Problem Set 1: Riemann Curvature

Problem 1. Let (M, g) be a Riemannian manifold.
(a) Let (U, x^i) and $(\tilde{U}, \tilde{x}^i)$ be two coordinate charts with non-empty overlap. In coordinates x^i, denote the metric by g_{ij}. In coordinates $\tilde{x}^i$, denote the metric by $\tilde{g}_{ij}$. Write down the relation between g_{ij} and $\tilde{g}_{ij}$.
(b) Let Γ^k_{ij} denote the Christoffel symbols in coordinates x^i. Let $\tilde{\Gamma}^k_{ij}$ denote the Christoffel symbols in coordinates $\tilde{x}^i$. Show that

$$\tilde{\Gamma}^k_{ij} = \frac{\partial \tilde{x}^k}{\partial x^p} \Gamma^p_{rs} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} - \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial^2 \tilde{x}^k}{\partial x^r x^s}.$$

We see that Christoffel symbols do not transform like a section of a vector bundle due to the additional second term.
(c) You can use this to show that the Levi-Civita connection is a well-defined connection. This means that if $V = V^i \partial_i$ is a vector field, then

$$W^i{}_k := \nabla_k V^i, \quad \nabla_k V^i = \frac{\partial}{\partial x^k} V^i + \Gamma^i_{kp} V^p$$

defines a section $\nabla V := W \in \Gamma(M, TM \otimes T^*M)$. For ∇V to define a section, it must satisfy

$$\tilde{\nabla}_\mu \tilde{V}^\nu = \frac{\partial x^a}{\partial \tilde{x}^\mu} \frac{\partial \tilde{x}^\nu}{\partial x^b} \nabla_a V^b$$

on overlaps of (U, x^i) and $(\tilde{U}, \tilde{x}^\mu)$. Here

$$\tilde{\nabla}_\mu \tilde{V}^\nu = \frac{\partial}{\partial \tilde{x}^\mu} \tilde{V}^\nu + \tilde{\Gamma}^\nu_{\mu\rho} \tilde{V}^\rho$$

is the expression for the connection over the coordinate chart $\tilde{U}$, and

$$\tilde{V}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^i} V^i$$

is the representation of the vector field over the chart $\tilde{U}$.

Problem 2. Compute the Riemann curvature, Ricci curvature, and scalar curvature of the Poincaré metric

$$g_{ij} = \frac{1}{y^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

on the upper half-plane $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.

Problem 3. Compute the Riemann curvature, Ricci curvature, and scalar curvature of $\mathbb{S}^2$ with the metric induced by Euclidean space in spherical coordinates

$$f(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi).$$

Problem 4. Let (M, g) be a Riemannian manifold of dimension $n = 2$. Suppose that there is a coordinate system where the metric g can locally be written as

$$g_{ij} = e^{f(x,y)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Show that

$$R_{ij} = -\frac{(f_{xx} + f_{yy})}{2}\delta_{ij}.$$

Problem 5. Equip the punctured sphere $S^2 \backslash \{N, S\}$ with spherical coordinates so that the metric appears as

$$g = d\varphi d\varphi + \sin^2\varphi d\theta d\theta.$$

Show that the change of coordinates

$$u = \log\tan\frac{\varphi}{2}, \quad v = \theta$$

makes the metric conformal to the flat cylinder $\mathbb{R} \times S^1$.

Problem 6. The tangent bundle TS^2 of the 2-sphere $S^2 \subset \mathbb{R}^3$ can be described by

$$TS^2 = \{(x, v) \in S^2 \times \mathbb{R}^3 : \langle v, x\rangle = 0\}.$$

Here $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^3$. A vector field on S^2 is a map

$$V : S^2 \to \mathbb{R}^3,$$

such that $\langle V(x), x\rangle = 0$. We can define a covariant derivative by

$$\nabla V = dV - \langle dV, x\rangle x.$$

We showed in lectures that ∇ is the Levi-Civita connection on (S^2, g), where g is the metric induced on S^2 by the Euclidean metric in $\mathbb{R}^3$. Show that the Riemann curvature acts by

$$R(V, W)Z = \langle Z, W\rangle V - \langle Z, V\rangle W,$$

where V, W, Z are tangent vectors at x, meaning $\langle V, x\rangle = 0$.

