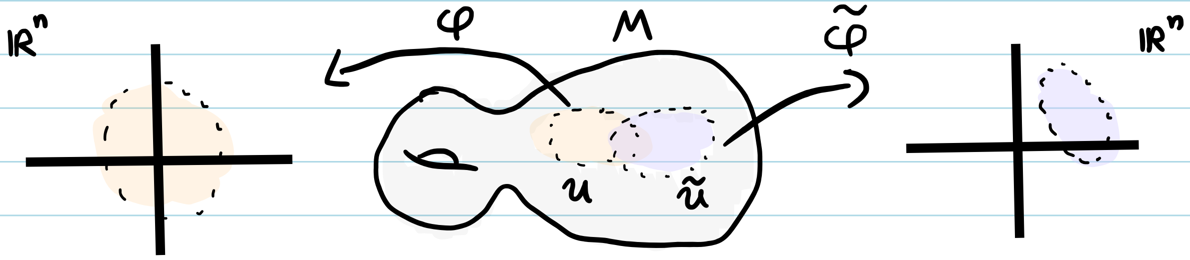
Review of Differential Forms

• Let M be a smooth manifold. Then:

$M = \bigcup_\alpha \mathcal{U}_\alpha$ where: each $\mathcal{U}_\alpha \subseteq M$ is open and comes with $\varphi_\alpha : \mathcal{U}_\alpha \to \varphi_\alpha(\mathcal{U}_\alpha) \subseteq \mathbb{R}^n$ homeomorphism.

On overlaps of $(\mathcal{U}, \varphi)$, $(\tilde{\mathcal{U}}, \tilde{\varphi})$, then $\tilde{\varphi} \circ \varphi^{-1}$ is smooth.



Notation: We will no longer explicitly refer to φ and instead use coordinates:

$$\varphi(p) = (x^1, \dots, x^n) \in \mathbb{R}^n$$

On overlaps $(\mathcal{U}, x^i)$, $(\tilde{\mathcal{U}}, \tilde{x}^i)$, write:

$$\tilde{x}^i = f^i(x^1, \dots, x^n) \quad \text{where } f = \tilde{\varphi} \circ \varphi^{-1} \text{ smooth.}$$

Def: $\omega \in \Omega^k(M)$ is a differential form if over a coord chart $(\mathcal{U}, x^i)$, then

$$\omega \overset{\text{loc}}{=} \frac{1}{k!} \omega_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

(Summation convention: omit Σ sum)

with $\omega_{i_1 \cdots i_k}(x)$ smooth functions anti-symmetric in $i_1 \cdots i_k$, (e.g. $\omega_{ij} = -\omega_{ji}$)

and on overlaps $(\mathcal{U}, x^i)$, $(\tilde{\mathcal{U}}, \tilde{x}^i)$ then:

$$\tilde{\omega}_{i_1 \cdots i_k} = \frac{\partial x^{p_1}}{\partial \tilde{x}^{i_1}} \cdots \frac{\partial x^{p_k}}{\partial \tilde{x}^{i_k}} \omega_{p_1 \cdots p_k}. \qquad (*)$$

Recall: 0) $\Omega^0(M) := C^\infty(M)$ = smooth functions on M.

1) $dx^i \wedge dx^j = -dx^j \wedge dx^i$

2) The point of $(*)$ is s.t. $\omega(V_1, \dots, V_k) \in C^\infty(M)$
for vector fields $V_1, \dots, V_k$.

$$\omega(V_1, \dots, V_k) := \omega_{i_1 \dots i_k} V_1^{i_1} \cdots V_k^{i_k}, \qquad V = V^i \frac{\partial}{\partial x^i} \text{ over } (\mathcal{U}, x^i)$$

More details on 2):

- $V \in \Gamma(TM)$ is a vector field if over a coord chart $(\mathcal{U}, x^i)$,

$$V \overset{loc}{=} V^i(x) \frac{\partial}{\partial x^i}, \quad \text{with } V^i(x) \text{ smooth functions,}$$

and on overlaps $(\mathcal{U}, x^i)$, $(\tilde{\mathcal{U}}, \tilde{x}^i)$, then

$$\tilde{V}^i = \frac{\partial \tilde{x}^i}{\partial x^p} V^p.$$

(ensures $V(f) = V^i \frac{\partial}{\partial x^i} f$ is well-defined action on $f \in C^\infty(M)$)

