Index Notation Review: $E \to M$ rank k.

- $s \in \Gamma(M, E)$, $\quad (\mathcal{U}, s^i), (\tilde{\mathcal{U}}, \tilde{s}^i)$

$\tilde{s}^i = Q^i{}_p s^p$ $\quad$ Q = transition function on $\mathcal{U} \cap \tilde{\mathcal{U}}$.

$\tilde{s} = Q s$

$$s = \begin{pmatrix} s^1 \\ \vdots \\ s^k \end{pmatrix}, \quad Q = \begin{pmatrix} Q^1{}_1 & Q^1{}_2 & \cdots & Q^1{}_k \\ Q^2{}_1 & Q^2{}_2 & & \vdots \\ \vdots & & & \\ Q^k{}_1 & \cdots & & Q^k{}_k \end{pmatrix}$$

- $\phi \in \Gamma(M, E^*)$

$\tilde{\phi}_i = \phi_p (Q^{-1})^p{}_i$ $\quad$ $\phi = (\phi_1 \ \phi_2 \ \cdots \ \phi_k)$

$\tilde{\phi} = \phi Q^{-1}$

- $h \in \Gamma(M, E \otimes E^*)$

$\tilde{h}^i{}_j = Q^i{}_p (Q^{-1})^q{}_j h^p{}_q$

$\tilde{h} = Q h Q^{-1}$

- $H \in \Gamma(M, E^* \otimes E^*)$

$\tilde{H}_{ij} = (Q^{-1})^p{}_i (Q^{-1})^q{}_j \tilde{H}_{pq}$

$\tilde{H} = (Q^{-1})^T H Q^{-1}$

Metrics

A metric on a vector bundle $E \to M$ is $H \in \Gamma(M, E^* \otimes E^*)$ s.t. $H|_p$ restricts to inner product on $E|_p \times E|_p$.

Local expression:

$M = \bigcup \mathcal{U}_\alpha$, metric is collection $(\mathcal{U}_\alpha, H^{\mathcal{U}_\alpha}_{ij})$
with $H^{\mathcal{U}}_{ij}(x)$ symmetric, positive-definite matrices

$$H^{\mathcal{U}}_{ij} : \mathcal{U} \to GL(k, \mathbb{R})$$

satisfying the transformation law:

$$H^{\mathcal{U}}_{ij} = H^{V}_{\alpha\beta}\, c_{V\mathcal{U}}{}^{\alpha}{}_{i}\, c_{V\mathcal{U}}{}^{\beta}{}_{j} \quad \text{on } \mathcal{U} \cap V.$$

$H^{\mathcal{U}} = c_{V\mathcal{U}}{}^{T} H^{V} c_{V\mathcal{U}}$ matrix notation

obtain inner product on sections $\phi, \psi \in \Gamma(M, E)$

$\langle \phi, \psi \rangle_H = H_{ij}\, \phi^i \psi^j . = \phi^T H \psi$ matrix notation

well-defn: $\langle \phi, \psi \rangle_H$ produces the same number using $(\mathcal{U}, \phi_{\mathcal{U}}, \psi_{\mathcal{U}}, H^{\mathcal{U}})$ or (V, ϕ_V, ψ_V, H^V):

i.e. $H^{\mathcal{U}}_{ij}\, \psi_{\mathcal{U}}{}^{i}\, \phi_{\mathcal{U}}{}^{j} = H^{V}_{\alpha\beta}\, \psi_V{}^{\alpha} \phi_V{}^{\beta}$.

check: $\psi \mapsto Q\psi$
$\phi \mapsto Q\phi$
$H \mapsto (Q^{-1})^T H Q^{-1}$.

$$\langle \phi, \psi \rangle_H = \phi^T H \psi \mapsto (Q\phi)^T (Q^{-1})^T H Q^{-1} (Q\psi)$$

$$= \phi^T \cancel{Q^T (Q^T)^{-1}} H \cancel{Q^{-1} Q} \psi$$

invariant under triv change ✓

Note: Metric provides isomorphism between E and E^*

$s^i \in \Gamma(M, E) \rightsquigarrow s_i = H_{ij} s^j \in \Gamma(M, E^*)$ index notation

$s \in \Gamma(M, E) \rightsquigarrow \langle s, \cdot \rangle_H \in \Gamma(M, E^*)$ invariant notation

Check that s_i transforms correctly: $s^{\mathcal{U}}_i = s^V_j\, c_{V\mathcal{U}}{}^{j}{}_{i}$.

