Problem Set 2: Submanifolds and Geodesics

Problem 1. Compute the scalar curvature of a graph $\Gamma \subseteq \mathbb{R}^3$ given by $\Gamma = \{(x, y, f(x, y))\}$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function.

Problem 2. Suppose (M, g) is a Riemannian manifold and $\sigma : M \to M$ is an isometric involution (satisfying $\sigma \circ \sigma(p) = p$). Let $S \subseteq M$ be the fixed point set of σ.

$$S = \{p \in M : \sigma(p) = p\}.$$

Show that each connected component of S is a smoothly embedded totally geodesic submanifold of M. Show that if there exists $p \in M$ such that $\sigma_*|_p = id$, then σ is the trivial map.

Problem 3. Let (M^4, g) be a Lorentzian manifold of dimension $n = 4$. This means for each $p \in M$, there is an orthogonal frame $\{e_\mu\}_{\mu=0}^3$ defined on a neighborhood of p such that $|e_0|_g^2 = -1$ and $|e_i|_g^2 = 1$ for $i = 1, 2, 3$. Let $T_{\mu\nu}$ be a given tensor (the stress-energy tensor), and suppose that the metric tensor $g_{\mu\nu}$ solves the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}.$$

Let (S, g_S) be a space-like hypersurface. This is a hypersurface $S \subset M^4$ such that the metric tensor $g_{\mu\nu}$ restricts to a positive-definite Euclidean metric g_S.

(a) At a point $p \in S \subset M$, choose an orthogonal frame $\{e_\mu\}$ of (M, g) such that $e_1|_p, e_2|_p, e_3|_p$ span T_pS and $|e_0|_g^2 = -1$. We call $T_{00} = T(e_0, e_0) := \rho$ the energy density of nongravitational fields. Show that S must satisfy the constraint equation

$$2\rho = R_S + (\operatorname{Tr}_{g_S} h)^2 - |h|_{g_S}^2,$$

where h is the second fundamental form. Conclude that for a space-like hypersurface $(S, g_S) \subset (\mathbb{R}^{3,1}, \eta_{\mu\nu})$ in Minkowski space with metric $\eta_{\mu\nu} = \operatorname{diag}(-1, 1, 1, 1)$, then $0 = R_S + (\operatorname{Tr}_{g_S} h)^2 - |h|_{g_S}^2$.

(b) Define the momentum density by the 1-form $J(X) = T(e_0, X)$ for X tangent to S. Show

$$J_n = \nabla_n H - \nabla_m h^m{}_n$$

where H is the mean curvature and indices m, n are coordinates on S.

Note: when doing this calculation, watch out for a sign change in the Gauss-Codazzi equations. For example,

$$R_{abba} = R^S_{abba} + h_{aa}h_{bb} - h^2_{ab}$$

with Lorentz metric. This is because if $\tilde{\nabla}$ is the connection on M, $\{e_a\}_{a=1}^3$ a local frame on S with $\delta_{ab} = \langle e_a, e_b \rangle$, and e_0 a normal vector, then

$$\tilde{\nabla}_{e_b} e_a = (\tilde{\nabla}_{e_b} e_a)^T - \langle \tilde{\nabla}_{e_b} e_a, e_0 \rangle e_0$$

since $\langle e_0, e_0 \rangle = -1$.

Problem 4. Let $p \in M$ be a point on a Riemannian manifold (M, g). In this problem, we will show that the volume of a geodesic ball $B_r(p)$ of radius $r > 0$ centered at p is given by

$$\mathrm{Vol}(B_r(p) \subset M) = \mathrm{Vol}(B_r(0) \subset \mathbb{R}^n)\left[1 - \frac{R(p)}{6(n+2)}r^2 + \mathcal{O}(r^4)\right].$$

This can be used to give an alternate definition of scalar curvature.

We will work using normal coordinates x^i such that $(0,0)$ corresponds to the point $p \in M$. Recall that in these coordinates, we have

$$g_{ij}(0) = \delta_{ij}, \quad \partial_k g_{ij}(0) = 0,$$

and

$$x^j = \sum_r x^r g_{rj}(x),$$

by the Gauss Lemma.

(a) Use Taylor expansion to show that near the origin, we have

$$\sqrt{\det g_{ij}}(x) = 1 + \frac{1}{4}(g^{pq}\partial_i\partial_j g_{qp})(0)x^i x^j + \mathcal{O}(|x|^3).$$

(b) Show that in normal coordinates, this can be written as

$$\sqrt{\det g_{ij}}(x) = 1 - \frac{1}{6}R_{ij}(0)x^i x^j + \mathcal{O}(|x|^3).$$

(c) Compute the area of a geodesic ball $\int_{B_r} d\mathrm{Vol}$ centered at p.