- $\alpha \in \Omega^1(M)$, $V \in \Gamma(TM)$ pair together to give: $\alpha(V) \in C^\infty(M)$.

$$\alpha(V) \overset{loc}{=} \alpha_i V^i$$

On overlaps $(\mathcal{U}, x^i)$, $(\tilde{\mathcal{U}}, \tilde{x}^i)$:

$$\tilde{\alpha}_i \tilde{V}^i = \left(\frac{\partial x^p}{\partial \tilde{x}^i} \alpha_p\right)\left(\frac{\partial \tilde{x}^i}{\partial x^k} V^k\right)$$

$$= \underbrace{\left(\frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial \tilde{x}^i}{\partial x^k}\right)}_{= \delta^p_k} \alpha_p V^k \qquad = \frac{\partial}{\partial x^k} x^p \text{ chain rule}$$

$$= \alpha_p V^p.$$

$\Rightarrow$ $\alpha_p V^p$ is well-defined (indep of choice of coords) ex) $\mathbb{C}\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$

points denoted: $[z_0, \dots, z_n] \in \mathbb{C}\mathbb{P}^n$

$[z_0, \dots, z_n] \sim [x_0, \dots, x_n] \iff \begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$ for some $\lambda \in \mathbb{C}^*$.

$\mathbb{C}\mathbb{P}^n$ is a smooth mfd of dim $2n$ (atlas given later)
is a complex mfd of dim n (defn of complex mfd will come later)

For now: $\mathbb{C}\mathbb{P}^1 = \mathcal{U}_0 \cup \mathcal{U}_1$

$\mathcal{U}_0 = \{ z_0 \neq 0 \}$
coords: $w = x + iy$ with $w = \dfrac{z_1}{z_0}$, $(x, y) \in \mathbb{R}^2$

$\mathcal{U}_1 = \{ z_1 \neq 0 \}$
coords: $\tilde{w} = \tilde{x} + i\tilde{y}$ with $\tilde{w} = \dfrac{z_0}{z_1}$, $(\tilde{x}, \tilde{y}) \in \mathbb{R}^2$

change of coords on $\mathcal{U}_0 \cap \mathcal{U}_1$: $\tilde{w} = 1/w$.

$$\tilde{x} + i\tilde{y} = \frac{x - iy}{x^2 + y^2}$$

$$\begin{cases} \tilde{x} = \dfrac{x}{x^2+y^2} \\ \tilde{y} = \dfrac{y}{x^2+y^2} \end{cases} \Rightarrow \mathbb{C}\mathbb{P}^1 \text{ is a smooth mfd of dim} = 2.$$

ex) $\omega = d\tilde{x}$ over $\mathcal{U}_1$. Looks harmless on $\mathbb{C}\mathbb{P}^1 \setminus \{[1,0]\}$
But does not extend to all of $\mathbb{C}\mathbb{P}^1$!

$\omega = \omega_1 \, dx + \omega_2 \, dy$ over $\mathcal{U}_0 \cap \mathcal{U}_1$

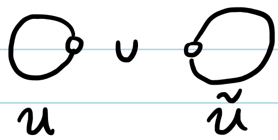
$$\omega_1 = \frac{\partial \tilde{x}^p}{\partial x^1}\tilde{\omega}_p = \frac{\partial \tilde{x}}{\partial x}\cdot 1. \qquad \tilde{\omega}_1 = 1, \quad \tilde{\omega}_2 = 0$$

$$= \frac{\partial}{\partial x}\left(x(x^2+y^2)^{-1}\right) = \frac{-x^2+y^2}{(x^2+y^2)^2}.$$

$$\omega_2 = \frac{\partial \tilde{x}^p}{\partial x^2}\tilde{\omega}_p = \frac{\partial}{\partial y}\left(x(x^2+y^2)^{-1}\right) = \frac{-2xy}{(x^2+y^2)^2}$$

$$\omega = \left(\frac{-x^2+y^2}{(x^2+y^2)^2}\right)dx - \frac{2xy}{(x^2+y^2)^2}dy.$$ Problem at $(x,y) = (0,0) \in \mathcal{U}_0$.

ex) $S^1 = \{e^{i\theta} : \theta \in [0, 2\pi]\}$

$S^1 =$  ,

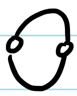
$\mathcal{U} = \{e^{i\theta} : \theta \in (0, 2\pi)\}$

coord θ

$\tilde{\mathcal{U}} = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}$

coord $\tilde{\theta}$

on $\mathcal{U} \cap \tilde{\mathcal{U}}$: $\tilde{\theta} = \begin{cases} \theta & \text{if } 0 < \theta < \pi \\ \theta - 2\pi & \text{if } \pi < \theta < 2\pi \end{cases}$



$d\theta \in \Omega^1(S^1)$ is global 1-form.