Note: Any vector bundle has a metric.

$M = \bigcup \mathcal{U}_\alpha$, $\{\psi_\alpha\}$ partition of unity wrt $\mathcal{U}_\alpha$

Define $\langle v, w \rangle_H = \sum \psi_\alpha \langle v_{\mathcal{U}_\alpha}, w_{\mathcal{U}_\alpha} \rangle_{(\mathbb{R}^k,\, g_{Euc})}$

ex) Metric on TM denoted g_{ij} ← metric tensor
Riemannian metric

Transformation Law:

$$\tilde{g}_{ij} = \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} g_{pq} \quad \text{overlap } (\mathcal{U}, x^i),\ (\tilde{\mathcal{U}}, \tilde{x}^i)$$

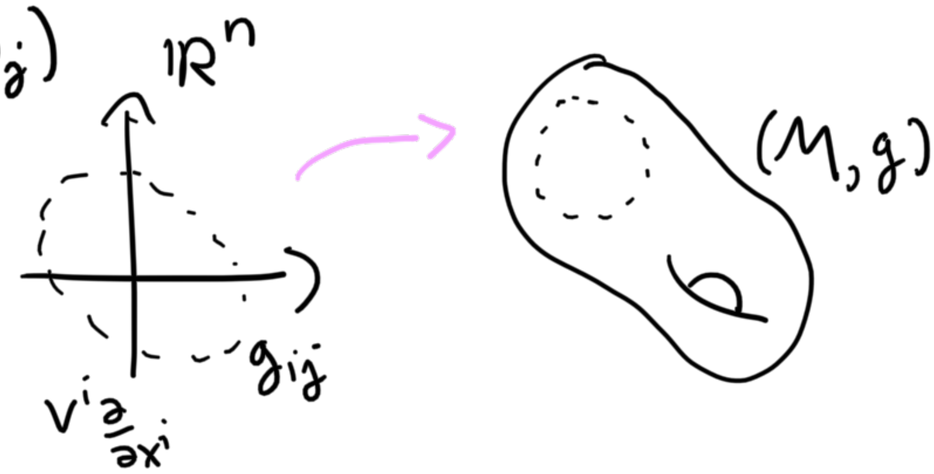
Notation:

$$g \overset{loc}{=} g_{ij}\, dx^i \otimes dx^j = g_{ij}\, dx^i dx^j$$

$$g(V, W) = \langle V, W \rangle_g = g_{ij} V^i W^j, \qquad V = V^i \frac{\partial}{\partial x^i}, \quad W = W^i \frac{\partial}{\partial x^i}$$

$$g(\partial_i, \partial_j) = g_{ij}$$

$$\begin{aligned} g(v, w) &= g(v^i \partial_i, w^j \partial_j) \\ &= v^i w^j g(\partial_i, \partial_j) \\ &= v^i w^j g_{ij}. \end{aligned}$$



g^{ij} for inverse of g_{ij}:

$$g^{ik} g_{kj} = \delta^i{}_j$$

Inverse transformation:

$$\tilde{g}^{ij} = \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial \tilde{x}^j}{\partial x^q} g^{pq}$$

← g^{pq} defines section of $TM \otimes TM$.

$\Rightarrow$ Inverse metric g^{ij} defines metric on T^*M.

$$\langle \alpha, \beta \rangle_g = g^{ij} \alpha_i \beta_j .$$

Can raise index: convert 1-form to vector field.

