# Riemann Curvature

Let $\nabla$ be Levi-Civita connection on $TM$.

Christoffel symbols $\Gamma_{ij}^k$. John Lee conventions

$$R_{ijk}{}^{l} = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{ip}^l \Gamma_{jk}^p - \Gamma_{jp}^l \Gamma_{ik}^p$$

Special case of $E \to M$, $\nabla = d + A$ with $E = TM$,

$A_i{}^k{}_j = \Gamma_{ij}^k$, $\quad F_{ij}{}^p{}_q = R_{ijq}{}^p$. ← difference in conventions to match Lee's book

Riemann curv tensor

$$R = R_{\underbrace{ij}_{\text{2-form indices}}\underbrace{k^{l}}_{\text{End}\,TM\text{ indices}}} \in \Omega^2(\mathrm{End}\,TM)$$

More notation: for $X, Y, Z \in \Gamma(TM)$,

plug into 2-form act by endomorphism

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^l \partial_l.$$

Consistency check:

$$\begin{aligned}
R(\partial_i, \partial_j)\partial_k &= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \\
&= \nabla_{\partial_i}(\Gamma_{jk}^n \partial_n) - (i \leftrightarrow j) \\
&= \partial_i \Gamma_{jk}^n \partial_n + \Gamma_{jk}^n \Gamma_{in}^r \partial_r - (i \leftrightarrow j) \\
&= \left(\partial_i \Gamma_{jk}^l + \Gamma_{ip}^l \Gamma_{jk}^p - (i \leftrightarrow j)\right)\partial_l \quad ✓ \\
&= R_{ijk}{}^l \partial_l
\end{aligned}$$

## Second covariant derivatives:

$$V = V^k \partial_k$$

$$\nabla_{\partial_i} V = (\nabla_i V^k)\partial_k$$

$$\nabla_i V^k = \partial_i V^k + \Gamma_{ij}^{k} V^j$$

Lee's book: $V^k{}_{/p} = \nabla_p V^k$

Then $T_i{}^k := \nabla_i V^k$ defines $T \in \Gamma(TM \otimes T^*M)$.

$$\nabla_j T_i{}^k = \partial_j T_i{}^k + \Gamma_{jp}^{k} T_i{}^p - T_p{}^k \Gamma_{ji}^{p}$$

covariant derivative

Write: $\nabla_j \nabla_i V^k := \nabla_j T_i{}^k$

Lee's book: $V^k{}_{/ij} = \nabla_j \nabla_i V^k$

$$\nabla_{\partial_i} \nabla_{\partial_j} V \neq (\nabla_i \nabla_j V^k)\partial_k .$$

In these notes: $\nabla_{\partial_i}$ acts on vectors: $\nabla_{\partial_i} V = (\nabla_i V^k)\partial_k$

$\nabla_i$ acts on components of tensors: $\nabla_i V^k = \partial_i V^k + \Gamma_{ij}^{k} V^j$

$$\begin{aligned}
\nabla_{\partial_i} \nabla_{\partial_j} V &= \nabla_{\partial_i}(\nabla_j V^k \partial_k) \\
&= \partial_i(\nabla_j V^k)\partial_k + \nabla_j V^k \nabla_{\partial_i}\partial_k \\
&= \left(\partial_i(\nabla_j V^k) + \Gamma_{il}^{k} \nabla_j V^l\right)\partial_k .
\end{aligned}$$

vs:

$$\nabla_i \nabla_j V^k = \partial_i(\nabla_j V^k) \underbrace{- \Gamma_{ij}^{l} \nabla_l V^k}_{\text{new term}} + \Gamma_{il}^{k} \nabla_j V^l$$

However, it is nevertheless true that

$$\nabla_i \nabla_j V^k - \nabla_j \nabla_i V^k = R_{ijp}{}^k V^p ,$$

Check: $[\nabla_i, \nabla_j] V^k$

$$= \nabla_i \nabla_j V^k - \overset{\text{switch}}{(i\,j)}$$

$$= \partial_i \nabla_j V^k + \Gamma_{ip}^{k} \nabla_j V^p - \nabla_p V^k \underbrace{\cancel{\Gamma_{ij}^{p}}}_{\text{sym}} - \underbrace{(i\,j)}_{\text{antisym}} \to 0$$

$$= \partial_i (\cancel{\partial_j V^k} + \Gamma_{jp}^{k} V^p) + \Gamma_{ip}^{k} (\partial_j V^p + \Gamma_{j\ell}^{p} V^\ell) - (i\,j)$$

$$= \partial_i \Gamma_{j\ell}^{k} V^\ell + \underbrace{\Gamma_{jp}^{k} \partial_i V^p + \Gamma_{ip}^{k} \partial_j V^p}_{\to 0 \text{ sym in } (i,j)} + \Gamma_{ip}^{k} \Gamma_{j\ell}^{p} V^\ell - (i\,j)$$

$$= \left( \partial_i \Gamma_{j\ell}^{k} + \Gamma_{ip}^{k} \Gamma_{j\ell}^{p} - (i\,j) \right) V^\ell \quad \checkmark$$

