Euler class of the tangent bundle

Thm: Let M be a compact oriented mfd.
Let $V \in \Gamma(TM)$ be a vector field with isolated zeroes $\{p_i\}_{i=1}^{k}$.

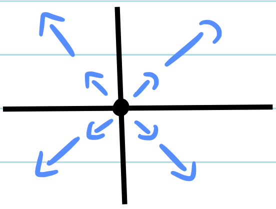
$$\int_M e(TM) = \sum_{i=1}^{k} \mathrm{ind}_{p_i}(V).$$

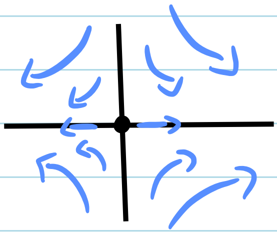
Index: $p_i \in \mathcal{U}$ a zero of $V \in \Gamma(TM)$, $\mathcal{U}$ chart with $p_i = 0$,

$$V\big|_{\mathcal{U}} = V^i(x)\frac{\partial}{\partial x^i}, \quad \text{let } F : \partial B_\varepsilon(0) \to S^{n-1}, \quad F^i(x) = \frac{V^i(x)}{|V^i(x)|}.$$

$$\mathrm{ind}_{p_i}(V) = \deg(F).$$

ex) z^k = vector field index k on $\mathbb{R}^2$
$\bar{z}^k$ = vector field index $-k$ on $\mathbb{R}^2$

ex)  $+1$

ex)  -1

$z = x+iy \leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = V$

$F = \frac{V}{|V|} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad F : S^1 \to S^1$

$$\deg(F) = \frac{1}{2\pi}\int_0^{2\pi} F^* d\theta$$

In local S^1 coords:
$F(\theta) \overset{loc}{=} \theta$

$$\deg(F) = \frac{1}{2\pi}\int_0^{2\pi} d\theta = +1$$

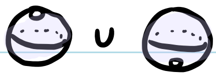
$\bar{z} = x-iy \leftrightarrow \begin{pmatrix} x \\ -y \end{pmatrix} = V$

$F = \frac{V}{|V|} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, \quad F : S^1 \to S^1$

$$\deg(F) = \frac{1}{2\pi}\int_0^{2\pi} F^* d\theta$$

In local S^1 coords:
$F(\theta) \overset{loc}{=} -\theta$

$$\deg(F) = \frac{1}{2\pi}\int_0^{2\pi} -d\theta = -1$$

ex) $S^2 =$  stereographic coords

$\mathcal{U}$ (u,v) $\quad\cup\quad$ $\tilde{\mathcal{U}}$ $(\tilde{u},\tilde{v})$

$$(\tilde{u},\tilde{v}) = \frac{1}{u^2+v^2}(u,v).$$

Suppose W is a vector field on S^2 s.t.

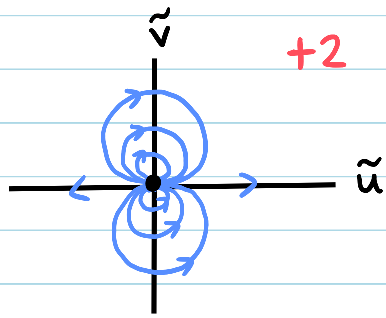
$W\big|_{\mathcal{U}} = \frac{\partial}{\partial u}$. Show $W=0$ at the North pole and compute index.

$$\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)} = \begin{pmatrix} \frac{\partial\tilde{u}}{\partial u} & \frac{\partial\tilde{u}}{\partial v} \\ \frac{\partial\tilde{v}}{\partial u} & \frac{\partial\tilde{v}}{\partial v} \end{pmatrix} \underset{\text{exercise}}{=} \begin{pmatrix} \tilde{v}^2-\tilde{u}^2 & -2\tilde{u}\tilde{v} \\ -2\tilde{u}\tilde{v} & \tilde{u}^2-\tilde{v}^2 \end{pmatrix}$$

$$\tilde{W}^i = \frac{\partial\tilde{x}^i}{\partial x^p} W^p$$

$$\begin{pmatrix}\tilde{W}^1\\ \tilde{W}^2\end{pmatrix} = \left(\frac{\partial\tilde{x}^i}{\partial x^j}\right)_{2\times2}\begin{pmatrix}W^1\\ W^2\end{pmatrix} \Rightarrow \begin{pmatrix}\tilde{W}^1\\ \tilde{W}^2\end{pmatrix} = \begin{pmatrix} \tilde{v}^2-\tilde{u}^2 & -2\tilde{u}\tilde{v} \\ -2\tilde{u}\tilde{v} & \tilde{u}^2-\tilde{v}^2 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$W\big|_{\tilde{\mathcal{U}}} = (\tilde{v}^2-\tilde{u}^2)\frac{\partial}{\partial\tilde{u}} - 2\tilde{u}\tilde{v}\frac{\partial}{\partial\tilde{v}}.$$



$W\big|_{(0,0)} = 0$ at $(\tilde{u},\tilde{v}) = (0,0)$.