Given $\alpha = \alpha_i dx^i$ $\rightsquigarrow$ obtain $\alpha^i = g^{ij} \alpha_j$ transforms like vector field

$$\alpha^\# = \alpha^i \partial/\partial x^i$$

Invariant way: Induced VF acts on functions by

$$\alpha^\#(f) := \langle \alpha, df \rangle_g .$$

Conventions for differential forms:

$\alpha \in \Omega^k(M)$ denotes $\alpha \in \Gamma(M, (T^*M)^{\otimes k})$ which is anti-symmetric.

$$\alpha \overset{\text{loc}}{=} \frac{1}{k!} \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where components $\alpha_{i_1 \cdots i_k}$ anti-symmetric.

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{p \in S_k} \operatorname{sgn}(p)\, dx^{i_{p(1)}} \otimes \cdots \otimes dx^{i_{p(k)}} .$$

recall

e.g. $dx^1 \otimes dx^2 \otimes dx^4 (V, W, X) = V^1 W^2 X^4$.

e.g. $dx^1 \wedge dx^2 \wedge dx^4 (V, W, X) = V^1 W^2 X^4 - V^1 W^4 X^2 - V^2 W^1 X^4 + V^2 W^4 X^1 - V^4 W^2 X^1 + V^4 W^1 X^2$

$$\alpha(V_1, \ldots, V_k) = \alpha_{i_1 \cdots i_k} V_1^{i_1} \cdots V_k^{i_k},$$

anti-sym $\alpha(V, W) = -\alpha(W, V)$

$$\langle \alpha, \beta \rangle_g = g^{i_1 j_1} \cdots g^{i_k j_k} \alpha_{i_1 \cdots i_k} \beta_{j_1 \cdots j_k} .$$

ex) 2-form $\alpha \in \Omega^2(M)$. $\alpha = \frac{1}{2}\alpha_{ij}\, dx^i \wedge dx^j$

reason for $\frac{1}{2}$:

$$\alpha(V,W) = \frac{1}{2}\alpha_{ij}\, dx^i \wedge dx^j (V,W) = \frac{1}{2}\alpha_{ij}(v^i w^j - v^j w^i)$$

$$= \left(\frac{1}{2}+\frac{1}{2}\right)\alpha_{ij} V^i W^j$$ since $\alpha_{ij} = -\alpha_{ji}$ antisymmetry

Agrees with $\alpha(V,W) = \alpha_{ij} V^i W^j$.

Inner prod on 2-forms:

$$\langle \alpha, \beta \rangle_g = g^{ij} g^{kl} \alpha_{ik} \beta_{jl} = \alpha_{ik}\beta^{ik}.$$

<u>Prop</u>: $E \to M$ vector bundle with metric $\langle \cdot, \cdot \rangle_H$. Can trivialize $M = \cup \mathcal{U}_\alpha$ s.t. transition functions are valued in $O(k)$.

<u>Pf</u>: In each triv $\mathcal{U} \times \mathbb{R}^k$, we have a local frame $\{e_a\}$. Using Gram-Schmidt at each $p \in \mathcal{U}$, construct an orthonormal frame wrt $\langle \cdot, \cdot \rangle_H$.

$f_a(p) = A^b{}_a(p)\, e_b(p)$, $\langle f_a, f_b \rangle = \delta_{ab}$.

On overlap $\mathcal{U} \cap \tilde{\mathcal{U}}$, can write:

$\tilde{f}_a = c_{\mathcal{U}\tilde{\mathcal{U}}}{}^b{}_a f_b$ for some linear combo $c_{\mathcal{U}\tilde{\mathcal{U}}}{}^b{}_a$.

