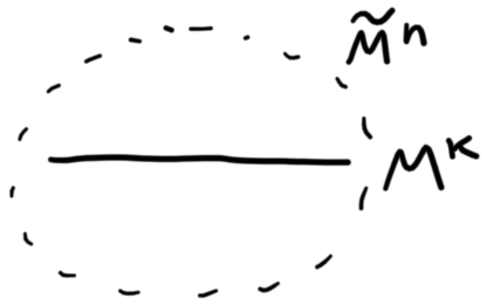
Geometry of Submanifolds

- $(\tilde{M}, \tilde{g})$ ambient space, $\tilde{\nabla}$ metric connection
- $f: M \to \tilde{M}$ submanifold

submfd theory: $\forall p \in M$, $\exists$ nbhd $\tilde{U} \subseteq \tilde{M}$ with



coords on $\tilde{M}$: $\{x^\mu\}_{\mu=1}^{n}$ s.t.

$\tilde{U} \cap M = \{x^{k+1} = \cdots = x^n = 0\}$

Coords on M : $\{x^i\}_{i=1}^{k}$

$f(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$.

Let $T\tilde{M}|_M = f^* T\tilde{M} := \tilde{E}$.

$\tilde{E} \to M^k$ is a rank n bundle over M^k.

$\tilde{E}|_p = \text{span}\left\{ \frac{\partial}{\partial x^\mu}\Big|_p \right\}_{\mu=1}^{n}$.

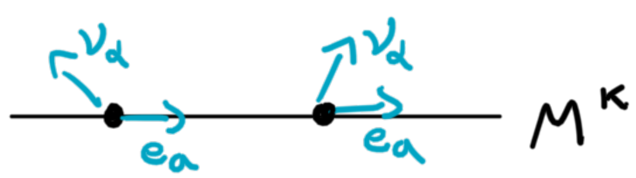
- From $\tilde{\nabla}$ on $T\tilde{M}$, obtain $\tilde{\nabla}$ on $\tilde{E}$:

$$\tilde{\nabla}_{\partial_i}(V^\mu \partial_\mu) = (\partial_i V^\mu + \Gamma_{i\beta}^{\ \ \mu} V^\beta)\, \partial_\mu .$$

- Use $\tilde{g}$ to split $\tilde{E} = TM \oplus TM^\perp$.

Denote $\{e_a\}$ orthonormal frame for TM
$\{\nu_\alpha\}$ orthonormal frame for $TM^\perp$

$\tilde{E}|_p = \text{span}\{e_1, \dots, e_k, \nu_1, \dots, \nu_{n-k}\}$



metric $\tilde{g}_{ij} \neq g_{Euc}$

$$\tilde{\nabla}_{\partial_i} e_a = A_i{}^b{}_a\, e_b + B_i{}^\mu{}_b\, \nu_\mu$$
$$\tilde{\nabla}_{\partial_i} \nu_\alpha = C_i{}^b{}_\alpha\, e_b + A^\perp_i{}^\mu{}_\alpha\, \nu_\mu$$

- Obtain connection ∇ on $TM \subseteq \tilde{E} = T\tilde{M}|_M$

$TM = \text{span}\{e_1, \dots, e_n\}$

$\nabla_{\partial_i} e_a := \left[(\tilde{\nabla}_{\partial_i} e_a) \right]^T$ check this defines connection

$\nabla_{\partial_i} e_a = A_i{}^b{}_a e_b$ $\quad \nabla = d + A$

$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y)$ $\quad X = X^a e_a$, $Y = Y^a e_a$

$B(X,Y) = X^a Y^b B_a{}^\alpha{}_b \nu_\alpha$.