$\omega = \begin{cases} d\theta & \text{over } \mathcal{U} \\ d\tilde{\theta} & \text{over } \tilde{\mathcal{U}} \end{cases}$ defines $\omega \in \Omega^1(S^1)$ since

$$1 = \frac{\partial \tilde{\theta}}{\partial \theta}\cdot 1 \quad \checkmark$$

<u>Wedge Product</u>: $\omega \in \Omega^k(M)$
$\tau \in \Omega^\ell(M)$

$$\omega \wedge \tau = \frac{1}{k!}\frac{1}{\ell!}\,\omega_{i_1\cdots i_k}\tau_{j_1\cdots j_\ell}\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_\ell}$$

<u>Note</u>: $\omega \wedge \gamma = (-1)^{k\ell} \gamma \wedge \omega$, $\quad \omega \in \Omega^k(M)$, $\gamma \in \Omega^\ell(M)$.

ex) $\omega, \gamma \in \Omega^1(M)$, $\quad \omega \overset{loc}{=} \omega_i \, dx^i$
$\gamma \overset{loc}{=} \gamma_i \, dx^i$

$$\omega \wedge \gamma = \omega_i \gamma_k \, dx^i \wedge dx^k$$
$$= \frac{1}{2} (\omega_i \gamma_k - \omega_k \gamma_i) \, dx^i \wedge dx^k$$
$$\omega \wedge \gamma = \frac{1}{2} (\omega \wedge \gamma)_{ik} \, dx^i \wedge dx^k$$
$$\Rightarrow (\omega \wedge \gamma)_{ik} = \omega_i \gamma_k - \omega_k \gamma_i$$

Well-defn? $\quad (\tilde{\omega}_i \tilde{\gamma}_k - \tilde{\omega}_k \tilde{\gamma}_i) = \dfrac{\partial x^p}{\partial \tilde{x}^i} \dfrac{\partial x^q}{\partial \tilde{x}^k} (\omega_p \gamma_q - \omega_q \gamma_p)$

<u>Pullback</u>: $f: M \to N$ smooth map, $\dim M = m$, $\dim N = n$,
$\omega \in \Omega^k(N)$
$$\omega \overset{loc}{=} \frac{1}{k!} \omega_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$f^* \omega \in \Omega^k(M)$ defined by:

$$f^* \omega \overset{loc}{=} \frac{1}{k!} \omega_{i_1 \cdots i_k}(f(y)) \frac{\partial f^{i_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial f^{i_k}}{\partial y^{\alpha_k}} \, dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k}$$

where (U, x^i) chart on N
(V, y^i) chart on M
$f \overset{loc}{=} (f^1(y^1, \ldots, y^m), \ldots, f^n(y^1, \ldots, y^m))$

Exercise: $f^* \omega$ well-defined.
e.g. 1-forms
$$(\widetilde{f^* \omega})_k \overset{?}{=} \frac{\partial y^p}{\partial \tilde{y}^k} (f^* \omega)_p$$

$$\omega_i \frac{\partial f^i}{\partial \tilde{y}^k} = \frac{\partial y^p}{\partial \tilde{y}^k} \omega_i \frac{\partial f^i}{\partial y^p}$$

Exterior Derivative

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

$$\alpha \overset{loc}{=} \frac{1}{k!} \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$d\alpha \overset{loc}{=} \frac{1}{k!} \partial_p \alpha_{i_1 \cdots i_k} dx^p \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Why is $d\alpha \in \Omega^{k+1}(M)$ well-defined? Check for $\alpha \in \Omega^1(M)$.

