DIFFERENTIAL GEOMETRY II: PROBLEM SET 4

Due April 5

1. Let (M, g) and $(N, \tilde{g})$ be Riemannian manifolds. Let

$$f : (M, g) \to (N, \tilde{g})$$

be an isometry. Let $\gamma(t)$ be a geodesic on (M, g). Show that $c = f \circ \gamma$ is a geodesic on $(N, \tilde{g})$.

To relate the Christoffel symbols Γ_{ij}^k to $\tilde{\Gamma}_{ij}^k$, you can use the formula

$$\tilde{\Gamma}_{ij}^k = \frac{\partial \tilde{x}^k}{\partial x^p} \Gamma_{rs}^p \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} - \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial^2 \tilde{x}^k}{\partial x^r x^s}.$$

derived in a previous problem set.

2. In this problem, we study geodesics on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Recall that $T\mathbb{S}^2$ can be identified as the set $(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $|p| = 1$ and $p \cdot v = 0$. The round metric on $\mathbb{S}^2$ is induced by the usual Euclidean metric in $\mathbb{R}^3$. In other words, the inner product of (p, v) and (p, w) is $v \cdot w$.

(a) Consider spherical coordinates

$$(\theta, \varphi) \mapsto (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi).$$

Write down the metric g_{ij} in spherical coordinates.

(b) Write down the geodesic equation in these coordinates.

(c) Find one geodesic.

(d) Note that $SO(3)$ acts on $\mathbb{S}^2$ by isometries, and recall that isometries take geodesics to geodesics. Find all geodesics on $\mathbb{S}^2$.

3. The Poincaré upper half-plane is the set

$$H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

equipped with the metric

$$g_{ij} = \frac{1}{y^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) We may represent an element $(x, y) \in H$ by a complex number $z = x + iy \in \mathbb{C}$. An element $A \in SL(2, \mathbb{R})$, given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

acts on H via

$$A \cdot z = \frac{az + b}{cz + d}.$$

Show that $A : H \to H$ is an isometry.

(b) Write down the geodesic equation.

(c) Show that any unit speed geodesic on H is given by

$$\gamma(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \cdot i,$$

for some

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}.$$

Hint: argue by uniqueness, and use that isometries take geodesics to geodesics.

(d) Interpret geodesics in the upper half plane geometrically as either vertical straight lines or half circles with center on the real axis. Conclude that any two points $p, q \in H$ can be joined by a unique geodesic.

4. Let $p \in M$ be a point on a Riemannian manifold (M, g). In this problem, we will show that the volume of a geodesic ball $B_r(p)$ of radius $r > 0$ centered at p is given by

$$\mathrm{Vol}(B_r(p) \subset M) = \mathrm{Vol}(B_r(0) \subset \mathbb{R}^n) \left[1 - \frac{R(p)}{6(n+2)} r^2 + \mathcal{O}(r^4) \right].$$

This can be used to give an alternate definition of scalar curvature.

We will work using normal coordinates x^i such that $(0, 0)$ corresponds to the point $p \in M$. Recall that in these coordinates, we have

$$g_{ij}(0) = \delta_{ij}, \quad \partial_k g_{ij}(0) = 0,$$

and

$$x^j = \sum_r x^r g_{rj}(x),$$

by the Gauss Lemma.

(a) Use Taylor expansion to show that near the origin, we have

$$\sqrt{\det g_{ij}}(x) = 1 + \frac{1}{4} (g^{pq} \partial_i \partial_j g_{qp})(0) x^i x^j + \mathcal{O}(|x|^3).$$

(b) Show that in normal coordinates, this can be written as

$$\sqrt{\det g_{ij}}(x) = 1 - \frac{1}{6} R_{ij}(0) x^i x^j + \mathcal{O}(|x|^3).$$

(c) Compute the area of a geodesic ball $\int_{B_r} d\mathrm{Vol}$ centered at p.