$(\mathcal{U} \cap \tilde{\mathcal{U}}, c_{\mathcal{U}\tilde{\mathcal{U}}})$ form new trans fun

Notice: $\delta_{ab} = \langle \tilde{f}_a, \tilde{f}_b \rangle = \langle f_i\, c_{\mathcal{U}\tilde{\mathcal{U}}}{}^i{}_a, f_j\, c_{\mathcal{U}\tilde{\mathcal{U}}}{}^j{}_b \rangle$

$\delta_{ab} = c_{\mathcal{U}\tilde{\mathcal{U}}}{}^i{}_a\, c_{\mathcal{U}\tilde{\mathcal{U}}}{}^i{}_b$ $\qquad A = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix}$

$\mathrm{Id} = c_{\mathcal{U}\tilde{\mathcal{U}}}^T\, c_{\mathcal{U}\tilde{\mathcal{U}}}$. ✓ $\qquad (AB)^i{}_j = A^i{}_k B^k{}_j$

$\square$

$GL(k)$

<u>Extra details</u>: OG bundle has trans fun $(\mathcal{U}\cap\tilde{\mathcal{U}}, \hat{c}_{\mathcal{U}\tilde{\mathcal{U}}})$

Gram-Schmidt $\rightsquigarrow$ new $(\mathcal{U}\cap\tilde{\mathcal{U}}, c_{\mathcal{U}\tilde{\mathcal{U}}})$.

$O(k)$

why are these isomorphic?

$A = \{(\mathcal{U}, (A^{\mathcal{U}})^b{}_a\}$ defines bundle isomorphism.

$\tilde{f}_a = \tilde{A}^b{}_a \tilde{e}_b = \tilde{A}^b{}_a \hat{c}_{\mathcal{U}\tilde{\mathcal{U}}}{}^c{}_b e_c$

$\tilde{f}_a = c_{\mathcal{U}\tilde{\mathcal{U}}}{}^b{}_a f_b = c_{\mathcal{U}\tilde{\mathcal{U}}}{}^b{}_a A^c{}_b e_c$.

$\Rightarrow \hat{c}_{\mathcal{U}\tilde{\mathcal{U}}} \tilde{A} = A c_{\mathcal{U}\tilde{\mathcal{U}}}$ $\Rightarrow$ correct transformation law for $A \in \Gamma(M, \mathrm{Hom}(E, \hat{E}))$.

<u>Pullback Metric</u>:

$f: M \to (N, g)$ with $\ker[f_*|_p] = 0 \ \forall p \in M$.

f^*g pullback metric: x^i coords on M, y^α coords on N, $f^\alpha = y^\alpha \circ f$

$$(f^*g)_{ij}(p) = g_{\alpha\beta}(f(p)) \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}.$$

Invariant way: $(f^*g)_p(X, Y) = g_{f(p)}(f_* X, f_* Y)$

Recall: push forward VF. $f_*|_p : T_pM \to T_{f(p)}M$

$f^*g(X,X) > 0$ if $X \neq 0$ since $\ker f_* = 0$.

$V = V^i \frac{\partial}{\partial x^i}$, $(f_*V)^\alpha(f(p)) = \frac{\partial f^\alpha}{\partial x^i}(p) V^i(p)$

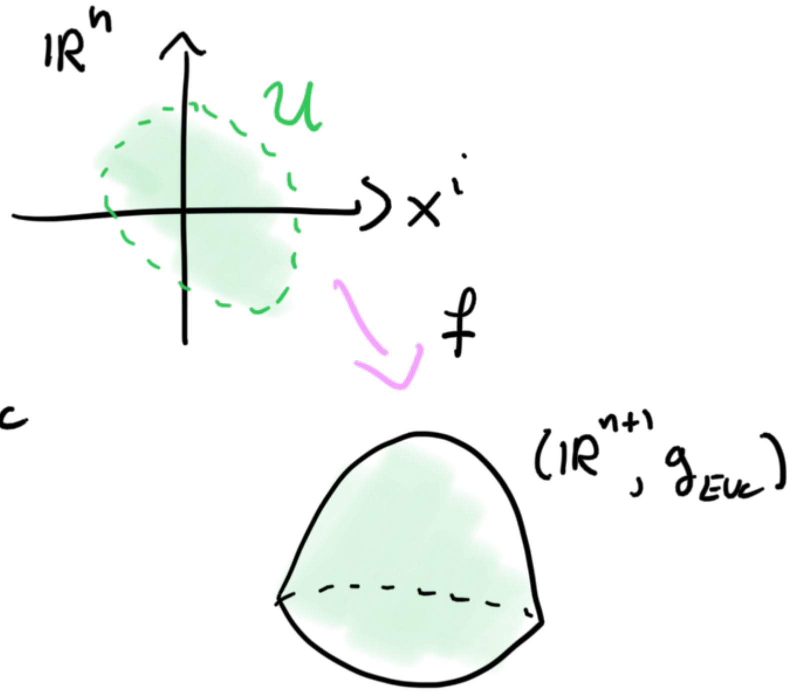
or: $(f_*V)(\phi) = V(\phi \circ f)$.

