# de Rham cohomology

Let $M$ be mfd dim $n$.

$$H^k(M,\mathbb{R}) = \frac{\operatorname{Ker}\{d : \Omega^k(M) \to \Omega^{k+1}(M)\}}{\operatorname{Im}\{d : \Omega^{k-1}(M) \to \Omega^k(M)\}}.$$

ex) $H^0(M,\mathbb{R}) = \{ f \in C^\infty(M,\mathbb{R}) : df = 0 \}$

$= \{ \text{constant functions} \}$

$\Rightarrow H^0(M,\mathbb{R}) \cong \mathbb{R}.$

<u>Thm</u>: Let $M, N$ be homotopic smooth mfd.
Then $H^k(M,\mathbb{R}) \cong H^k(N,\mathbb{R})$.

Recall: $M \simeq N$ are homotopic if $\exists$ smooth maps
$f: M \to N$, $g: N \to M$ s.t. $f \circ g \simeq id$
$g \circ f \simeq id$.

Recall: smooth maps $\phi, \psi : M \to N$ are homotopic
$(\phi \simeq \psi)$ if $\exists$ smooth $F : M \times [0,1] \to N$ s.t.
$F(x,0) = \phi$
$F(x,1) = \psi$.

<u>Pf of Thm:</u>

From $\phi: M \to N$, can define

$\phi^* : H^k(N,\mathbb{R}) \to H^k(M,\mathbb{R})$ by:

$\phi^*[\alpha] = [\phi^*\alpha]$.

if $d\alpha = 0$, then
$d\phi^*\alpha = \phi^* d\alpha = 0$.

<u>Claim</u>: given $\phi, \psi : M \to N$, $\phi \simeq \psi$, then:
$\phi^*, \psi^* : H^k(N,\mathbb{R}) \to H^k(M,\mathbb{R})$ are equal.

Assuming claim, we prove the thm.

Take $f: M \to N$ with $f \circ g \simeq id$
$g: N \to M$ $g \circ f \simeq id$

Claim $\Rightarrow$ $(f \circ g)^* = id$ $\Rightarrow$ $g^* f^* = id$
$(g \circ f)^* = id$ $f^* g^* = id$

$\Rightarrow f^*: H^k(N, \mathbb{R}) \to H^k(M, \mathbb{R})$ is isomorphism.

Proof of claim: let $f_t = F(\cdot, t)$ be a family of maps with $f_0 = \phi$, $f_1 = \psi$.

Let $\alpha \in \Omega^k(N)$
$d\alpha = 0$.

$$f_1^*\alpha - f_0^*\alpha = \int_0^1 \frac{d}{dt}(f_t^*\alpha)\,dt$$

$$= \int_0^1 f_t^* L_X \alpha \, dt$$

defn of Lie derivative
$X = \frac{d}{dt} f_t$

$$= \int_0^1 f_t^* d\iota_X \alpha \, dt$$

Cartan's formula
$L_X = d\iota_X + \iota_X d$

$$= d\left(\int_0^1 f_t^* \iota_X \alpha \, dt\right)$$

$\Rightarrow f_1^*\alpha - f_0^*\alpha = d\beta$

$\Rightarrow [f_1^*\alpha] = [f_0^*\alpha] \Rightarrow f_1^* = f_0^*: H^k(N) \to H^k(M)$.

$\square$

ex) Let $B$ = ball radius $R$ in $\mathbb{R}^n$.

$H^k(B, \mathbb{R}) = 0 \quad \forall k \geq 1$ since $B \simeq \{pt\}$.

$\Rightarrow$ If $d\alpha = 0$, then $\alpha = d\beta$.

<u>Prop</u>: Let $M$ be compact, orientable.

$\Rightarrow H^n(M, \mathbb{R}) \cong \mathbb{R}$, and

$$[\alpha] \mapsto \int_M \alpha \quad \text{is an isomorphism.}$$

<u>Pf:</u>

Let $g_{ij}$ be a metric on $M$, with volume $dvol_g = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$.

Since $\int_M dvol_g \neq 0$, $[\alpha] \mapsto \int_M \alpha$ is surjective onto $\mathbb{R}$ by taking $\alpha = \lambda\, dvol_g$.

To show injectivity, suppose $\alpha \in \Omega^n(M)$ satisfies $\int_M \alpha = 0$. Will show $\alpha = d\beta$ so $[\alpha] = 0$.

PDE Thm: can solve $\Delta_g f \, dvol_g = \alpha$ for a smooth $f \in C^\infty(M, \mathbb{R})$ iff $\int_M \alpha = 0$.

Assuming this, can write

$$\begin{aligned}
\alpha &= \Delta_g f \, dvol_g \\
&= \nabla_p \nabla^p f \, dvol_g \\
&= \nabla_p X^p \, dvol_g, \qquad X^p = \nabla^p f \\
&= d\, \iota_X \, dvol_g. \quad \Rightarrow [\alpha] = 0.
\end{aligned}$$

$\square$

ex) $H^0(S^1, \mathbb{R}) = \mathbb{R}$

$H^1(S^1, \mathbb{R}) = \mathbb{R}$

<u>Mayer-Vietoris Sequence:</u>

Let $M = U \cup V$ union of open sets.