$B \in \Omega^1(M, \text{Hom}(TM, TM^\perp))$ is 2^{nd} fund form

check B well-defn: $B_i(e_a) = \tilde{\nabla}_{\partial_i} e_a - \nabla_{\partial_i} e_a$

$B_i(V^a(x) e_a) = V^a(x) B_i(e_a)$

In frame $\{e_1, \dots, e_k, \nu_1, \dots, \nu_{n-k}\}$, $\tilde{\nabla} = d + \tilde{A}$ becomes:

$$\tilde{A} = \begin{array}{cc} \begin{pmatrix} A & C \\ B & A^\perp \end{pmatrix} & \begin{array}{l} TM \\ TM^\perp \end{array} \end{array}$$

$\nabla = d + A$ connection TM

$\nabla^\perp = d + A^\perp$ connection $TM^\perp$

$B \in \Omega^1(M, \text{Hom}\,TM, TM^\perp)$

$$\left[\begin{array}{l} \tilde{\nabla}_i e_b = A_i{}^a{}_b e_a + B_i{}^\alpha{}_b \nu_\alpha \\ \tilde{\nabla}_i \nu_\alpha = C_i{}^a{}_\alpha e_a + (A_i^\perp)^\mu{}_\alpha \nu_\mu \end{array} \right.$$

Metric compatibility:

$0 = \partial_i \langle e_a, \nu_\alpha \rangle$

$= \langle \tilde{\nabla}_{\partial_i} e_a, \nu_\alpha \rangle + \langle e_a, \tilde{\nabla}_{\partial_i} \nu_\alpha \rangle$

$0 = B_i{}^\alpha{}_a + C_i{}^a{}_\alpha$

$$\Rightarrow \quad \tilde{A} = \begin{pmatrix} A & -B^* \\ B & A^\perp \end{pmatrix}$$

Compute curvature: $\tilde{F} = d\tilde{A} + \tilde{A} \wedge \tilde{A}$

$$\tilde{A} \wedge \tilde{A} = \begin{pmatrix} A & -B^* \\ B & A^\perp \end{pmatrix} \begin{pmatrix} A & -B^* \\ B & A^\perp \end{pmatrix}$$

$$= \begin{pmatrix} A \wedge A - B^* \wedge B & -A \wedge B^* - B^* \wedge A^\perp \\ B \wedge A + A^\perp \wedge B & A^\perp \wedge A^\perp - B \wedge B^* \end{pmatrix}$$

$$\tilde{F} = \begin{pmatrix} F - B^* \wedge B & -dB^* - A \wedge B^* - B^* \wedge A^\perp \\ dB + B \wedge A + A^\perp \wedge B & F^\perp - B \wedge B^* \end{pmatrix}$$

Generalized Gauss - Codazzi eqn

First entry: $p : T\tilde{M} \to TM$, $\iota : TM \hookrightarrow T\tilde{M}$

$$p \circ \tilde{F} \circ \iota = F - B^* \wedge B$$

with indices: a, b, c : $\{e_a\}$ tangential dir
α, β, γ : $\{\nu_\alpha\}$ normal dir
i, j, k : coords on M

$$(1) \quad \tilde{F}_{ij}{}^a{}_b = F_{ij}{}^a{}_b - \sum_\alpha B_i{}^\alpha{}_a B_j{}^\alpha{}_b + \sum_\alpha B_j{}^\alpha{}_a B_i{}^\alpha{}_b$$

$$(F^a{}_b) = \begin{pmatrix} F^1{}_1 & F^1{}_2 \\ F^2{}_1 & F^2{}_2 \end{pmatrix}$$

$$(2) \quad \tilde{F}_{ij}{}^\alpha{}_b = \partial_i B_j{}^\alpha{}_b + A^\perp{}_i{}^\alpha{}_\beta B_j{}^\beta{}_b + B_i{}^\alpha{}_a A_j{}^a{}_b - (i \leftrightarrow j).$$

<u>Special Case</u>: $M \subseteq (\tilde{M}, \tilde{g})$ codim 1.
only one normal direction ν.

Denote: $h_{ib} = B_i{}^\nu{}_b$.

