Curvature and Cohomology Classes

- More general setup: consider

$E \to M$ a complex vector bundle.

Same defn as before, but $E|_p \cong \mathbb{C}^k$

$c_{uv} \in GL(k, \mathbb{C})$.

- $\nabla = d + A$ connection,
$F = dA + A \wedge A$ curvature, $F \in \Omega^2(M, \operatorname{End} E)$.

1. $\operatorname{Tr} F \in \Omega^2(M)$, since in triv $U \cap V$:

$$\operatorname{Tr} F_U = \operatorname{Tr} c_{uv} F_V c_{uv}^{-1} = \operatorname{Tr} F_V.$$

2. $d \operatorname{Tr} F = 0$.

Bianchi $\Rightarrow d_\nabla F = 0 \Rightarrow dF + A \wedge F - F \wedge A = 0$

$$d \operatorname{Tr} F = \operatorname{Tr} dF = \operatorname{Tr} F \wedge A - \operatorname{Tr} \underbrace{A}_{\text{1-form}} \wedge \underbrace{F}_{\text{2-form}}$$

$$= 0 \quad \text{since } \operatorname{Tr} AB = \operatorname{Tr} BA.$$

3. Let A, A' be two connections on $E \to M$.
Then $\operatorname{Tr} F - \operatorname{Tr} F' = d\alpha$, $\alpha \in \Omega^1(M)$.

$$\operatorname{Tr} F - \operatorname{Tr} F' = \operatorname{Tr} (dA + A \wedge A - dA' - A' \wedge A')$$

$$= d \operatorname{Tr} (A - A'), \quad \text{since } \operatorname{Tr} A \wedge A \xrightarrow{} 0 \;\; = -\operatorname{Tr} A \wedge A$$

<u>claim</u>: $A = A' + a$ with $a \in \Omega^1(M, \operatorname{End} E)$

$\Rightarrow \operatorname{Tr} F - \operatorname{Tr} F' = d \operatorname{Tr} a$

4. Let $$H^p(M) = \frac{\{\ker d : \Omega^p(M) \to \Omega^{p+1}(M)\}}{\{\operatorname{Im} d\, \Omega^{p-1}(M)\}}.$$

Then $[\mathrm{Tr} F] = [\mathrm{Tr} F']$

$\Rightarrow [\mathrm{Tr} F] \in H^2(M)$ indep of choice of connection on $E \to M$.

To finish, check claim:

By $A_u = c_{uv} A_v c_{uv}^{-1} - dc_{uv} c_{uv}^{-1}$, then

$$A'_u - A_u = c_{uv} (A'_v - A_v) c_{uv}^{-1}$$

$\Rightarrow A' - A \in \Omega^1(M, \mathrm{End} E)$

<u>More generally</u>: Given $\nabla = d + A$, define

$$ch_m(F) = \left(\frac{i}{2\pi}\right)^m \mathrm{Tr}(F \wedge \cdots \wedge F) \in \Omega^{2m}(M)$$

To explain factors of i:

Let $\langle \cdot, \cdot \rangle$ be Hermitian metric on E:

$$\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle$$
$$\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$$
$$\langle u, u \rangle \geq 0$$
$$\langle u, v \rangle = \overline{\langle v, u \rangle}$$

Let ∇ be metric compatible:

$$\partial_i \langle \varphi, \psi \rangle = \langle \nabla_i \varphi, \psi \rangle + \langle \varphi, \nabla_i \psi \rangle.$$

Let $\{e_a\}$ be an orthonormal frame

$$\nabla_i e_a = A_i{}^b{}_a e_b$$

$$0 = \partial_i \langle e_a, e_b \rangle = \langle A_i{}^c{}_a e_c, e_b \rangle + \langle e_a, A_i{}^c{}_b e_c \rangle$$

$$\Rightarrow 0 = A_i{}^b{}_a + \overline{A_i{}^a{}_b}$$

$\Rightarrow A = A_i\, dx^i$ valued in $u(n)$ = skew-hermitian

Note $F_{jk} = \partial_j A_k - \partial_k A_j + A_j A_k - A_k A_j$.