Problem 5. Let (M, g) and $(N, \tilde{g})$ be Riemannian manifolds. Let

$$f : (M, g) \to (N, \tilde{g})$$

be an isometry. Let $\gamma(t)$ be a geodesic on (M, g). Show that $c = f \circ \gamma$ is a geodesic on $(N, \tilde{g})$.

To relate the Christoffel symbols Γ^k_{ij} to $\tilde{\Gamma}^k_{ij}$, you can use the formula

$$\tilde{\Gamma}^k_{ij} = \frac{\partial \tilde{x}^k}{\partial x^p} \Gamma^p_{rs} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} - \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial^2 \tilde{x}^k}{\partial x^r x^s}.$$

derived in a previous problem set.

Problem 6. In this problem, we study geodesics on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Recall that $T\mathbb{S}^2$ can be identified as the set $(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $|p| = 1$ and $p \cdot v = 0$. The round metric on $\mathbb{S}^2$ is induced by the usual Euclidean metric in $\mathbb{R}^3$. In other words, the inner product of (p, v) and (p, w) is $v \cdot w$.

(a) Consider spherical coordinates

$$(\theta, \phi) \mapsto (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi).$$

Write down the metric g_{ij} in spherical coordinates.

(b) Write down the geodesic equation in these coordinates.

(c) Find one geodesic.

(d) Note that $SO(3)$ acts on $\mathbb{S}^2$ by isometries, and recall that isometries take geodesics to geodesics. Find all geodesics on $\mathbb{S}^2$.

Problem 7. The Poincaré upper half-plane is the set

$$H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

equipped with the metric

$$g_{ij} = \frac{1}{y^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) We may represent an element $(x, y) \in H$ by a complex number $z = x + iy \in \mathbb{C}$. An element $A \in SL(2, \mathbb{R})$, given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

acts on H via

$$A \cdot z = \frac{az+b}{cz+d}.$$

Show that $A : H \to H$ is an isometry.

(b) Write down the geodesic equation.

(c) Show that any unit speed geodesic on H is given by

$$\gamma(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \cdot i,$$

for some

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}.$$

Hint: argue by uniqueness, and use that isometries take geodesics to geodesics.

(d) Interpret geodesics in the upper half plane geometrically as either vertical straight lines or half circles with center on the real axis. Conclude that any two points $p, q \in H$ can be joined by a unique geodesic.

Problem 8.

(a) Let $f_t : M \to M$ be a family of diffeomorphisms parametrized by $t \in (-\epsilon, \epsilon)$. Suppose the local expressions $f_t = f^\alpha(x,t)$ depend smoothly on the parameter t, and $f_0 = id$. Denote

$$V^\alpha(x) = \frac{\partial f^\alpha}{\partial t}(x, 0).$$

Show that $V = V^\alpha \partial_\alpha$ defines a vector field on M, i.e. if $\tilde{f}^p = \tilde{x}^p \circ f$ and $f^p = x^p \circ f$ are two choices of coordinates, then

$$\tilde{V}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^p} V^p$$

which is the transformation law for a vector field.

(b) Let η be a 1-form on M. We define the Lie derivative as the linearization of the action of f_t on η.

$$\mathcal{L}_V \eta = \left.\frac{d}{dt}\right|_{t=0} f_t^* \eta$$

where the pullback $f^* : \Lambda^1_{f(x)} M \to \Lambda^1_x M$ is

$$(f^*\eta)_\alpha(x) = \frac{\partial f^\mu}{\partial x^\alpha}(x) \eta_\mu(f(x)).$$

Show that

$$(\mathcal{L}_V \eta)_\alpha = V^p \partial_p \eta_i + \eta_p \partial_i V^p$$

Note for later: similar calculations show that defining

$$(\mathcal{L}_V Y)|_p = \left.\frac{d}{dt}\right|_{t=0} (f_t^{-1})_* Y|_{f_t(p)}$$

for a vector field Y leads to

$$\mathcal{L}_V Y = [V, Y], \tag{1}$$

and defining

$$\mathcal{L}_V h = \frac{d}{dt}\bigg|_{t=0} f_t^* h$$

for $h \in \Gamma(M, T^*M \otimes T^*M)$ gives

$$(\mathcal{L}_V h)_{ij} = V^p \partial_p h_{ij} + h_{pj} \partial_i V^p + h_{ip} \partial_j V^p. \tag{2}$$