$F = \frac{W}{|W|} : S^1 \to S^1$, $\quad F = (\sin^2\theta - \cos^2\theta, -2\sin\theta\cos\theta)$

$\tilde{u} = r\cos\theta$

$\tilde{v} = r\sin\theta$

$\Rightarrow F = (-\cos 2\theta, -\sin 2\theta)$. In S^1-coords, $F(\theta) \overset{\text{loc}}{=} 2\theta + \pi$.

$$\deg(F) = \frac{1}{2\pi}\int_0^{2\pi} F^* d\theta = \frac{1}{2\pi}\int_0^{2\pi} 2\,d\theta = +2.$$

$$\Rightarrow \int_{S^2} e(TS^2) = 2.$$

Cor: Every vector field on S^2 must vanish.

ex) S^1, $V = \frac{\partial}{\partial\theta}$ global vector field without zeroes.

$$\int_{S^1} e(TS^1) = 0.$$

ex) $T^2 = S^1 \times S^1$, $V = \frac{\partial}{\partial\theta^1}$. $\Rightarrow \int_{T^2} e(TM) = 0.$

Proof of Theorem: We only prove the case $\dim M = 2$.
See Bott-Tu for general case.

Equip TM with a metric g.
$\Rightarrow$ Transition functions of $TM \to M$ can be reduced to $SO(2) \cong U(1)$.
$\Rightarrow$ Can understand $TM \to M$ as a $U(1)$-bundle.

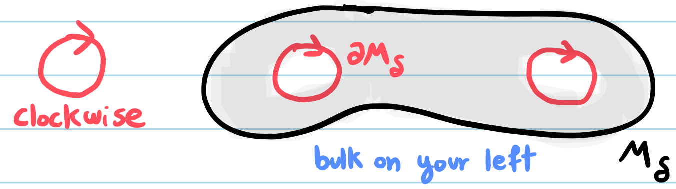
$$\int_M e(TM) = \lim_{\delta \to 0} \int_{M_\delta} e(TM), \quad \text{where } M_\delta = M \setminus \cup B_\delta(p_i), \ p_i \text{ zeroes of } V \in \Gamma(TM)$$

$$= \lim_{\delta\to 0} \int_{M_\delta} V^* \pi^* e \qquad V: M \to TM, \ \pi: TM \to M$$
$$V|_{\mathcal{U}} = (x^1, x^2, V^1(x), V^2(x))$$

$$= \lim_{\delta\to 0} -\int_{M_\delta} V^* d\psi \qquad \psi \in \Omega^1(TM^\circ) \text{ angular form}$$
$$d\psi = -\pi^* e$$
$$\psi = \frac{1}{2\pi}(d\theta - i\pi^* A) \quad (*)$$

$$-\int_{M_\delta} V^* d\psi \overset{\text{Stokes}}{=} -\int_{\partial M_\delta} V^* \psi$$

$$\overset{\text{orientation of } \partial M}{=} +\sum_i \int_{\partial D_\delta(p_i)} V^* \psi$$



$$\overset{(*)}{=} \sum_i \frac{1}{2\pi} \int_{\partial D_\delta(p_i)} V^* d\theta - \sum_i \frac{i}{2\pi} \overset{\text{Stokes}}{\int_{D_\delta(p_i)}} \pi^* dA$$

$dA = F \in \Omega^2(M)$

$D_\delta(p_i) \leftarrow$ goes to zero as $\delta \to 0$

$$\therefore \int_M e(TM) = \sum_i \frac{1}{2\pi} \int_{\partial D_\delta(p_i)} V^* d\theta = \sum_i \operatorname{ind}(p_i). \qquad \square$$

Thm: Let M be a compact oriented mfd.

$$\int_M e(TM) = \chi(M).$$

Cor: (Poincaré - Hopf)
Let M be a compact oriented mfd.
Let $V \in \Gamma(TM)$ be a vector field with isolated zeroes $\{p_i\}_{i=1}^{k}$.

$$\chi(M) = \sum_i \operatorname{ind}_{p_i}(V).$$

Proof of Thm:

Let $\{\omega_i\}$ basis for $H^*(M)$
$\{\tau_i\}$ dual basis under Poincaré duality:

$$\int_M \omega_i \wedge \tau_j = \delta_{ij}$$

Consider $M \times M$ with projections $\pi : M \times M \to M$
$\rho : M \times M \to M$

Künneth: $H^*(M \times M)$ has basis $\pi^* \omega_i \wedge \rho^* \tau_j$.

