DIFFERENTIAL GEOMETRY II: PROBLEM SET 3

Due March 17

1. Let $\mathbb{R}^{1,3}$ be the space $\mathbb{R}^4 = \{(t, x, y, z)\}$ equipped with the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

Let $P = \mathbb{R}^{3,1} \times U(1)$ be the trivial bundle. Show that the Yang-Mills equations together with the Bianchi identity

$$\nabla^\nu F_{\mu\nu} = 0, \quad dF = 0,$$

is equivalent to the Maxwell equations

$$\nabla \cdot E = 0, \quad \nabla \cdot B = 0,$$

$$\partial_t B + \nabla \times E = 0, \quad \partial_t E - \nabla \times B = 0,$$

via the identification

$$F_{\mu\nu} = i \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}.$$

2. In this question, we will derive the $SU(2)$ Yang-Mills instanton of Belavin-Polyakov-Schwarz-Tyupkin.

First, let $g \in SU(2)$. Show that g can be identified with a point $\nu = (\nu^0, \nu^1, \nu^2, \nu^3) \in \mathbb{S}^3 \subset \mathbb{R}^4$ via

$$g = (\nu^0 I - \nu^i \tau_i),$$

where

$$\tau_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \ \tau_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \tau_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Note the identities

$$\tau_i \tau_j = -\tau_j \tau_i, \ \tau_i^2 = -I, \ \tau_i \tau_j = -\varepsilon_{ijk} \tau_k.$$

Verify that the inverse of g is the quaternionic conjugate

$$g^{-1} = \nu^0 I + \nu^i \tau_i.$$

We now construct a Yang-Mills connection over a rank 2 bundle over $\mathbb{R}^4$. Let

$$r^2 = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2.$$

To define the connection, we first take a point $x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$, then consider $x/r \in S^3 \cong SU(2)$ and define

$$A = f(r) g^{-1} dg, \quad g = \frac{1}{r}(x^0 I - x^i \tau_i)$$

for a radial function $f(r)$ to be determined. This is a reasonable radial ansatz, given that $g^{-1}dg$ restricted to $SU(2)$ is well-known as the Maurer-Cartan form and we are extending it radially to reach all of $\mathbb{R}^4$. Show that

$$A = \frac{f(r)}{r^2} \alpha^k \tau_k,$$

where

$$\alpha^k = x^k dx^0 - x^0 dx^k + x^i dx^j \varepsilon_{ijk}.$$

Using notation from class, we have

$$d\alpha^i = -2\omega_-^i, \quad \omega_-^i = dx^0 dx^i - dx^j dx^k, \quad \star\omega_-^i = -\omega_-^i.$$

Compute the field strength $F = dA + A \wedge A$, and write it as the sum of an anti-self-dual part spanned by the ω_-^i, and another part spanned by $[dr^2 \wedge \alpha^k]\tau_k$. This other part will vanish for the right choice of $f(r)$. To be precise, show that

$$\star F = -F$$

if and only if

$$rf' - 2f + 2f^2 = 0.$$

This differential equation has solution

$$f(r) = \frac{r^2}{r^2 + \lambda^2}$$

where λ is a scale parameter. Take $\lambda = 1$. Show that

$$F = -\frac{2}{(r^2+1)^2} \omega_-^k \tau_k$$

and compute the instanton charge

$$Q = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \operatorname{Tr} F \wedge F.$$