$$

Polarization argument: for $g_1 = g_2$ just need $|v|_{g_1} = |v|_{g_2}$ $\forall v$

NB: $w = \dot\gamma(0)$, $\gamma(t)$ on $\{r = b\}$

$\pi_* w = \frac{d}{dt} \pi \circ \gamma = \frac{w}{b}$, $\pi(x) = \frac{x}{|x|}$

$\hat{g}(w,w) = |\pi_* w|_{g_{Euc}}^2$

Second Variation Formula

Let $\gamma(t) : [a,b] \to (M,g)$ be unit speed geodesic.

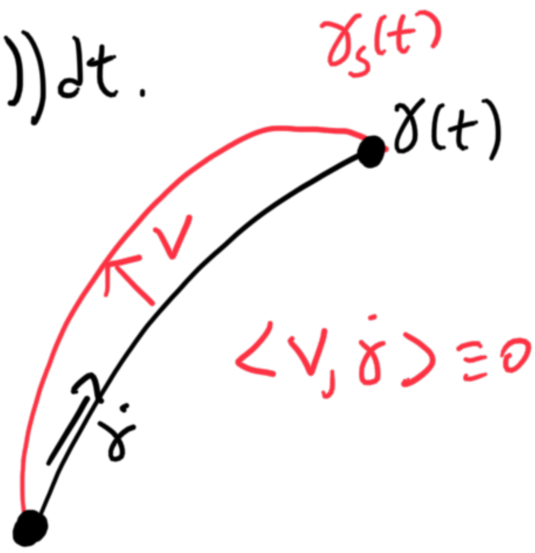
Let $\gamma_s(t)$ be a family of curves for $t \in [a,b]$, $s \in (-\varepsilon, \varepsilon)$, with $\gamma_0 = \gamma$.

Let $V = \frac{d}{ds}\Big|_{s=0} \gamma_s$ and assume $\langle V, \dot{\gamma}_0 \rangle_g (t) \equiv 0$.

Assume $\gamma_s(a) = p$, $\gamma_s(b) = q$ $\forall s$.

Then:

$$\frac{d^2}{ds^2}\Big|_{s=0} L_g(\gamma_s) = \int_a^b \left(|D_t V|^2 + Rm(V, \dot{\gamma}, V, \dot{\gamma}) \right) dt.$$



Proof:

Recall: $\gamma = \gamma_s(t)$

$\dot{\gamma} = \frac{d}{dt}\gamma_s$, $V = \frac{d}{ds}\gamma_s$

$D_t X^k = \partial_t X^k + \dot{\gamma}^i \Gamma_{il}^k X^l$

$D_s X^k = \partial_s X^k + V^i \Gamma_{il}^k X^l$

$$\Rightarrow \frac{d}{ds} |\dot{\gamma}|^2 = 2 \langle D_s \dot{\gamma}, \dot{\gamma} \rangle$$

$$\overset{(*)}{=} 2 \langle D_t V, \dot{\gamma} \rangle$$

$D_s \dot{\gamma} = D_t V$ since $\Gamma_{il}^k = \Gamma_{li}^k$

Next, recall: $[\nabla_i, \nabla_j] W^l = R_{ijp}{}^l W^p$

$[D_s, D_t] W^l = V^i \dot{\gamma}^j R_{ijp}{}^l W^p$

$$\Rightarrow \frac{d^2}{ds^2}\langle\dot\gamma,\dot\gamma\rangle = 2\langle D_s D_t V,\dot\gamma\rangle + 2\langle D_t V, D_s\dot\gamma\rangle$$

$$= 2\langle D_t D_s V,\dot\gamma\rangle + 2 g_{lk} V^i \dot\gamma^j R_{ijp}{}^{l} V^p \dot\gamma^k + 2|D_t V|^2$$

$$\frac{d^2}{ds^2}\langle\dot\gamma,\dot\gamma\rangle \overset{(**)}{=} 2\frac{d}{dt}\langle D_sV,\dot\gamma\rangle + 2|D_tV|^2 + 2R(V,\dot\gamma,V,\dot\gamma).$$

$$\frac{d}{ds}\int_a^b|\dot\gamma| = \int_a^b \frac{1}{2|\dot\gamma|}\frac{d}{ds}\langle\dot\gamma,\dot\gamma\rangle$$