J. Lee conventions: $R_{ijq}{}^p = F_{ij}{}^p{}_q$,
$\langle R(\partial_i, \partial_j) e_b, e_a \rangle = R_{ijba}$

(For us: $\langle F(\partial_i, \partial_j) e_b, e_a \rangle = F_{ij}{}^a{}_b$)

(1) $\tilde{R}_{ijba} = R_{ijba} - h_{ia}h_{jb} + h_{ja}h_{ib}$

$A^{\perp\alpha}{}_{\alpha} = 0$

(2) $\tilde{R}_{ijb\alpha} = \partial_i h_{jb} + 0 + h_{ia} A_j{}^{a}{}_{b} - \partial_j h_{ib} - h_{ja} A_i{}^{a}{}_{b}$

$- \underbrace{\Gamma_{ij}^{k}}_{} h_{kb}$ ← killed by skew-sym $(i \leftrightarrow j)$

$$\tilde{R}_{ijb\alpha} = \nabla_i h_{ja} - \nabla_j h_{ia}$$

acts now on both indices:
j with Levi-Civita Γ_{ij}^{k} ← is actually also LC
a with induced $A_j{}^{a}{}_{b}$.

<u>Prop</u>: If $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is Levi-Civita connection,

then induced (M, g, ∇) is Levi-Civita connection.

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\,\nu$$

for all $X, Y \in \Gamma(M, TM)$.

<u>Pf</u>: Extend X, Y arbitrarily to $\tilde{X}, \tilde{Y} \in \Gamma(\tilde{M}, T\tilde{M})$

claim:
$\nabla_X Y := (\tilde{\nabla}_{\tilde{X}} \tilde{Y})^{T}$ is LC and

does not depend on extension.

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y}(p) = \Big(\underbrace{X^i(p)\,\partial_i \tilde{Y}^k}_{\frac{d}{dt}\big|_{t=0} Y^k(\gamma(t)),} + \tilde{\Gamma}_{ij}^{k}(p)\, X^i(p)\, Y^j(p)\Big)\,\partial_k$$

$\gamma(0) = p$
$\dot{\gamma}(0) = X(p)$
$\gamma(t) \in M$

Will show: metric compatibility + torsion-free.

Metric compatibility:

Let $X, Y, Z \in \Gamma(TM)$.

$$\nabla_X \langle Y, Z \rangle = \langle \tilde{\nabla}_X Y, \overset{\text{tangent}}{Z} \rangle + \langle \overset{\text{tangent}}{Y}, \tilde{\nabla}_X Z \rangle$$

$$= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

Torsion free: $X, Y \in \Gamma(TM)$

$$\nabla_X Y - \nabla_Y X = (\tilde{\nabla}_X Y - \tilde{\nabla}_Y X)^T$$

$$= [X, Y]^T = [X, Y].$$

why $[X,Y]^T = [X,Y]$: M submfd $\Rightarrow \exists$ coords s.t. $M = \{x^1 = \cdots x^k = 0\}$.

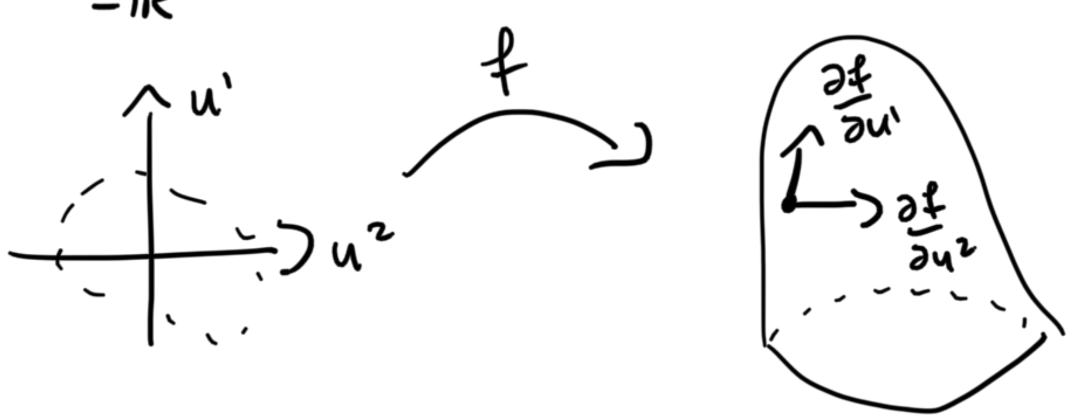
$$T\tilde{M} \overset{\text{loc}}{=} \text{span}\left(\frac{\partial}{\partial x^1}, \cdots \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^{k+1}}, \cdots, \frac{\partial}{\partial x^n}\right)$$

$$TM \overset{\text{loc}}{=} \text{span}\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}\right)$$

obv: $X, Y \in TM \Rightarrow [X,Y] \in TM.$ $\square$

Special Case: $M \subseteq (\mathbb{R}^3, g_{Euc})$ hypersurface.