$\Rightarrow F$ is 2-form locally valued in $u(n)$.

$\Rightarrow F^\dagger = -F$ $\quad F^\dagger = \overline{F^T}$ dagger for adjoint

$$\Rightarrow \overline{ch_m(F)} = (-1)^m \left(\frac{i}{2\pi}\right)^m Tr(\bar{F} \wedge \cdots \wedge \bar{F})$$

$$= (-1)^m (-1)^m ch_m(F).$$

$\therefore ch_m(F) \in \Omega^{2m}(M, \mathbb{R})$ real

NB: $E \to M$ real bundle

$\nabla = d + A$ metric compatible

same argument $\Rightarrow A$ valued in $o(n)$ skew-symmetric matrices

$\Rightarrow F^T = -F$

$\Rightarrow Tr F = -Tr F = 0$.

Nothing interesting here, but $Tr\, F \wedge F$ non-trivial.

<u>Thm</u>: $[ch_m(F)] \in H^{2m}(M)$

indep of choice of connection. Just write $[ch_m(E)]$.

<u>Pf</u>: First, check $d\, Tr\, F \wedge \cdots \wedge F = 0$.

This follows from:

$$d\, Tr\, F \wedge \cdots \wedge F = Tr\, d_\nabla F \wedge \cdots \wedge F + \cdots + Tr\, F \wedge \cdots \wedge d_\nabla F$$

combined with Bianchi identity: $d_\nabla F = 0$.

Check:

$$d \,\mathrm{Tr}\, F\wedge\cdots\wedge F = \mathrm{Tr}\, dF\wedge\cdots\wedge F + \cdots + \mathrm{Tr}\, F\wedge\cdots\wedge dF$$
$$= \mathrm{Tr}\, d_\nabla F\wedge\cdots\wedge F + \cdots + \mathrm{Tr}\, F\wedge\cdots\wedge d_\nabla F$$
$$- \mathrm{Tr}\, A\wedge F\wedge\cdots\wedge F + \mathrm{Tr}\, \underbrace{F}\wedge \underbrace{A\wedge\cdots\wedge F} \quad \text{line cancels}$$
$$- \mathrm{Tr}\, \underbrace{F\wedge\cdots\wedge A}\wedge \underbrace{F} + \mathrm{Tr}\, F\wedge\cdots\wedge F\wedge A \quad \text{line cancels}$$

$$d_\nabla F = dF + A\wedge F - F\wedge A$$

Next: $[\mathrm{ch}_m(F)]$ indep of A

$$\mathrm{Tr}\, F'\wedge\cdots\wedge F' - \mathrm{Tr}\, F\wedge\cdots\wedge F = \int_0^1 \frac{d}{dt}\, \mathrm{Tr}\, F_t\wedge\cdots\wedge F_t \, dt$$

with
$$A_t = tA' + (1-t)A$$
$$F_t = dA_t + A_t\wedge A_t \; .$$

Compute:
$$\frac{d}{dt} F_t = d\dot{A} + \dot{A}\wedge A + A\wedge \dot{A}$$
$$= d_\nabla \dot{A}, \qquad d_\nabla b = db + A\wedge b + b\wedge A$$
for $b \in \Omega^1(\mathrm{End}\,E)$.

Note: $d_\nabla b = \nabla_i b_\kappa{}^\alpha{}_\beta \, dx^i\wedge dx^\kappa$

$$= \left(\begin{array}{l} \partial_i b_\kappa{}^\alpha{}_\beta + A_i{}^\alpha{}_\mu b_\kappa{}^\mu{}_\beta - b_\kappa{}^\alpha{}_\mu A_i{}^\mu{}_\beta \\ \quad - \underbrace{\Gamma^\rho_{i\kappa}}_{\text{sym in } (i,\kappa)} b_\rho{}^\alpha{}_\beta \end{array} \right) dx^i\wedge dx^\kappa$$

$$= db + (A_i dx^i)\wedge(b_\kappa dx^\kappa) - (b_\kappa dx^i)\wedge(A_i dx^\kappa) \quad (-1)\ \text{switch}$$
$$= db + A\wedge b + b\wedge A \, .$$