$\langle V,\dot\gamma\rangle\equiv 0$

$$\frac{d^2}{ds^2}\int_a^b|\dot\gamma| \overset{(*)\,(**)}{=} \int_a^b -|\dot\gamma|^{-3/2}\underbrace{\langle D_tV,\dot\gamma\rangle}_{=0}{}^2 + \int_a^b \frac{1}{|\dot\gamma|}\left(\frac{d}{dt}\langle D_sV,\dot\gamma\rangle + |D_tV|^2 + R(V,\dot\gamma,V,\dot\gamma)\right)$$

$=0$ since $\frac{d}{dt}\langle V,\dot\gamma\rangle = \langle D_tV,\dot\gamma\rangle + \langle V,\cancel{D_t\dot\gamma}\rangle^{0}$

Use: $|\dot\gamma|=1$,

$$\frac{d^2}{ds^2}\int_a^b|\dot\gamma| = \Big|_a^b \underbrace{\langle D_sV,\dot\gamma\rangle}_{=0} + \int_a^b |D_tV|^2 + R(V,\dot\gamma,V,\dot\gamma).$$

□

$=0$ since $D_sV = 0$ @ a,b

$V(a)=0$, $\frac{d}{ds}\Big|_{a=0} V = 0$

$\gamma_s(a)=p \quad \forall s$

Denote: $I(V,W) = \int_a^b \left(\langle D_t V, D_t W \rangle + R(V, \dot\gamma, W, \dot\gamma) \right) dt$

for V, W vector fields along $\gamma(t)$.

Thm: Let $\gamma(t)$ be unit speed geodesic from p to q. Suppose γ has an interior conjugate pt. Then $\gamma(t)$ is not length minimizing.

from defn of conj pt

Pf: Write $\gamma : [a,b] \to M$, and $J(t)$ a normal Jacobi field with $J(a)=0$, $J(c)=0$, $a<c<b$.

Goal: construct X normal vector field along γ s.t. $I(X,X)<0$. Then family $\gamma_s(t) = \gamma(t) + sX$ is s.t.

$$\frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_s) \overset{\text{2nd var formula}}{=} I(X,X) < 0. \Rightarrow \text{contradicts that } \gamma \text{ is minimizer of length.}$$

Consider: $V(t) = \begin{cases} J(t) & a \le t \le c \\ 0 & c \le t \le b \end{cases}$

normal: if $\langle J, \dot\gamma \rangle = 0 \Rightarrow \langle D_t J, \dot\gamma \rangle = 0$

$W(t)$ = normal VF s.t.: $W(c) = -D_t J(c)$.
$W(a) = 0$
$W(b) = 0$

$$X_\varepsilon(t) = V + \varepsilon W.$$

Then: $I(X_\varepsilon, X_\varepsilon) = I(V,V) + 2\varepsilon I(V,W) + \varepsilon^2 I(W,W)$.

① $I(V,V) = \int_a^c \langle D_t J, D_t J \rangle + R(J, \dot\gamma, J, \dot\gamma)$

$$= \Big|_a^c \langle D_t J, \underset{\substack{\| \\ 0 \\ \text{conjugate pts}}}{J} \rangle - \int_a^c \langle \underbrace{D_t^2 J + R(J,\dot\gamma)\dot\gamma}_{=0 \quad \text{Jacobi field eqn}}, J \rangle$$

$$= 0.$$

② $$I(V,W) = \Big|_a^c \langle D_t J, W \rangle - \int_a^c \langle \underbrace{D_t^2 J + R(J,\dot\gamma)\dot\gamma}_{=0}, W \rangle$$

$$= -|D_t J(c)|^2$$

$W(c) = -D_t J(c)$
$W(a) = 0$

$$\Rightarrow I(X_\varepsilon, X_\varepsilon) = \varepsilon \left[-2|D_t J(c)|^2 + \varepsilon I(W,W) \right]$$

$$< 0 \quad \text{for } \varepsilon \text{ small.}$$

$\square$

$|D_t J(c)| \neq 0$ because if $J(c) = 0$, $D_t J(c) = 0$ then backwards uniqueness of ODE $\Rightarrow J \equiv 0$.