$f: \underset{\subseteq \mathbb{R}^2}{U} \to M \subseteq \mathbb{R}^3$ local parametrization



$\subseteq \mathbb{R}^3$

$$T_pM = \text{span}\left\{ \frac{\partial f}{\partial u^1}\bigg|_p, \frac{\partial f}{\partial u^2}\bigg|_p \right\}$$

Tangent vectors: $X(f(u)) = X^i(u) \dfrac{\partial f}{\partial u^i}(u)$

Let $X, Y \in \Gamma(TM)$.

$$\tilde{\nabla}_X Y = D_X Y$$ directional derivative in $\mathbb{R}^3$

$$= \frac{d}{dt}\Big|_{t=0} Y(f(\gamma(t))),$$

$c(t) = f \circ \gamma(t)$ path on M

$\dot{c} = \partial_i f \dot{\gamma}^i$

if $\dot{\gamma}^i(0) = X^i$, then

$\dot{c}(0) = X^i \partial_i f = X$

$$= X^i \frac{\partial}{\partial u^i}\left(Y^k(u) \frac{\partial f}{\partial u^k} \right)$$

$$= X^i \partial_i Y^k \underbrace{\partial_k f}_{\text{tangent}} + X^i Y^k \partial_i \partial_k f$$

$$h(X, Y) = \langle \tilde{\nabla}_X Y, \nu \rangle = X^i Y^k \langle \partial_i \partial_k f, \nu \rangle$$

$$\quad \shortparallel$$

$$h_{ij} X^i Y^j$$

$$h_{ij} = \langle \partial_i \partial_j f, \nu \rangle.$$ Note $h_{ij} = h_{ji}$

Gauss-Codazzi eqn with $\tilde{R} \equiv 0$:

$$\begin{cases} R_{ijba} = h_{ia} h_{jb} - h_{ja} h_{ib} \\ \nabla_i h_{ja} = \nabla_j h_{ia} \end{cases}$$

$$R = R_{ij}{}^{ji} = h_i{}^i h_j{}^j - h_j{}^i h_i{}^j$$

$$R = (\operatorname{Tr} h)^2 - \operatorname{Tr} h^2$$

view $h_i{}^j = h_{ik} g^{kj} = h^j{}_i$

endomorphism, trace well-defn

Dimension $n = 2$

$$R = (\lambda_1 + \lambda_2)^2 - (\lambda_1^2 + \lambda_2^2)$$ λ_i eigenvalues of $h^i{}_j$

$$= 2\lambda_1 \lambda_2$$

$$\Rightarrow R = 2 \underbrace{\det h^i{}_j}_{\text{"Gauss curvature"}} = 2K$$

Gauss Thm Egregium

$\Rightarrow$ R_{ijkl}, $F_{ij}{}^{p}{}_{q}$ etc, all generalizations of Gauss's curvature on $M \subseteq \mathbb{R}^3$.

<u>NB</u>: Mean curvature is $H := \mathrm{Tr}\, h^i{}_j$.
<u>NB</u>: $K = \det h_{ij} / \det g_{ij}$

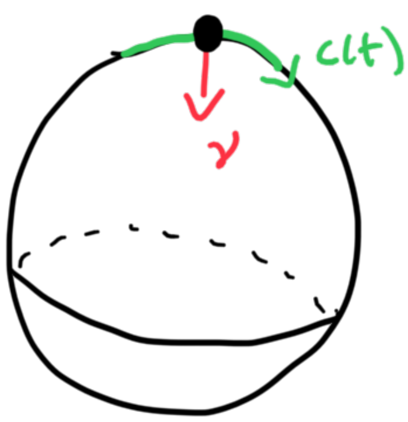
<u>Visualizing Gauss curvature:</u>

$$h(X,X) = \langle \ddot{c}, \nu \rangle (0), \quad X = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