Recall that the index notation means that in local coordinates then $h = h_{ij} dx^i \otimes dx^j$ and $\mathcal{L}_X h = (\mathcal{L}_X h)_{ij} dx^i \otimes dx^j$. You do not need to write-up the proof of (1) and (2) as the computation is similar to $(\mathcal{L}_V \eta)_\alpha$.
(c) For vectors $Y = Y^i \partial_i$ and $Z = Z^i \partial_i$, using (2) derive the invariant formula

$$(\mathcal{L}_V h)(Y, Z) = \partial_V h(Y, Z) - h([V, Y], Z) - h(Y, [V, Z]).$$

(d) Let (M, g) be a Riemannian manifold. A Killing field is a vector field X satisfying

$$\mathcal{L}_X g = 0.$$

Show that a Killing field satisfies

$$\nabla_i X_j + \nabla_j X_i = 0.$$

Here we write $X_i = g_{ip} X^p$. This equation is called the Killing equation.
(d) Let $\gamma(t)$ be a geodesic on (M, g) and X a Killing field. Show the conservation law

$$\frac{d}{dt} g(X, \dot\gamma)(t) = 0.$$

Therefore $g(X, \dot\gamma(t))$ is a constant function of time.

Problem 9. Let $G = SO(n)$. This is a smooth manifold and the tangent space at the identity is

$$T_I G := \mathrm{Lie}(G) = \{X \in \mathrm{Mat}(n \times n) : X^T = -X\}.$$

This can be obtained by differentiating $AA^T = I$: recall that for a manifold M, then $X \in T_p M$ is represented by a path $\gamma(t)$ with $\gamma(0) = p$ and $\dot\gamma(0) = X$. The tangent space at another point $A \in G$ is

$$T_A G = \{AX : X \in \mathrm{Lie}(G)\}$$

which can be seen by writing any path $M(t)$ with $M(0) = A$ as $A(A^{-1}M(t))$ so that $A^{-1}M(t)$ is a path going through I at $t = 0$. Given $X \in \mathrm{Lie}(G)$, we can define a vector field V on all G by

$$V|_A = AX.$$

Such a V is called a left-invariant vector field.

(a) We will now show that for $X, Y \in \mathrm{Lie}(G)$, $A \in G$, then

$$[AX, AY] = A[X, Y]$$

where the left-hand side is the commutator of the vector fields $V|_A = AX$, $W|_A = AY$, and the right-hand side is the regular matrix commutator $[X, Y] = XY - YX$ using matrix multiplication. To do this, use the Lie derivative formula

$$[V, W]|_A = (\mathcal{L}_V W)|_A = \frac{d}{dt}\bigg|_{t=0} (f_t^{-1})_* W|_{f_t(A)}, \quad f_t(A) = Ae^{tX}.$$

Here the matrix exponential is as usual $e^{tX} = I + tX + \frac{t^2}{2!}X^2 + \dots$.

- First compute $(f_t^{-1})_* W|_{f_t}$. Recall that the pushforward in general is

$$f_* W|_p = \frac{d}{ds}\bigg|_{s=0} f \circ \gamma(s), \quad \dot{\gamma}(0) = W,\ \gamma(0) = p.$$

- Next, expand $(f_t^{-1})_* W|_{f_t}$ in orders of t and show that the linear order in t is $A[X, Y]$.

(b) Verify that

$$\langle X, Y \rangle = -\mathrm{Tr} XY$$

is an inner product on $\mathrm{Lie}(G)$. Note that $MXM^{-1} \in \mathrm{Lie(G)}$ for $M \in G$, $X \in \mathrm{Lie}(G)$. Verify that this inner product is "Ad-invariant", meaning that

$$\langle MXM^{-1}, MYM^{-1} \rangle = \langle X, Y \rangle, \quad M \in G,\ X, Y \in \mathrm{Lie}(G).$$

Differentiate

$$\langle e^{tX} Y e^{-tX}, e^{tX} Z e^{-tX} \rangle = \langle Y, Z \rangle.$$

to obtain

$$0 = \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle.$$

We extend $\langle \cdot, \cdot \rangle$ to a Riemannian meric g on all of G by declaring the inner product on $T_A G$ to be

$$g|_A(AX, AY) = \langle X, Y \rangle.$$

(c) Let $V|_A = AX$ be a left-invariant vector field. Show V is a Killing field, i.e.

$$(L_V g)(W, U) = V(g(W, U)) - g([V, W], U) - g(V, [W, U]) = 0$$

for all W, U left-invariant, and hence $L_V g = 0$.

(d) Use the conservation law for Killing fields and compute

$$\frac{d}{dt} g(V - \dot{\gamma}, V - \dot{\gamma})$$

to show that the geodesic through the identity I with initial velocity X satisfies

$$\dot{\gamma}|_{\gamma(t)} = \gamma(t) X.$$

Conclude by uniqueness of solutions to this ODE that this geodesic is given by

$$\gamma(t) = \exp(tX).$$

Here exp is the matrix exponential map. We see that the matrix exponential map and the Riemannian exponential map coincide in this case.