Let $\eta_\Delta \in H^n(M \times M)$ be the Poincaré dual of $\Delta \subseteq M \times M$,
$\Delta = \{(p,p) : p \in M\}$.

Step 1: $\eta_\Delta = \sum_i (-1)^{\deg \omega_i} \pi^* \omega_i \wedge \rho^* \tau_i$

Step 2: $\int_\Delta \eta_\Delta = \chi(M)$

Step 3: $\int_\Delta \eta_\Delta = e(TM)$

Proof of step 1: $\eta_\Delta = c_{ij}\, \pi^*\omega_i \wedge \rho^*\tau_j$ linear combo of basis.

Test η_Δ against $\pi^*\tau_k \wedge \rho^*\omega_\ell$:

$$\int_{M\times M} \pi^*\tau_k \wedge \rho^*\omega_\ell \wedge \eta_\Delta = \int_{M\times M} c_{ij}\, \pi^*\tau_k \wedge \rho^*\omega_\ell \wedge \pi^*\omega_i \wedge \rho^*\tau_j$$

$$= (-1)^{(\deg\omega_i)(\deg\tau_k + \deg\omega_\ell)}\, c_{ij} \int_{M\times M} \pi^*(\omega_i \wedge \tau_k)\, \rho^*(\omega_\ell \wedge \tau_j)$$

$$= (-1)^{(\deg\omega_k)(\deg\tau_k + \deg\omega_\ell)}\, c_{k\ell}.$$

Property of Poincaré duality:

$$\int_{M\times M} \pi^*\tau_k \wedge \rho^*\omega_\ell \wedge \eta_\Delta = \int_\Delta \pi^*\tau_k \wedge \rho^*\omega_\ell$$

$$= \int_M \tau_k \wedge \omega_\ell = (-1)^{(\deg\tau_k)(\deg\omega_\ell)}\, \delta_{k\ell}.$$

$$\Rightarrow c_{k\ell} = (-1)^{\deg\omega_k}\, \delta_{k\ell}.$$

Proof of step 2:

$$\int_\Delta \eta_\Delta = \sum_i (-1)^{\deg\omega_i} \int_\Delta \pi^*\omega_i \wedge \rho^*\tau_i$$

$$= \sum_i (-1)^{\deg\omega_i} \underbrace{\int_M \omega_i \wedge \tau_i}_{=1}$$

$$= \sum_q (-1)^q \dim H^q(M) = \chi(M).$$

Proof of step 3: We claim $T\Delta \to \Delta$ and $N_\Delta \to \Delta$ are isomorphic bundles,

and $T\Delta \cong TM$, $\Delta \cong M$

Assuming this,

$$\int_\Delta \eta_\Delta = \int_{\{\text{zero section}\}} \Phi(N_\Delta)$$

Poincaré dual of $S \subseteq M$ is Thom class of $N \to S$

$$= \int_{\{\text{zero section}\}} \Phi(T\Delta) \qquad T\Delta \cong N_\Delta$$

$$= \int_{\{\text{zero section}\}} \Phi(TM) \qquad T\Delta \cong TM$$

$$= \int_M e(TM).$$

Proof of claim:

Let $E = T(M \times M)\big|_\Delta$, $\quad E \to \Delta$

$$E = \{ (p,p,v,w) : p \in M,\ v,w \in T_pM \}$$

$$T\Delta = \{ (p,p,v,v) : p \in M,\ v \in T_pM \}$$

$E = T\Delta \oplus N_\Delta$, where $N_\Delta = \{ (p,p,v,-v) : p \in M, v \in T_pM \}$

As a bundle, trivializations on $T\Delta \to \Delta$: use coords on Δ:

$$\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha \times \mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^n$$

$$(x, x, v^i \partial_i, v^i \partial_i) \mapsto (x, v^i)$$

$$\begin{array}{c} \mathcal{U}_\alpha \times \mathcal{U}_\alpha \subseteq \Delta \\ \downarrow \\ x \in \mathcal{U}_\alpha \subseteq M \end{array}$$

Trivializations on $N_\Delta \to \Delta$:

$$\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha \times \mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^n$$

$$(x, x, v^i \partial_i, -v^i \partial_i) \mapsto (x, v^i)$$

$\therefore$ $\begin{array}{l} T\Delta \to \Delta \\ N_\Delta \to \Delta \end{array}$ are isomorphic bundles via $(p,p,v,v) \mapsto (p,p,v,-v)$.