$$\alpha = \alpha_i dx^i$$

$$d\alpha = \partial_k \alpha_i dx^k \wedge dx^i$$

$$= \frac{1}{2} (\partial_k \alpha_i - \partial_i \alpha_k) dx^k \wedge dx^i$$

$$\Rightarrow (d\alpha)_{ki} = \partial_k \alpha_i - \partial_i \alpha_k$$

Need to show: $(d\alpha)_{ki} = \frac{\partial \tilde{x}^p}{\partial x^k} \frac{\partial \tilde{x}^q}{\partial x^i} (d\alpha)_{pq}$ on $(\mathcal{U}, x) \cap (\tilde{\mathcal{U}}, \tilde{x})$

$$(d\alpha)_{ki} = \frac{\partial}{\partial x^k} \alpha_i - \frac{\partial}{\partial x^i} \alpha_k$$

$$= \frac{\partial}{\partial x^k} \left(\frac{\partial \tilde{x}^q}{\partial x^i} \tilde{\alpha}_q \right) - \frac{\partial}{\partial x^i} \left(\frac{\partial \tilde{x}^q}{\partial x^k} \tilde{\alpha}_q \right)$$

Key cancellation!

$$= \frac{\partial \tilde{x}^q}{\partial x^i} \frac{\partial}{\partial x^k} \tilde{\alpha}_q(\tilde{x}) - \frac{\partial \tilde{x}^q}{\partial x^k} \frac{\partial}{\partial x^i} \tilde{\alpha}_q(\tilde{x})$$

$\partial_i \partial_k \tilde{x} = \partial_k \partial_i \tilde{x}$

$$= \frac{\partial \tilde{x}^q}{\partial x^i} \frac{\partial \tilde{x}^p}{\partial x^k} \left(d\alpha \right)_{pq} \checkmark$$

$\frac{\partial}{\partial x^k} = \frac{\partial \tilde{x}^p}{\partial x^k} \frac{\partial}{\partial \tilde{x}^p}$

An identical calculation shows:

Prop: $f : M \to N$ smooth map. Then $f^* d\alpha = d f^* \alpha \quad \forall \alpha \in \Omega^k(N)$

e.g. check for 1-forms $\alpha \in \Omega^1(N)$.

$$\alpha = \alpha_i dx^i$$

$$(f^* d\alpha)_{ki}(x) = \frac{\partial f^p}{\partial x^k}\frac{\partial f^q}{\partial x^i}(\partial_p \alpha_q - \partial_q \alpha_p)(f(x))$$

$$(d f^* \alpha)_{ki}(x) = \frac{\partial}{\partial x^k}\left(\frac{\partial f^q}{\partial x^i}\alpha_q(f(x))\right) - (k \leftrightarrow i)$$

Show these are equal by cancelling $\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^i} f = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k} f$.

Prop: $d^2 = 0$.

Pf: $$d^2\alpha = \frac{1}{k!}\underbrace{\partial_p \partial_q}_{\text{sym}} \alpha_{i_1\cdots i_k} \underbrace{dx^p \wedge dx^q}_{\text{anti-sym}} \wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$= 0.$$

$\square$

Prop: $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^k \alpha\wedge d\beta$, $\quad \alpha \in \Omega^k(M)$

Pf: $$d\left(\frac{1}{k!}\frac{1}{l!}\alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_l}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_l}\right)$$

$$= \frac{1}{k!\,l!}\partial_p \alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_l}\, dx^p\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_l}$$

$$+ \frac{1}{k!\,l!}\alpha_{i_1\cdots i_k}\partial_p \beta_{j_1\cdots j_l}\, \underbrace{dx^p\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}}_{\text{pick up } (-1)^k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_l}$$

$\square$

ex) $f \in C^\infty(\mathbb{R}^3)$

$\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz \in \Omega^1(\mathbb{R}^3)$

$\beta = \beta_3\, dx\wedge dy + \beta_2\, dz\wedge dx + \beta_1\, dy\wedge dz \in \Omega^2(\mathbb{R}^3)$

$df = (\partial_1 f)dx + (\partial_2 f)dy + (\partial_3 f)dz$ "∇f"

$d\alpha = (\partial_1\alpha_2 - \partial_2\alpha_1)\,dx\wedge dy - (\partial_1\alpha_3 - \partial_3\alpha_1)\,dz\wedge dx$
$\quad + (\partial_2\alpha_3 - \partial_3\alpha_2)\,dy\wedge dz$ "curl α"

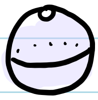
$d\beta = (\partial_1\beta_1 + \partial_2\beta_2 + \partial_3\beta_3)\,dx\wedge dy\wedge dz$ "div β"

Def: $\alpha \in \Omega^k(M)$ is : closed if $d\alpha = 0$

: exact if $\alpha = d\beta$ for $\beta \in \Omega^{k-1}(M)$.