Note: $\dot{A}_t = (A' - A) \in \Omega^1(\operatorname{End} E)$. showed earlier

$$\Rightarrow \operatorname{Tr} F' \wedge \cdots \wedge F' - \operatorname{Tr} F \wedge \cdots \wedge F = m \int_0^1 \operatorname{Tr} d_\nabla \dot{A} \wedge F \wedge \cdots \wedge F \, dt.$$

Claim: $\forall\, b \in \Omega^1(\operatorname{End} E)$

$$\begin{aligned} d \operatorname{Tr} b \wedge F \wedge \cdots \wedge F = {} & \operatorname{Tr} d_\nabla b \wedge F \wedge \cdots \wedge F \\ & - \operatorname{Tr} b \wedge d_\nabla F \wedge \cdots \wedge F \\ & - \operatorname{Tr} b \wedge F \wedge \cdots \wedge d_\nabla F. \end{aligned}$$

Given claim $+\, d_\nabla F = 0$,

$$\operatorname{Tr} F' \wedge \cdots \wedge F' - \operatorname{Tr} F \wedge \cdots \wedge F = m\, d \int_0^1 \operatorname{Tr} \dot{A} \wedge F \wedge \cdots \wedge F \, dt$$

$$\Rightarrow [\operatorname{Tr} F' \wedge \cdots \wedge F'] = [\operatorname{Tr} F \wedge \cdots \wedge F].$$

Verifying the claim when $m = 3$:

$$\begin{aligned} d \operatorname{Tr} b \wedge F \wedge F &= \operatorname{Tr} db \wedge F \wedge F - \operatorname{Tr} b \wedge dF \wedge F - \operatorname{Tr} b \wedge F \wedge dF \\ &= \operatorname{Tr} d_\nabla b \wedge F \wedge F \quad - \operatorname{Tr} b \wedge d_\nabla F \wedge F \quad - \operatorname{Tr} b \wedge F \wedge d_\nabla F \\ & \quad - \operatorname{Tr} A \wedge b \wedge F \wedge F \quad - \operatorname{Tr} b \wedge A \wedge F \wedge F \\ & \quad + \operatorname{Tr} b \wedge A \wedge F \wedge F \quad - \operatorname{Tr} b \wedge F \wedge A \wedge F \quad + \\ & \quad + \operatorname{Tr} b \wedge F \wedge A \wedge F \quad - \operatorname{Tr} b \wedge F \wedge F \wedge A. \end{aligned}$$

$(d_\nabla F = dF + A \wedge F - F \wedge A)$

$\square$

<u>Naturality</u>: Let $g : E \to \tilde{E}$ be bundle isomorphism.
Then: $[\operatorname{ch}_m(\tilde{E})] = [\operatorname{ch}_m(E)]$.

Indeed: given $\nabla = d + A$ on E, define $\tilde{\nabla} = g \nabla g^{-1}$ on $\tilde{E}$. $\tilde{\nabla} s = g \tilde{\nabla}(g^{-1} s)$.

$$\begin{aligned}\tilde{\nabla} s &= g(d+A) g^{-1} s \\ &= g\, dg^{-1} s + g g^{-1} ds + g A g^{-1} s \\ &= -g g^{-1} dg\, g^{-1} s + ds + g A g^{-1} s\end{aligned}$$

$\Rightarrow \tilde{\nabla} = d + \tilde{A}$, $\tilde{A} = g A g^{-1} - dg\, g^{-1}$.

Earlier calculation: $\tilde{F} = g F g^{-1}$.

$\Rightarrow ch_m(F) = ch_m(\tilde{F})$ ✓ $\operatorname{Tr} g A g^{-1} = \operatorname{Tr} A$.