<u>Def</u>: $R_{ijp\ell} = g_{\ell k} R_{ijp}{}^{k}$

$$Rm(X, Y, Z, W) = \langle R(X,Y)Z, W \rangle_g$$
$$= R_{ijp\ell} X^i Y^j Z^p W^\ell.$$

<u>Remark</u>: Given $p \in M$, can choose coords s.t.
$g_{ij}(p) = \delta_{ij}$ identity.

Indeed: if $G = [g_{ij}(p)]$, take square root of matrix:

$G = Q^T Q \Rightarrow$ new coords $\tilde{x}^i = Q^i{}_j x^j$

$$g_{ij} \mapsto \frac{\partial x^p}{\partial \tilde{x}^i} g_{pq} \frac{\partial x^q}{\partial \tilde{x}^j}.$$

$$g_{ij}(p) \mapsto (Q^{-1})^T G\, Q^{-1} = I.$$

Q: Can we find coords s.t.

$$g_{ij}(x) \equiv \delta_{ij} \text{ in a nbhd of } p?$$

Thm: $Rm(g) \equiv 0 \iff \forall p \in M, \exists$ local coords $(\mathcal{U}, x^i)$ s.t. $g_{ij} \equiv \delta_{ij}$ in $\mathcal{U}$.

pf: $(\Leftarrow)$ trivial

$(\Rightarrow)$ We proved earlier that $F \equiv 0 \Rightarrow \exists$ ON frame $\{e_a\}$ on $\mathcal{U}$ with $\nabla e_a = 0$.

Since torsion $= 0$,
$[e_a, e_b] = \nabla_{e_a} e_b - \nabla_{e_b} e_a = 0$.

By integrability thm, $\exists$ coords $x^i$ near $p$ with $\frac{\partial}{\partial x^i} = e_i$.

$$g_{ij} = g(\partial_i, \partial_j) = g(e_i, e_j) = \delta_{ij}. \qquad \square$$

## Local Expression:

- Let $a \in M$. Suppose $(x^i)$ are local coords s.t.

$$g_{ij}(a) = \delta_{ij}, \quad \partial_k g_{ij}(a) = 0$$

(e.g. take normal coords at p)

$$\Rightarrow R_{ijp\ell}(a) = g_{\ell k} \partial_i \Gamma^k_{jp} - g_{\ell k} \partial_j \Gamma^k_{ip}$$

$$= \frac{1}{2} \partial_i \left( -\partial_\ell g_{jp} + \partial_j g_{\ell p} + \partial_p g_{j\ell} \right) - (i\ j)$$

Recall: $\Gamma^k_{jp} = \frac{g^{kq}}{2}(-\partial_q g_{jp} + \partial_j g_{qp} + \partial_p g_{jq})$

(*)

$$R_{ijp\ell}(a) = \frac{1}{2}\left(-\partial_i \partial_\ell g_{jp} + \partial_i \partial_p g_{j\ell} + \partial_j \partial_\ell g_{ip} - \partial_j \partial_p g_{i\ell}\right)$$

## Symmetries:

- $R_{ijpl} = -R_{jipl}$
- $R_{ijpl} = -R_{ijlp}$
- $R_{ijpl} = R_{plij}$
- $R_{ijpl} + R_{pijl} + R_{jpil} = 0$

Check: directly in normal coords $(\mathcal{U}, x^i)$ $(*)$

For arbitrary coords $(V, \tilde{x}^i)$, transition fun $\frac{\partial x}{\partial \tilde{x}}$,

$$(R_V)_{ijpl} = \frac{\partial x^a}{\partial \tilde{x}^i} \frac{\partial x^b}{\partial \tilde{x}^j} \frac{\partial x^c}{\partial \tilde{x}^p} \frac{\partial x^d}{\partial \tilde{x}^l} \underbrace{(R_{\mathcal{U}})}_{}{}_{abcd}$$

if symmetry here, then symmetry for $(R_V)_{ijkl}$.

