DIFFERENTIAL GEOMETRY II: PROBLEM SET 1

Due Feb 6

1. Let $\mathbb{RP}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$, where $\sim$ is the equivalence relation

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$$

if

$$(x_0, \ldots, x_n) = (\lambda y_0, \ldots, \lambda y_n)$$

for some $\lambda \in \mathbb{R}\backslash\{0\}$. The equivalence class of a point $x = (x_0, \ldots, x_n)$ will be denoted

$$[x_0, \ldots, x_n],$$

with square brackets. We can cover $\mathbb{RP}^n$ by the open sets

$$U_i = \{[x_0, \ldots, x_n] : x^i \neq 0\}.$$

Recall that $\mathbb{RP}^n$ is a manifold with coordinates on U_i given by

$$(u^1, \ldots, u^n) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

The tautological line bundle $L \to \mathbb{RP}^n$ is

$$L = \{([x], w) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : w \in [x]\}.$$

Define the bundle projection $\pi : L \to \mathbb{RP}^n$ by $\pi([x], w) = [x]$.

We can view L as a line bundle by trivializing L over U_i. A local generator for the fibers over U_i is given by

$$e^{U_i}([x]) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

so that any $w \in L|_{[x]}$ can be written on U_i as $w = \lambda e^{U_i}([x])$ for $\lambda \in \mathbb{R}$. In the notation from class, here e^{U_i} is the local frame and λ is the local component in that frame.

(a) Compute the transition function c_{ij} on $U_i \cap U_j$.

(b) Show that the linear functions $x_0, x_1, \ldots, x_n$ on $\mathbb{R}^{n+1}$, and more generally,

$$s = \sum_k a_k x_k, \quad a_i \in \mathbb{R}$$

define sections of the dual bundle L^*. To do this, from s we obtain local functions $s_i : U_i \to \mathbb{R}$ on U_i by setting $s_i = s/x_i$ (write these local functions in u^i coordinates), and show that s transforms on $U_i \cap U_j$ by

$$s_i = c^*_{ij} s_j,$$

where c^*_{ij} are the transition functions of (L^*, U_i).

2.

(a) Let $E \to M$ and $\tilde{E} \to M$ be two vector bundles. Suppose E and $\tilde{E}$ can be trivialized by the same cover $M = \bigcup U_\alpha$. Denote the transition functions by

$$c_{\mu\nu} : U_\mu \cap U_\nu \to GL(k, \mathbb{R}), \quad \tilde{c}_{\mu\nu} : U_\mu \cap U_\nu \to GL(k, \mathbb{R}).$$

Show that E is isomorphic to $\tilde{E}$ if and only if there exists local functions $h_\mu : U_\mu \to GL(k, \mathbb{R})$ such that

$$\tilde{c}_{\mu\nu} = h_\mu c_{\mu\nu} h_\nu^{-1},$$

on $U_\mu \cap U_\nu$.

(b) Show that a line bundle $L \to M$ is isomorphic to the trivial bundle $\mathbb{R} \times M$ if and only if it admits a nowhere vanishing section.

3. Let $E \to M$ be a vector bundle with trivializing cover $M = \bigcup U_\mu$ and transition functions $c_{\mu\nu} : U_\mu \cap U_\nu \to GL(r, \mathbb{R})$. Recall that E is orientable if it is isomorphic to a bundle with trivializations satisfying $\det c_{\mu\nu} > 0$. The determinant bundle associated to E, denoted $\det E$, is a line bundle over M with the same trivialization $M = \bigcup U_\mu$ and transition functions $\det c_{\mu\nu}$.

(a) Let M^n be an orientable manifold with Riemannian metric g_{ij}. Show that

$$\mathrm{dVol}_g = \sqrt{\det g_{ij}}$$

defines a nowhere vanishing section of $\det T^*M$. This section is sometimes written

$$\mathrm{dVol}_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n,$$

since $\det T^*M = \Omega^n(M)$.

(b) Show that a vector bundle $E \to M$ is orientable if and only if $\det E$ is isomorphic to the trivial bundle $M \times \mathbb{R}$.