Then $\exists$ long exact sequence:

$$\cdots \to H^k(M) \to H^k(\mathcal{U}) \oplus H^k(V) \to H^k(\mathcal{U}\cap V) \to H^{k+1}(M) \to \cdots$$

Exact seq: Kernel = image of map behind it.

Homological algebra: long exact seq comes from short exact seq:

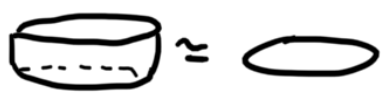
$$0 \to \Omega^p(M) \to \Omega^p(\mathcal{U}) \oplus \Omega^p(V) \to \Omega^p(\mathcal{U}\cap V) \to 0.$$

ex) Cohomology of $S^2$.

Already know $H^2(S^2, \mathbb{R}) = \mathbb{R}$, $H^0(S^2, \mathbb{R}) = \mathbb{R}$. Will compute $H^1(S^2, \mathbb{R})$.

$\mathcal{U} \simeq pt$

$V \simeq pt$

$\mathcal{U}\cap V =$  $= S^1$

Mayer-Vietoris:

$$\begin{tikzcd}
0 \arrow[r] & H^0(S^2) \arrow[r] & H^0(\mathcal{U}) \oplus H^0(V) \arrow[r] & H^0(\mathcal{U}\cap V) \arrow[r] & H^1(S^1) \arrow[d] \\
 & & & & H^1(\mathcal{U}) \oplus H^1(V)
\end{tikzcd}$$

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \to H^1(S^1) \to 0$$

For $T: V \to W$ linear map,
$\dim V = \dim \ker T + \dim \operatorname{Im} T$.

$$0 \to \underset{\substack{\dim\ker = 0 \\ \dim\mathrm{Im} = 1}}{\mathbb{R}} \to \underset{\substack{\dim\ker = 1 \\ \dim\mathrm{Im} = 1}}{\mathbb{R}^2} \to \underset{\substack{\dim\ker = 1 \\ \dim\mathrm{Im} = 0}}{\mathbb{R}} \to \underset{\substack{\dim\ker = 0 \\ \dim\mathrm{Im} = 0}}{H^1(S^1)} \to 0$$

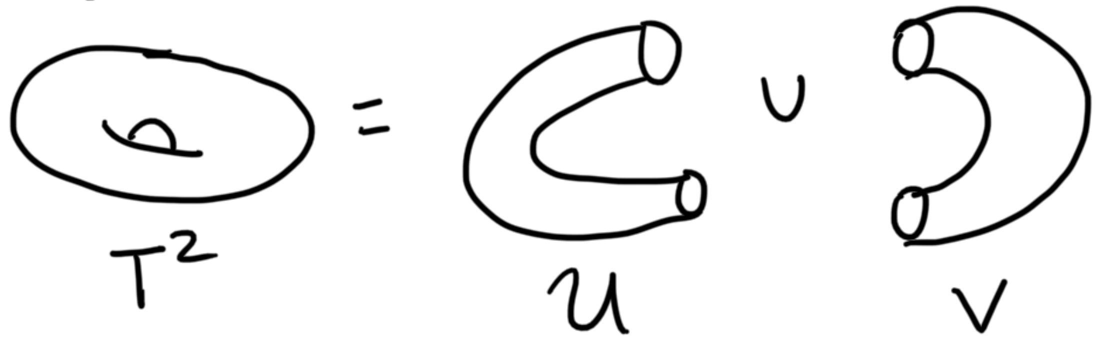
$\Rightarrow \dim H^1(S^1, \mathbb{R}) = 0.$

ex) Cohomology of $T^2$

Already know $H^0(T^2) \cong \mathbb{R}$
$H^2(T^2) \cong \mathbb{R}$

Will compute $H^1(T^2) \cong \mathbb{R}^2$.

Mayer-Vietoris:




$\mathcal{U} \simeq \bigcirc = S^1$
$V \simeq \bigcirc = S^1$

$\mathcal{U} \cap V =$  $\simeq S^1 \sqcup S^1$

$$\to H^0(T^2) \to H^0(\mathcal{U}) \oplus H^0(V) \to H^0(\mathcal{U}\cap V) \to$$
$$\to H^1(T^2) \to H^1(\mathcal{U}) \oplus H^1(V) \to H^1(\mathcal{U}\cap V) \to H^2(T^2)$$
$$\downarrow$$
$$H^2(\mathcal{U}) \oplus H^2(\mathcal{U})$$

dimKer=0 dimKer=1 dimKer=1
dimIm=1 dimIm=1 dimIm=1

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^2 \to H^1(T^2) \to \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R} \to 0$$

dimKer=1 dimKer=1 dimKer=1
dimIm=1 dimIm=1 dimIm=1

$\Rightarrow \dim H^1(T^2) = 1 + 1$
$\Rightarrow H^1(T^2) \cong \mathbb{R}^2$.

## de Rham Thm:

$$H^k_{dR}(M, \mathbb{R}) \cong H^k_{sing}(M, \mathbb{R})$$

singular cohomology from algebraic topology