Note: push forward/pullback formulas vs change of coords

<u>Def</u>: A diffeomorphism $f: (M, \tilde{g}) \to (N, g)$ is an isometry if $f^*g = \tilde{g}$.

ex) Let $f : \underset{\subseteq \mathbb{R}^n}{\mathcal{U}} \to \mathbb{R}^{n+1}$ be a parametrized hypersurface. (Df rank n)

Classical metric tensor
(AKA 1st fund form)



$$g = f^* g_{Euc}$$

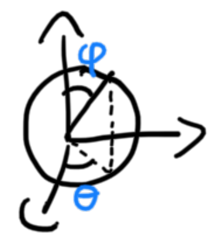
$$g_{ij} = \left\langle \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle_{g_{Euc}}$$

$$= \delta_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}$$

e.g. for spherical coords

$$f(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$$

$$g_{S^2} = (\sin\varphi)^2 d\theta d\theta + d\varphi d\varphi$$



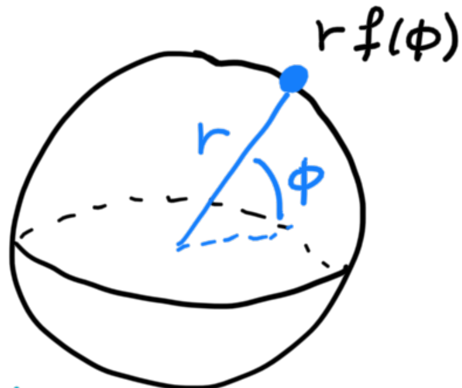
g_{Euc} in polar coords:

$$g_{Euc} = dr^2 + r^2 g_{S^{n-1}}(\phi)$$

Explanation: polar coords are of form

$$x^i = r f^i(\phi), \quad f : \underbrace{\mathcal{U}}_{\text{angle coords}} \to S^{n-1} \text{ param of sphere.}$$

Transformation of metric:



$$\underbrace{\tilde{g}_{\alpha\beta}}_{\substack{\text{metric} \\ \text{in new} \\ \text{coords}}} = \frac{\partial x^i}{\partial \tilde{x}^\alpha} \underbrace{\delta_{ij}}_{g_{Euc} \text{ in Cartesian coords}} \frac{\partial x^j}{\partial \tilde{x}^\beta}$$

$$\tilde{x} = (r, \phi_1, \cdots, \phi_{n-1})$$

$$g_{rr} = f^i \delta_{ij} f^j = \underline{1} \quad f \text{ on sphere}$$

$$g_{r\phi_1} = f^i \delta_{ij} r \frac{\partial f^j}{\partial \phi_1} = r \left\langle f, \frac{\partial f}{\partial \phi_1} \right\rangle_{g_{Euc}}$$

$$= \frac{r}{2} \frac{\partial}{\partial \phi_1} \langle f, f \rangle_{g_{Euc}} = 0.$$

$$g_{\phi_\alpha \phi_\beta} = r^2 \frac{\partial f^i}{\partial \phi_\alpha} \delta_{ij} \frac{\partial f^j}{\partial \phi_\beta} \Rightarrow g_{\phi_\alpha \phi_\beta} = r^2 \underbrace{(f^* g_{Euc})}_{g_{S^{n-1}}(\phi)}{}_{\phi_\alpha \phi_\beta}$$