Total Chern class:

1. Linear algebra: for a diagonalizable matrix A,

$$S^{-1} A S = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

$$\begin{aligned}\det(I + A) &= \det\left(S \begin{bmatrix} 1+\lambda_1 & & \\ & \ddots & \\ & & 1+\lambda_n \end{bmatrix} S^{-1} \right) \\ &= (1+\lambda_1)\cdots(1+\lambda_n) \\ &= 1 + \sigma_1(\lambda) + \sigma_2(\lambda) + \cdots + \sigma_n(\lambda),\end{aligned}$$

$$\sigma_k = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

$\sigma_1(A) = \sum \lambda_i = \operatorname{Tr} A$

$\sigma_2(A) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left((\operatorname{Tr} A)^2 - \operatorname{Tr} AA \right)$

$\sigma_k(A) =$ combo of $\operatorname{Tr} A, \operatorname{Tr} A^2, \cdots \operatorname{Tr} A^k$.

2. Chern class. Formally:

$$c(F) = \det\left(I + \frac{iF}{2\pi}\right)$$

expand det replacing product with wedge product

view as matrix of 2-forms

$$= 1 + c_1(F) + c_2(F) + \cdots$$

$c_i(F) \in \Omega^{2i}(M)$ i^{th} Chern class

Following same pattern as before:

$$c_1(F) = \frac{i}{2\pi} \operatorname{Tr} F$$

$$c_2(F) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \left(\operatorname{Tr} F \wedge \operatorname{Tr} F - \operatorname{Tr} F \wedge F\right)$$

$\cdots$ $c_m(F)$ is a combination of $ch_m(F)$.

<u>Note</u>: $E = E_1 \oplus E_2$, $F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$

$$ch(F) = \operatorname{Tr}\left(\exp\left(\frac{i}{2\pi} F\right)\right)$$

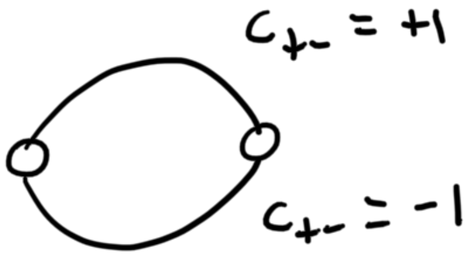
$$\Rightarrow ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2).$$

$$c(F) = \det\left(I + \frac{i}{2\pi} F\right)$$

$$\Rightarrow c(E_1 \oplus E_2) = c(E_1) \wedge c(E_2)$$

ex) Line bundle with $c_1(L) = 0$, but L non-trivial.

$L \to S^1$ Möbius bundle. Transition functions:



$\mathcal{U}_+ \cup \mathcal{U}_-$

$A_+ = 0$ $A_- = 0$

$A_+ = c A_- c^{-1} - dc\, c^{-1}$ ✓

$\nabla = d + A = d$, $F = 0$, $c_1(L) = 0 \in H^2(S^1, \mathbb{R})$.

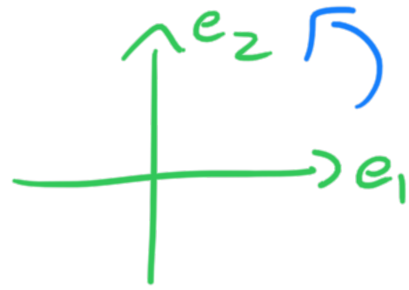
But L not trivial: any section must pass through zero.

ex) (Σ, g) oriented compact Riemann surface

can define almost complex structure $I = I(g)$:

Let $\{e_1, e_2\}$ local oriented orthonormal frame $T\Sigma$

Define $I(e_1) = e_2$
$I(e_2) = -e_1$



<u>Check:</u> if $\{f_1, f_2\}$ another ONB with

$$f_a = f^i{}_a e_i, \quad \det [f^i{}_a] > 0,$$

then $I(f_1) = f_2$
$I(f_2) = -f_1$ so I well-defn.

$$T_p^{1,0}\Sigma := \{ X - iIX : X \in T_p\Sigma \}$$

Motivation: I breaks $T_{\mathbb{C}}\Sigma$ into $+i, -i$ eigenspaces.
$T^{1,0}\Sigma$ is the $+i$ eigenspace:
$IX = iX \quad \forall X \in T^{1,0}\Sigma$.

