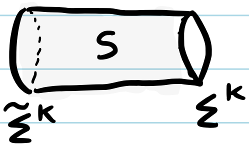
Intersection Product

Def: Let M be an oriented mfd of dim n.
Let Σ^k, Σ^{n-k} be compact oriented submfds
Let $\eta_{\Sigma^k} \in H^{n-k}(M)$, $\eta_{\Sigma^{n-k}} \in H^{n-k}(M)$ be Poincaré duals.

Define: $$\Sigma^k \cdot \Sigma^{n-k} = \int_M \eta_{\Sigma^{n-k}} \wedge \eta_{\Sigma^k}.$$

Note: If $\partial S = \Sigma^k - \tilde{\Sigma}^k$, $\partial T = \Sigma^{n-k} - \tilde{\Sigma}^{n-k}$ $\Rightarrow \Sigma^k \cdot \Sigma^{n-k} = \tilde{\Sigma}^k \cdot \tilde{\Sigma}^{n-k}$.



$[\eta_\Sigma] = [\eta_{\tilde{\Sigma}}]$ since:

$$\int_M \omega \wedge \eta_\Sigma = \int_\Sigma \omega = \int_{\tilde{\Sigma}} \omega + \underbrace{\int_{\partial S} \omega}_{=0} = \int_M \omega \wedge \eta_{\tilde{\Sigma}}$$

$\Rightarrow [\eta_\Sigma] = [\eta_{\tilde{\Sigma}}] \in \left(H^k_c(M)\right)^*$
$\Rightarrow [\eta_\Sigma] = [\eta_{\tilde{\Sigma}}] \in H^{n-k}(M)$. Poincaré duality

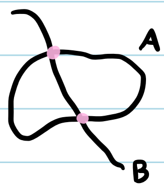
Note: $L = \mathcal{O}(D)$ hol'c line bundle over cplx mfd M
$D \subseteq M$ smooth analytic hypersurface
$C \subseteq M$ smooth holomorphic curve real dim $C = 2$

$$C \cdot D = \int_C c_1(L) = \int_C \eta_D$$

$c_1(L) = \left[\frac{-i}{2\pi} \partial\bar{\partial} \log h_L\right]$,

h_L metric on L.

Prop: Let M be an oriented mfd of dim n.
Let A, B be compact oriented submfds, $\dim A = k$, $\dim B = n-k$
Suppose A, B intersect transversely.



Then: $$A \cdot B = \sum_{p \in A \cap B} \operatorname{sgn}(p).$$

Pf: 1. We may assume $\operatorname{supp} \eta_A \subseteq B_\varepsilon(A) := T$ = small open tube containing A
$\operatorname{supp} \eta_B \subseteq B_\varepsilon(B)$

Indeed: $A \subseteq T$ defines a linear functional on $H^k(T)$.
By Poincaré duality, $\exists !\ \eta_A \in H_c^{n-k}(T)$ s.t.

$$\int_T \omega \wedge \eta_A = \int_A \omega \qquad \forall \omega \in H^k(T).$$

$\Rightarrow \eta_A \in H^{n-k}(M)$ is Poincaré dual to A.

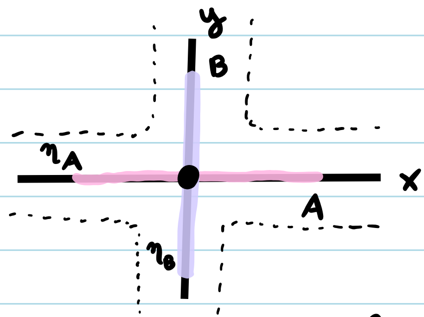
2. Let $p \in A \cap B$. We showed earlier that transversality gives coords on a nbhd $\mathcal{U}$ of p s.t.

- $A \cap \mathcal{U} = \{ (x,y) \in \mathcal{U} : y = 0 \}$
- $B \cap \mathcal{U} = \{ (x,y) \in \mathcal{U} : x = 0 \}$
- $\mathrm{sgn}(p)\, dx^1 \wedge \cdots \wedge dx^k \wedge dy^1 \wedge \cdots \wedge dy^{n-k}$ is an oriented top form.