This transformation rule follows from:

$$[\nabla_i, \nabla_j] V^k = R_{ijp}{}^k V^p$$

indices here transform correctly $\Rightarrow$ indices on $R_{ijp}{}^k$ transform correctly

$$[\nabla_i, \nabla_j] V^k \mapsto \frac{\partial x^a}{\partial \tilde{x}^i} \frac{\partial x^b}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^k}{\partial x^c} [\nabla_a, \nabla_b] V^c$$

## Bianchi identity:

$$\nabla_m R_{ijpl} + \nabla_l R_{ijmp} + \nabla_p R_{ijlm} = 0.$$

Claim: this is equiv to $d_A F = 0$ in general formalism $\nabla = d + A$ on bundle $E \to M$.

$$d_A F = \frac{1}{2} \nabla_k F_{ij}{}^{\alpha}{}_{\beta}\, dx^k \wedge dx^i \wedge dx^j . \quad (*)$$

Earlier, used $d_\nabla F = dF + A \wedge F - F \wedge A$.

For tensor, should use:

$$\nabla_k F_{ij}{}^{\alpha}{}_{\beta} = \partial_k F_{ij}{}^{\alpha}{}_{\beta} - \underbrace{\Gamma^p_{ki}}_{\text{sym}} F_{pj}{}^{\alpha}{}_{\beta} - \underbrace{\Gamma^p_{kj}}_{\text{sym}} F_{ip}{}^{\alpha}{}_{\beta} \quad (**)$$
$$+ A_k{}^{\alpha}{}_{\gamma} F_{ij}{}^{\gamma}{}_{\beta} - F_{ij}{}^{\alpha}{}_{\gamma} A_k{}^{\gamma}{}_{\beta} .$$

Sub $(**)$ into $(*)$:

$$d_A F = \frac{1}{2}\left(\partial_k F_{ij}{}^{\alpha}{}_{\beta} - 0 - 0 + A_k{}^{\alpha}{}_{\gamma} F_{ij}{}^{\gamma}{}_{\beta} - F_{ij}{}^{\alpha}{}_{\gamma} A_k{}^{\gamma}{}_{\beta}\right) dx^k \wedge dx^i \wedge dx^j$$

$\Rightarrow d_A F = dF + A \wedge F - F \wedge A$ consistent with $(**)$

Bianchi $\Rightarrow$ $0 = \frac{1}{2} \nabla_k F_{ij}{}^{\alpha}{}_{\beta}\, dx^k \wedge dx^i \wedge dx^j$

$$0 = \frac{1}{3!}\left(\nabla_k F_{ij}{}^{\alpha}{}_{\beta} + \nabla_j F_{ki}{}^{\alpha}{}_{\beta} + \nabla_i F_{jk}{}^{\alpha}{}_{\beta}\right) dx^k \wedge dx^j \wedge dx^j .$$

If $F_{ij}{}^{q}{}_{p} = R_{ijp}{}^{q}$, using $\nabla_k g_{ij} = 0$ then:

$$0 = \nabla_k R_{ijpl} + \nabla_j R_{kipl} + \nabla_i R_{jkpl}$$

$$\Rightarrow 0 = \nabla_k R_{plij} + \nabla_j R_{plki} + \nabla_i R_{pljk} .$$

Defn: Ricci curvature

$$R_{ij} = R_{pij}{}^{p}$$

basically just 1 way (up to sign) to contract without getting zero

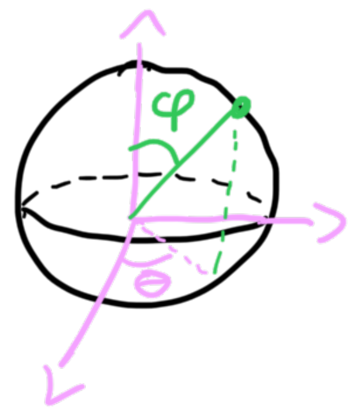
Defn: Scalar curvature

$$S = g^{ij} R_{ij}$$

Note: $R_{ij} = R_{ji}$, also $R_{ij} = R_{ip}{}^{p}{}_{j} = -R_{ipj}{}^{p}$.

Exercise: In spherical coords, metric on sphere of radius $a>0$ appears as

$$g = a \sin^2\varphi \, d\theta \otimes d\theta + a \, d\varphi \otimes d\varphi$$




Compute:

$$\Gamma_{ij}^{\theta} = \begin{bmatrix} 0 & \cos\varphi/\sin\varphi \\ \cos\varphi/\sin\varphi & 0 \end{bmatrix}$$

$$\Gamma_{ij}^{\varphi} = \begin{bmatrix} -\sin\varphi\cos\varphi & 0 \\ 0 & 0 \end{bmatrix}$$

$R_{\theta\varphi\varphi\theta} = a \sin^2\varphi$ (others determined by symmetry of curvature in 2D)

$$R_{ij} = \begin{bmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{bmatrix},$$ note: $R_{ij} = a^{-1} g_{ij}$ (Einstein metric)

$S = 2/a$ constant, positive scalar curvature

## Sectional curvature:

- $\Pi \subseteq T_pM$ 2D subspace, $\Pi = \text{span}(V, W)$

$$\sec(\Pi) = \sec(V, W) = \frac{R(V, W, W, V)}{\left(|V|^2|W|^2 - \langle V, W\rangle^2\right)}$$

obs: If $\sec \equiv c$, then:

$$Rm(X, Y, Z, W) = c\left(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\right)$$

Indeed: Let $T = (LHS) - (RHS)$.