$T^{1,0}\Sigma \to \Sigma$ complex line bundle

fiber now cplx, generated by
$e_1 - i e_2$
$e_2 + i e_1$
these are mult by i, so just one cplx direction

Note: $\int_\Sigma c_1(T^{1,0}\Sigma)$ indep of choice of $\nabla = d + A$

$$\int_\Sigma c_1(F) = \int_\Sigma c_1(F') + \cancelto{0}{\int_\Sigma dT} \qquad \text{Stokes' Thm}$$

The Gauss-Bonnet Thm states:

$$\int_\Sigma c_1(T^{1,0}\Sigma) = \chi(\Sigma)$$ ← Euler characteristic

$\chi(\Sigma) = 2-2g$ on genus g surf

classical Gauss-Bonnet

$$\int_\Sigma K\, dvol_g = 2\pi\, \chi(\Sigma),$$ (chap 9 in Lee)

↑ Gauss curvature

To reduce Chern class version to classical Gauss-Bonnet:

$(\Sigma, g) \rightsquigarrow \nabla^{LC}$ on $T\Sigma$ — Levi-Civita

① $\nabla = d + A$ on $T^{1,0}\Sigma$ — ∇^{LC} descends to $T^{1,0}\Sigma$

② $c_1(F) = \frac{i}{2\pi} \operatorname{Tr} F$

③ $i \operatorname{Tr} F = \frac{R}{2}\, dvol_g$ — Calculation linking scalar curvature

④ $\frac{R}{2} dA = K\, dvol_g$ — Gauss Thm Egregium

Details for ①: Write $\nabla^{LC} = d + A$

$$\nabla e_1 = \alpha \otimes e_2$$
$$\nabla e_2 = -\alpha \otimes e_1$$

$$A = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$$

$0 = d\langle e_1, e_1\rangle = 2\langle \nabla e_1, e_1\rangle$

$0 = d\langle e_1, e_2\rangle$

$\Rightarrow \langle \nabla e_1, e_2\rangle = -\langle e_1, \nabla e_2\rangle$

Local generator for $T^{1,0}\Sigma$: $\varepsilon = e_1 - i e_2$.

$$\nabla^{LC}\varepsilon = \alpha \otimes e_2 - i(-\alpha \otimes e_1) = i\alpha \otimes \varepsilon.$$

$\Rightarrow \nabla^{LC}$ preserves $T^{1,0}\Sigma \subseteq T_{\mathbb{C}}\Sigma$ and defines a connection ∇ on $T^{1,0}\Sigma$.

Details for ③: compute R for ∇^{LC}:

$$A = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}, \quad Rm = dA + A \wedge A$$
$$Rm = \begin{bmatrix} 0 & -d\alpha \\ d\alpha & 0 \end{bmatrix}$$

write $d\alpha = \kappa\, e^1 \wedge e^2$, $\quad R_{12} = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}$

Conventions: $R_{1212} = \kappa$, $\quad R = R_{ip}{}^{pi} = R_{12}{}^{21} + R_{21}{}^{12}$

$\Rightarrow R = -2\kappa$

Compute F for $(T^{1,0}\Sigma, \nabla)$: $\quad \nabla \varepsilon = i\alpha \otimes \varepsilon$
$A^{T^{1,0}\Sigma} = i\alpha$

$F = dA^{T^{1,0}\Sigma}$ line bundle
$= i\, d\alpha$
$= i\kappa\, e^1 \wedge e^2$

$$\Rightarrow i \operatorname{Tr} F = -\kappa\, e^1 \wedge e^2 = \frac{R}{2} \underbrace{e^1 \wedge e^2}_{d\mathrm{vol}_g}$$

<u>Hairy Ball Theorem:</u> Any smooth vector field on S^2 must vanish at some point.

Assume by contraction that $\exists\, V \in \Gamma(TS^2)$ nowhere vanishing.

Equip S^2 with metric g. As above, obtain almost-cplx str I.

$\Rightarrow V - iIV$ is nowhere vanishing section of $T^{1,0}S^2$.

$\Rightarrow T^{1,0}S^2$ is a trivial line bundle.

$\Rightarrow c_1(T^{1,0}S^2) = 0.$

But $\int_{S^2} c_1(T^{1,0}S^2) = 2$ by Gauss-Bonnet.

$\Rightarrow\Leftarrow$