More examples of manifolds :

ex) $S^n = \left\{ \sum_{i=1}^{n+1} x_i^2 = 1 \right\} \subseteq \mathbb{R}^{n+1}$

$S^n =$  $\cup$ 

$\mathcal{U} = S^n \setminus \{N\}$, $\tilde{\mathcal{U}} = S^n \setminus \{S\}$

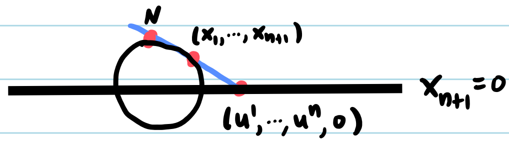
$N = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$
$S = (0, \cdots, 0, -1) \in \mathbb{R}^{n+1}$

Coords on $\mathcal{U}$:

$$\varphi(x_1, \cdots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \cdots, \frac{x_n}{1 - x_{n+1}} \right)$$

stereographic projection

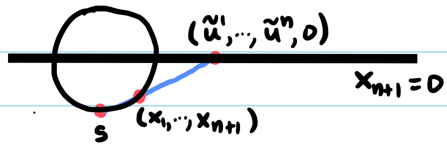
$$= (u^1, \cdots, u^n)$$



Coords on $\tilde{\mathcal{U}}$:

$$\tilde{\varphi}(x_1, \cdots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \cdots, \frac{x_n}{1 + x_{n+1}} \right)$$

$$= (\tilde{u}^1, \cdots, \tilde{u}^n)$$

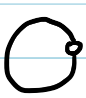
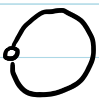
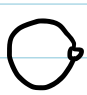
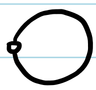


Change of coords:

$$\tilde{u}^i = \frac{u^i}{(u^1)^2 + \cdots + (u^n)^2}.$$

ex) $T^2 = S^1 \times S^1$ 

Open cover:

$T^2 = \{$  $\cup$  $\} \times \{$  $\cup$  $\}$

S^1 , S^1

$\mathcal{U}_1$, $\tilde{\mathcal{U}}_1$, $\mathcal{U}_2$, $\tilde{\mathcal{U}}_2$

$S^1 = \{\mathcal{U}, \theta\} \cup \{\tilde{\mathcal{U}}, \tilde{\theta}\}$, $\quad \mathcal{U} = \{e^{i\theta} : 0 < \theta < 2\pi\}$
$\tilde{\mathcal{U}} = \{e^{i\tilde{\theta}} : -\pi < \tilde{\theta} < \pi\}$

Charts for T^2:

$$\{\mathcal{U}_1 \times \mathcal{U}_2, (\theta^1, \theta^2)\}, \{\mathcal{U}_1 \times \tilde{\mathcal{U}}_2, (\theta^1, \tilde{\theta}^2)\}$$
$$\{\tilde{\mathcal{U}}_1 \times \mathcal{U}_2, (\tilde{\theta}^1, \theta^2)\}, \{\tilde{\mathcal{U}}_1 \times \tilde{\mathcal{U}}_2, (\tilde{\theta}^1, \tilde{\theta}^2)\}$$

e.g. $d\theta^1, d\theta^2 \in \Omega^1(T^2)$
e.g. $d\theta^1 \wedge d\theta^2 \in \Omega^2(T^2)$

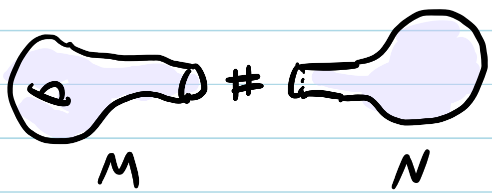
Check $d\theta^1 \wedge d\theta^2$ well-defn on overlap

e.g. $\{\mathcal{U}_1 \times \mathcal{U}_2, (\theta^1, \theta^2)\}$
$\{\tilde{\mathcal{U}}_1 \times \tilde{\mathcal{U}}_2, (\tilde{\theta}^1, \tilde{\theta}^2)\}$

$$\tilde{\theta}^1 = \begin{cases} \theta^1 \\ \theta^1 - 2\pi \end{cases} \qquad \tilde{\theta}^2 = \begin{cases} \theta^2 \\ \theta^2 - 2\pi \end{cases}$$

$$d\tilde{\theta}^1 \wedge d\tilde{\theta}^2 = d\theta^1 \wedge d\theta^2$$

ex) Connected Sums: M, N mfd, $\dim M = \dim N = n$.