$$c(t) = f \circ \gamma(t) \qquad \dot{\gamma}(0) = X$$

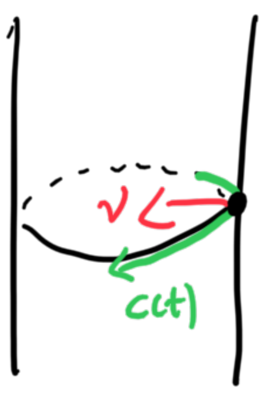
$$\dot{c}(t) = \partial_i f \; \dot{\gamma}^i(t)$$
$$\ddot{c}(t) = \partial_i \partial_j f \; \dot{\gamma}^i \dot{\gamma}^j + \partial_i f \; \ddot{\gamma}^i \quad \leftarrow \text{tangent}$$
$$\langle \ddot{c}, \nu \rangle (0) = \langle \partial_i \partial_j f, \nu \rangle X^i X^j$$
$$= h_{ij} X^i X^j \quad ✓$$

ex)

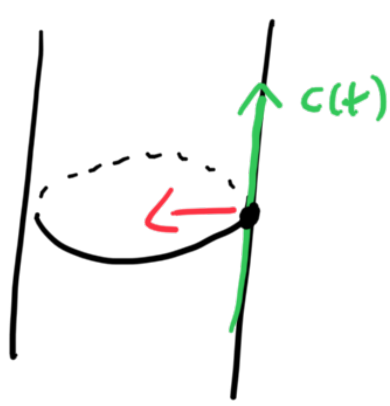


$K(p) > 0$ all test curves give positive $h(X,X)$

ex)

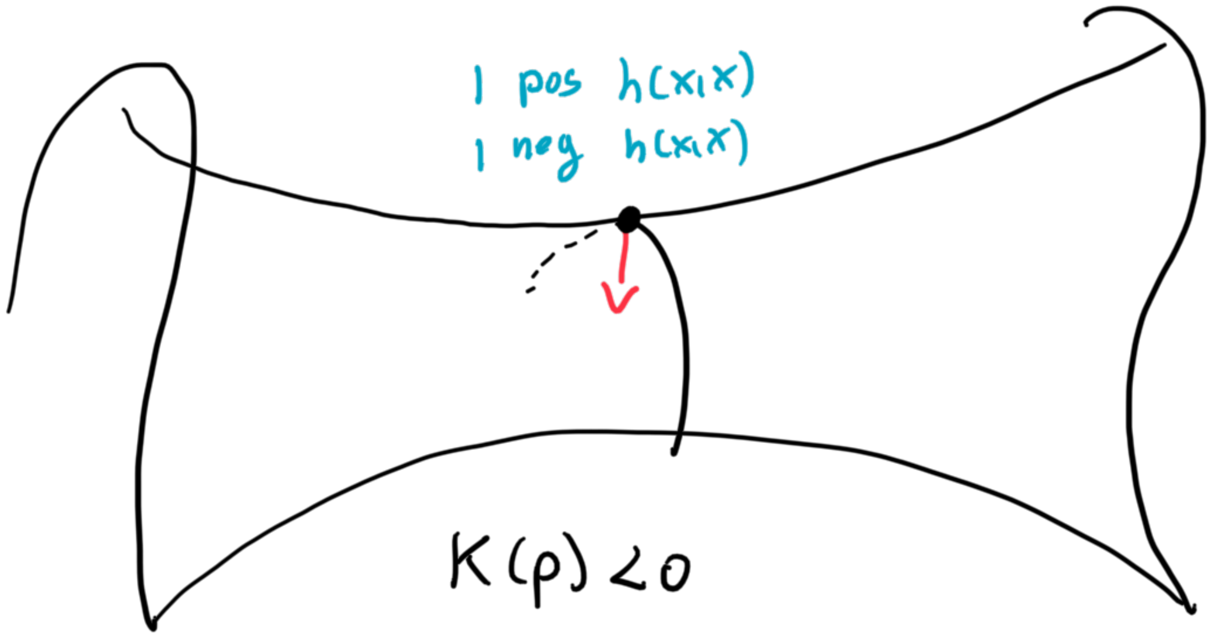


zero eigenvalue



$K(p) = 0$

ex)



$K(p) < 0$

ex)

$K(p) > 0$

$K(p) < 0$

$K(p) = 0.$