$\Rightarrow T(X, Y, Y, X) \equiv 0$ + $T$ satisfies symmetries of Riemann curv tensor.

$$0 = T(X+W, Y, Y, X+W)$$
$$= T(X, Y, Y, W) + T(W, Y, Y, X) \quad \text{Sym of Rm}$$
$$\Rightarrow 0 = T(X, Y, Y, W)$$

$$0 = T(X, Y+Z, Y+Z, W)$$
$$0 = T(X, Y, Z, W) + T(X, Z, Y, W) \quad *$$

Bianchi:

$$0 = T(X, Y, Z, W) + T(Z, X, Y, W) + T(Y, Z, X, W)$$
$$0 \overset{*}{=} T(X, Y, Z, W) - T(X, Z, Y, W) - T(Y, X, Z, W)$$
$$0 = 3T(X, Y, Z, W) \checkmark$$

Note: Same argument shows: if $\sec_{g_1} = \sec_{g_2}$, then $Rm_{g_1} = Rm_{g_2}$.

## Commutator Formulas:

We noted earlier that

$$[\nabla_i, \nabla_j] V^k = R_{ijp}{}^k V^p. \qquad V \in \Gamma(TM)$$

This implies

$$[\nabla_i, \nabla_j] \alpha_k = R_{ij}{}^p{}_k \alpha_p. \qquad \alpha \in \Omega^1(M)$$

Indeed:

$$\begin{aligned}
[\nabla_i, \nabla_j] \alpha_k &= [\nabla_i, \nabla_j] g_{kp} \alpha^p \\
&= g_{kp} [\nabla_i, \nabla_j] \alpha^p \\
&= R_{ij}{}^l{}_k \alpha_l.
\end{aligned}$$

__Note:__ For $f \in C^\infty(M, \mathbb{R})$, can define the Hessian of $f$:

$$\nabla_i \nabla_j f = \partial_i \partial_j f - \Gamma_{ij}^l \partial_l f.$$

Note
$\nabla_i \nabla_j f = \nabla_j \nabla_i f.$

The Laplace operator is $\Delta f = \nabla^p \nabla_p f$, or:

$$\Delta = g^{pq} \nabla_p \nabla_q.$$

Can apply $\Delta$ on functions: $\Delta : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$
Can also apply $\Delta$ on tensors

ex)
$$\begin{aligned}
\Delta \nabla_i f &= g^{pq} \nabla_p \nabla_q \nabla_i f \\
&= g^{pq} \nabla_p \nabla_i \nabla_q f \\
&= g^{pq} \nabla_i \nabla_p \nabla_q f + g^{pq} R_{pi}{}^l{}_q \nabla_l f \\
&= \nabla_i \Delta f + R_{il} \nabla^l f
\end{aligned}$$

apply $\Delta$ to 1-form $\alpha_i = \nabla_i f$.
$\Delta : \Omega^1(M) \to \Omega^1(M)$

$\alpha_i = \nabla_i f$
$\Delta \alpha_i := (\Delta \alpha)_i$

This type of calculation is used in "Bochner formulas".

More examples of commuting covariant derivatives:

ex) $[\nabla_i, \nabla_j] W^k{}_l = R_{ijp}{}^k W^p{}_l + R_{ij}{}^p{}_l W^k{}_p$

ex) $\nabla_i \nabla_j \nabla_k V^l$ commute $\nabla_i \; \nabla_j \; \nabla_k$.

$$
\begin{aligned}
\nabla_i \nabla_j \nabla_k V^l &= \nabla_i \left( \nabla_k \nabla_j V^l + R_{jkp}{}^l V^p \right) \\
&= \nabla_i \nabla_k \nabla_j V^l + \nabla_i \left( R_{jkp}{}^l V^p \right) \\
&= \nabla_k \nabla_i \nabla_j V^l + R_{ik}{}^p{}_j \nabla_p V^l \\
&\quad + R_{ikp}{}^l \nabla_j V^p + \nabla_i \left( R_{jkp}{}^l V^p \right).
\end{aligned}
$$