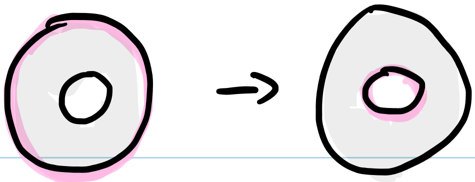
$M \# N$

$p \in M$ in chart $(\mathcal{U}, \varphi)$ with $\varphi(p) = 0$
$q \in N$ in chart (V, ψ) with $\psi(p) = 0$

$B_{2\varepsilon}(0) \subseteq \varphi(\mathcal{U})$, $\quad B_{2\varepsilon}(0) \subseteq \psi(V)$

$$g : \{\varepsilon < |x| < 2\varepsilon\} \longrightarrow \{\varepsilon < |x| < 2\varepsilon\}$$
$$x \longmapsto \frac{2\varepsilon^2}{|x|^2} x$$

note: $\{|x| = 2\varepsilon\} \longleftrightarrow \{|x| = \varepsilon\}$



$$M \# N = \Big(M \setminus \underset{\varphi(\mathcal{U})}{\{|x| \le \varepsilon\}} \sqcup N \setminus \underset{\psi(V)}{\{|x| \le \varepsilon\}} \Big) \Big/ \sim$$

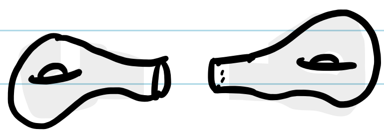
with glueing $\underset{\varphi(\mathcal{U})}{x} \sim \underset{\psi(V)}{g(x)}$.

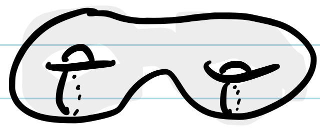
In terms of change of coords: $(\underset{\subseteq M}{\mathcal{U}}, x^i)$, $(\underset{\subseteq N}{V}, \tilde{x}^i)$

Declare: $\tilde{x}^i = \dfrac{2\varepsilon^2}{|x|^2} x^i$

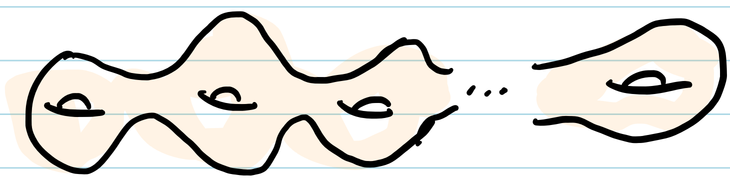
on region $\{\varepsilon < |x| < 2\varepsilon\} = \{\varepsilon < |\tilde{x}| < 2\varepsilon\}$.

ex) Genus 2 surface $= T^2 \# T^2$

$=$ 

$=:$ 

ex) $\Sigma_g = \underbrace{T^2 \# \cdots \# T^2}_{g}$ genus g surf



Def: M is a complex manifold if M is a smooth manifold together with an open cover

$$M = \bigcup_\alpha \mathcal{U}_\alpha \quad \text{with homeomorphisms} \quad z_\alpha : \mathcal{U}_\alpha \to V \subseteq \mathbb{C}^n$$

s.t. on overlaps $(\mathcal{U}, z)$, $(\tilde{\mathcal{U}}, \tilde{z})$, then

$$\tilde{z}^i = f^i(z^1, \dots, z^n), \qquad f = \tilde{z} \circ z^{-1}$$

with f hol'c with hol'c inverse.

Recall: $f : \underset{\subseteq \mathbb{C}^n}{\Omega} \to \mathbb{C}^k$ with $f \in C^1(\Omega)$ is hol'c if

$$f = (f^1, \dots, f^k), \quad \frac{\partial f^i}{\partial \bar{z}^k} = 0 \quad \forall i, k.$$

Recall: $z = x + iy \in \mathbb{C}$

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z})$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\left(\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y}\right)$$

ex) $\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\sim$ as before

Denote $[z_0, \dots, z_n] \in \mathbb{CP}^n$.