Problem 7. Let (M, g) be a Riemannian manifold. Recall the local expression for the Laplacian on functions:

$$\Delta f = g^{ij}\partial_i\partial_j f - g^{ij}\Gamma^k_{ij}\partial_k f.$$

Let (U, x^i) be a local coordinate chart. Suppose $\Delta x^i = 0$ for all i. Such coordinates are called harmonic coordinates, or the harmonic gauge condition. Fix i, j, let $g_{ij} = g(\partial_{x^i}, \partial_{x^j})$, and view $g_{ij}(x)$ as a local function on U. Show that in these coordinates

$$\Delta g_{ij} = -2R_{ij} + \mathcal{O}(g, \partial g).$$

This is why the Ricci tensor can be viewed as the "Laplacian of the metric". This expression was used by Choquet-Bruhat to apply PDE theory to the initial value problem in general relativity. This is also why the Ricci flow $\partial_t g = -2\mathrm{Ric}(g)$ can be understood as a heat flow for the metric tensor. One way to prove this is to first show the Bochner formula

$$\Delta\langle\nabla f, \nabla h\rangle = 2\langle\nabla^2 f, \nabla^2 h\rangle + \langle\nabla f, \nabla\Delta h\rangle + \langle\nabla\Delta f, \nabla h\rangle + 2R^{pq}\partial_q f\partial_p h.$$

and then apply it to $f = x^i$, $h = x^j$.

Problem 8. Let G be a Lie group with identity I. Denote for $g \in G$, denote the left action by

$$L_g : G \to G, \quad L_g h = gh.$$

Given a basis $e_1|_I, \dots, e_n|_I \in T_I G$, we obtain a global frame of vector fields $(L_g)_* e_1|_I, \dots (L_g)_* e_n|_I$, which we denote by $e_1, \dots, e_n$ for simplicity. The structure constants of this frame are given by

$$[e_i, e_j] = c^k{}_{ij} e_k.$$

In terms of the structure constants, the Jacobi identity $[V,[W,X]]+[W,[X,V]]+[X,[V,W]] = 0$ is

$$c^p{}_{ir} c^q{}_{jk} + c^p{}_{kr} c^q{}_{ij} + c^p{}_{jr} c^q{}_{ki} = 0$$

for all indices p, q, i, r, j, k.

(a) Show that

$$de^k = \frac{1}{2} c^k{}_{ij} e^j \wedge e^i,$$

where $\{e^i\}$ is the dual frame to $\{e_i\}$.

(b) We define a metric on the Lie group G by

$$g(e_i, e_j) = \delta_{ij}.$$

Recall that the Levi-Civita connection ∇ is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y]) \\ &\quad + g(Y,[Z,X]) + g(X,[Z,Y]). \end{aligned}$$

Compute the connection coefficients $A_i{}^k{}_j$, defined by

$$\nabla_{e_i} e_j = A_i{}^k{}_j e_k.$$

The connection form is $A = A_i{}^k{}_j e^i$.

(c) Further suppose that G is such that the structure constants are totally skew-symmetric. This means that in addition to the usual antisymmetry $c^k{}_{ij} = -c^k{}_{ji}$, we also have

$$c^k{}_{ij} = -c^i{}_{kj}.$$

Compute the components $F_{ij}{}^p{}_q$ of the curvature form $F = dA + A \wedge A$.

$$F = \frac{1}{2} F_{ij}{}^p{}_q \, e^i \wedge e^j.$$

Use the Jacobi identity to simplify your answer.

(d) The Lie group $SO(3)$ admits a global frame of vector fields $\{e_i\}_{i=1}^3$ which satisfies

$$[e_i, e_j] = \epsilon_{ijk} e_k,$$

where ϵ_{ijk} is the Levi-Civita symbol. Show that $SO(3)$ admits an Einstein metric. This is a metric which satisfies the curvature condition

$$R_{ij} = \lambda g_{ij},$$

where λ is a constant.