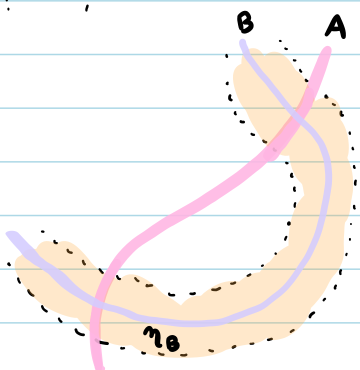
$x = (x^1, \dots, x^k) \in \mathbb{R}^k$
$y = (y^1, \dots, y^{n-k}) \in \mathbb{R}^{n-k}$

Then:

$$\int_M \eta_A \wedge \eta_B = \sum_{p \in A \cap B} \int_{\mathcal{U}_p} \eta_A \wedge \eta_B$$



3. Compute $\displaystyle\int_{\mathcal{U}} \eta_A \wedge \eta_B = \pm 1$



For simplicity: let's just check this when
$\dim A = 1$, $\dim B = 1$, $\dim M = 2$.

Claim: Let $\rho(x)$ be a bump function: $\displaystyle\int_{-\infty}^{\infty} \rho(x) = 1$.

$\forall \varphi \in \Omega^1(\mathbb{R}^2)$ with: $\mathrm{supp}\, \varphi \subseteq \{ |y| < R \}$
$d\varphi = 0$

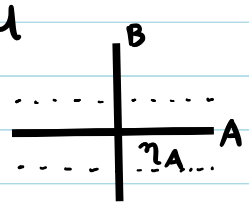
then:

$$\int_{\mathbb{R}^2} \rho(x)\, dx \wedge \varphi = \begin{cases} + \displaystyle\int_{\{x=0\}} \varphi & \text{if } dx \wedge dy \text{ oriented top form} \\ - \displaystyle\int_{\{x=0\}} \varphi & \text{if } dy \wedge dx \text{ oriented top form} \end{cases}$$

Assuming claim:

$$\int_{\mathcal{U}} \eta_A \wedge \eta_B = \int_{\substack{\{x=0\}\\ \| \\ B}} \eta_A = \operatorname{sgn}(p) \iint \rho(x)\, dx \wedge \eta_A$$

supp η_A $\subseteq \{|y| < \varepsilon\}$



$$= \operatorname{sgn}(p) \int_{\substack{\{y=0\}\\ \| \\ A}} \rho(x)\, dx = \pm 1$$

<u>**Pf of claim:**</u> Suppose $dx \wedge dy$ is oriented.

Write: $\varphi = \varphi_1\, dx + \varphi_2\, dy$.

$d\varphi = 0 \implies \partial_1 \varphi_2 = \partial_2 \varphi_1$

Consider:

$$\frac{d}{da} \int_{\{x=a\}} \varphi = \frac{d}{da} \int_{-\infty}^{\infty} \varphi_2(a,y)\, dy$$

$$= \int_{-\infty}^{\infty} \partial_1 \varphi_2(a,y)\, dy$$

$$= \int_{-\infty}^{\infty} \partial_2 \varphi_1(a,y)\, dy$$

$$= 0 \quad \text{since } \operatorname{supp} \varphi \subseteq \{|y| < R\}$$

$$\implies \int_{\{x=a\}} \varphi = \int_{\{x=0\}} \varphi.$$

$$\int_{\mathbb{R}^2} \rho(x)\, dx \wedge \varphi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x)\, \varphi_2(x,y)\, \underbrace{dx \wedge dy}_{\text{oriented}}$$

$$= (+1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x)\, \varphi_2(x,y)\, dx\, dy$$

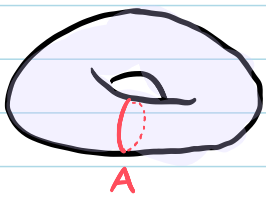
$$= \int_{-\infty}^{\infty} \rho(x) \left(\int_{-\infty}^{\infty} \varphi_2(x,y)\, dy \right) dx$$

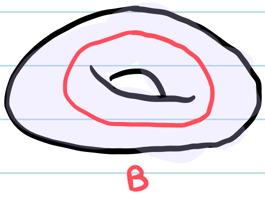
$$\| $$

$$\left(\int_{-\infty}^{\infty} \varphi_2(0,y)\, dy \right)$$

$$= \int_{-\infty}^{\infty} \rho(x)\, dx \int_{-\infty}^{\infty} \varphi_2(0,y)\, dy = \int_{\{x=0\}} \varphi. \qquad \square$$

ex) $T^2 = S^1 \times S^1$

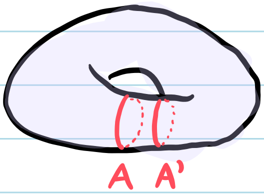




$\eta_A = \dfrac{d\theta^1}{2\pi}$ $\qquad\qquad$ $\eta_B = \dfrac{d\theta^2}{2\pi}$

$A \cdot B = 1$
$A \cdot A = 0$
$B \cdot B = 0$

Self-intersection:



$A' - A = \partial N$

$A \cdot A = A \cdot A' = 0.$

ex) $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$, $\{[z_0, z_1, 0] \in \mathbb{C}P^2\} = \mathbb{C}P^1.$

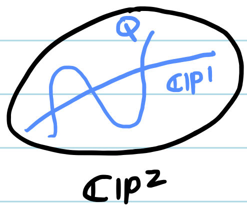
$$\mathbb{C}P^1 \cdot \mathbb{C}P^1 = \int_{\mathbb{C}P^2} \eta_{\mathbb{C}P^1} \wedge \eta_{\mathbb{C}P^1}$$

$$= \int_{\mathbb{C}P^2} \omega_{FS}^2 \qquad \eta_{\mathbb{C}P^1} = [\omega_{FS}]$$

$$= 1.$$

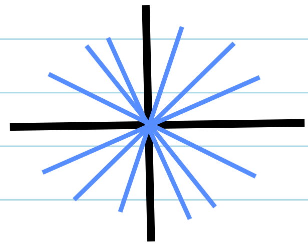
ex) $Q \subseteq \mathbb{C}P^2$, $\quad Q = \{ \sum_{i=0}^{2} z_i^3 = 0 \} \subseteq \mathbb{C}P^2$

$Q \cdot \mathbb{C}P^1 = 3$, since $\eta_Q = 3[\omega_{FS}]$.



ex) $\mathbb{C}P^{n-1} \subseteq \mathbb{C}P^n$, $\quad \mathbb{C}P^{n-1} \cdot \mathbb{C}P^1 = 1.$

ex) $Bl_0 \mathbb{C}^n = \{ (z, [u]) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} : x \in [u] \}$



e.g. $Bl_0 \mathbb{C}^2 = \{ (x,y), [u,v] \in \mathbb{C}^2 \times \mathbb{CP}^1 : \frac{x}{u} = \frac{y}{v} \}$

$p : Bl_0 \mathbb{C}^n \to \mathbb{C}^n$

$\quad (z,[u]) \mapsto z$

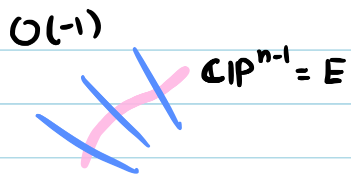
$p^{-1}(0) = \mathbb{CP}^{n-1}$

$p : Bl_0 \mathbb{C}^n \setminus \{\pi^{-1}(0)\} \to \mathbb{C}^n \setminus \{0\}$ bi hol'c.

<u>Exercise:</u> $\pi : Bl_0(\mathbb{C}^n) \to \mathbb{CP}^{n-1}$ is a vector bundle with transition
$\quad (z,[u]) \mapsto [u]$ functions $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{C}^*$

$\mathcal{U}_\alpha = \{ Z_\alpha \neq 0 \}, \quad g_{\alpha\beta} = \frac{Z_\alpha}{Z_\beta}$

$\therefore Bl_0 \mathbb{C}^n = \text{tot}\,(\mathcal{O}(-1) \to \mathbb{CP}^{n-1})$.



Notation: E = zero section
$\quad = p^{-1}(0)$
$\quad = \mathbb{CP}^{n-1}$
$\quad$ = "exceptional divisor"

ex) $E \subseteq Bl_0 \mathbb{C}^2, \quad E \cdot E = ?$

η_E = Thom class $\mathcal{O}(-1) \to \mathbb{CP}^1 = \Phi$

$$E \cdot E = \int_{\text{tot}\left(\begin{smallmatrix}\mathcal{O}(-1)\\ \downarrow \\ \mathbb{CP}^1\end{smallmatrix}\right)} \Phi \wedge \Phi = \int_{\mathbb{CP}^1} j^* \Phi, \qquad j : \mathbb{CP}^1 \to \mathcal{O}(-1) \text{ zero section}$$

$$= \int_{\mathbb{CP}^1} e \qquad \text{Euler class}$$

$$= (-1) \int_{\mathbb{CP}^1} \omega_{FS} \qquad e(\mathcal{O}(-1)) = -\omega_{FS}$$

$$= -1.$$

$\therefore E^2 = -1.$

$\Delta \subseteq M \times M$

ex) M compact oriented mfd.
$\Delta \subseteq M \times M$ submfd given by:
$\Delta = \{ (p,p) : p \in M \}$.

Will show later: $\Delta \cdot \Delta = \chi(M)$.

e.g. $\Delta \subseteq S^1 \times S^1$
$\Delta \cdot \Delta = 0$