$\mathbb{CP}^n = \mathcal{U}_0 \cup \dots \cup \mathcal{U}_n$

$\mathcal{U}_k = \{ z_k \neq 0 \}$

Coords on e.g. $\mathcal{U}_0$:

$$(w^1, \dots, w^n) = \left(\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0}, \dots, \frac{Z_n}{Z_0} \right).$$

<u>Exercise</u>: check change of coords are hol'c.

Differential Forms on complex manifolds:

- X complex mfd, local coords $z^k = x^k + iy^k$.

$\Rightarrow$ $dx^k = \frac{1}{2}(dz^k + d\bar{z}^k)$, $\quad dy^k = \frac{1}{2i}(dz^k - d\bar{z}^k)$.

$\Rightarrow$ $\alpha \in \Omega^1(X)$ can be written:

$$\alpha \overset{\text{loc}}{=} \alpha_i \, dz^i + \alpha_{\bar{i}} \, d\bar{z}^i.$$

Decompose: $\Omega^1_{\mathbb{C}}(X) = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X)$ where:

$$\Omega^{1,0}(X) = \left\{ \alpha \in \Omega^1_{\mathbb{C}}(X) : \alpha \overset{\text{loc}}{=} \alpha_i \, dz^i \right\}$$

$$\Omega^{0,1}(X) = \left\{ \alpha \in \Omega^1_{\mathbb{C}}(X) : \alpha \overset{\text{loc}}{=} \alpha_{\bar{i}} \, d\bar{z}^i \right\}.$$

<u>Check</u> well-defn: On overlap $(\mathcal{U}, z), (\tilde{\mathcal{U}}, \tilde{z})$,
if $\alpha = \alpha_i \, dz^i$, then can also write
$\alpha = \tilde{\alpha}_i \, d\tilde{z}^i$.

More generally: $\Omega^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X)$ where:

$$\Omega^{p,q}(X) = \left\{ \begin{array}{l} \alpha \in \Omega^{p+q}_{\mathbb{C}}(X) : \\ \alpha \overset{\text{loc}}{=} \frac{1}{p!} \frac{1}{q!} \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \underbrace{dz^{i_1} \wedge \cdots \wedge dz^{i_p}}_{p \text{ "} dz \text{"}} \wedge \underbrace{d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}}_{q \text{ "} d\bar{z} \text{"}} \end{array} \right\}$$

Exterior Derivative on complex manifolds:

ex) On $\mathbb{C}$, $z = x + iy$.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$ Change to $(z, \bar{z})$ coords:

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.$$

$$\partial f = \frac{\partial f}{\partial z} dz$$

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$$df := \partial f + \bar{\partial} f,$$

More generally: On cplx mfd X, write

$$d = \partial + \bar{\partial},$$ where:

$$\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \qquad \bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$$

$$\alpha = \frac{1}{p!}\frac{1}{q!} \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

$$\partial \alpha = \frac{1}{p!}\frac{1}{q!} \frac{\partial}{\partial z^{l}} \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{l} \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

$$\bar{\partial} \alpha = \frac{1}{p!}\frac{1}{q!} \frac{\partial}{\partial \bar{z}^{l}} \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} d\bar{z}^{l} \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

Exercise: $\partial^2 = 0$, $\bar{\partial}^2 = 0$, $\partial \bar{\partial} = -\bar{\partial} \partial$.

ex) $B_1(0) \subseteq \mathbb{C}$, $\omega = -i \partial \bar{\partial} \log(1 - |z|^2) \in \Omega^{1,1}(B_1(0))$

$$\omega = i \partial \left(\frac{z \, d\bar{z}}{(1 - |z|^2)} \right) = \frac{i \, dz \wedge d\bar{z}}{(1 - |z|^2)} + \frac{i z \bar{z}}{(1 - |z|^2)^2} dz \wedge d\bar{z}$$

$$= \frac{1}{(1 - |z|^2)^2} i \, dz \wedge d\bar{z}.$$

Note: $\bar{\omega} = \omega \Rightarrow \omega \in \Omega^2_{\mathbb{R}}(B_1(0))$

Note: ω nowhere vanishing top form