Also:
$$\begin{tikzcd} \left\{ (p,v) : \begin{array}{l} p \in M \\ v \in T_pM \end{array} \right\} = TM \arrow[r, "\sim"] \arrow[d] & T\Delta = \left\{ (p,p,v,v) : \begin{array}{l} p \in M \\ v \in T_pM \end{array} \right\} \arrow[d] \\ M \arrow[r, "\sim"] & \Delta \end{tikzcd}$$

Since $\Delta \cong M$ are diffeomorphic, their tangent bundles are isomorphic:

$T\Delta \cong TM$.

$\square$

Gauss - Bonnet Theorem

Thm: Let M be compact oriented with $\dim M = 2$.

$$\int_M \frac{R}{2}\, dvol_g = 2\pi\, \chi(M).$$

Riemannian Geometry: Let (M,g) be Riemannian mfd.

Let ∇ be the Levi-Civita connection of g.
For $X \in \Gamma(TM)$, have $\nabla_X : \Gamma(TM) \to \Gamma(TM)$ satisfying

$(\nabla_K = \nabla_{\partial_K})$

1) $\partial_K \langle X, Y \rangle_g = \langle \nabla_K X, Y \rangle_g + \langle X, \nabla_K Y \rangle_g$
2) $\nabla_X Y - \nabla_Y X = [X, Y]$.

Let $\{e_a\}$ be a local orthonormal frame for TM.
$\langle e_a, e_b \rangle_g = \delta_{ab}$.

Denote: $\nabla_K e_a = \omega_K{}^b{}_a\, e_b$ vier bein

Note: $0 = \partial_K \langle e_a, e_b \rangle_g = \langle \nabla_K e_a, e_b \rangle + \langle e_a, \nabla_K e_b \rangle$
$\Rightarrow \omega_i{}^a{}_b = -\omega_i{}^b{}_a$.

Note: If $\tilde{e}_a = e_b\, g^b{}_a$ for $g = [g^a{}_b] \in O(n)$, then:
$\nabla_K \tilde{e}_a = \tilde{\omega}_K{}^b{}_a\, \tilde{e}_b$

$$\tilde{\omega} = g^{-1} \omega g + g^{-1} dg.$$

$\left(\text{Expand } \nabla_K(\tilde{e}_a) = \nabla_K (e_b g^b{}_a) = (g^{-1}\omega_K g)^b{}_a\, \tilde{e}_b + (g^{-1}\partial_K g)^b{}_a\, \tilde{e}_b \right)$

Curvature: $Rm \in \Omega^2(\mathrm{End}\, TM)$
$Rm = d\omega + \omega \wedge \omega$
$Rm = \frac{1}{2} R_{ij}{}^K{}_\ell\, e^i \wedge e^j \otimes e_K \otimes e^\ell$.

We now restrict to $n=2$

$$\omega = \begin{pmatrix} 0 & \omega^1{}_2 \\ -\omega^1{}_2 & 0 \end{pmatrix}, \quad \text{write } A = -i\,\omega^1{}_2 \qquad \alpha \in \Omega^1(\mathcal{U})$$
$$\alpha = \omega^1{}_2$$

Note: A defines a $U(1)$ connection.

Recall this means: $\tilde{A} = A - i\,d\varphi$ when: $g = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$\therefore\ \tilde{\omega} = g\omega g^{-1} - dg\,g^{-1}$

$\tilde{v}^i = g^i{}_j v^j$

$\tilde{e}_a = e_b (g^{-1})^b{}_a$

$\tilde{v}^i \tilde{e}_i = v^i e_i$

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$- d\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & d\varphi \\ -d\varphi & 0 \end{pmatrix} \Rightarrow \begin{aligned} \tilde{\alpha} &= \alpha + d\varphi \\ \tilde{A} &= A - i\,d\varphi \end{aligned} \qquad \alpha = iA$$

$F = dA$ curvature of $U(1)$-bundle

$$Rm = d\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$$

$$Rm = \begin{pmatrix} 0 & iF \\ -iF & 0 \end{pmatrix} \overset{(*)}{=} \begin{pmatrix} 0 & R^1{}_2 \\ -R^1{}_2 & 0 \end{pmatrix}$$

Scalar curvature: $R = \sum_{p,i} R_{pi}{}^p{}_i$ orthonormal frame indices

Claim: $\frac{R}{2}\, dvol_g = iF \quad (\dim M = 2)$

$R = R_{1i}{}^1{}_i + R_{2i}{}^2{}_i = 2R_{12}{}^1{}_2 \overset{(*)}{=} 2iF_{12}$.

$dvol_g = e^1 \wedge e^2$

$\Rightarrow R\, dvol_g = 2iF_{12}\, e^1 \wedge e^2$.

$$\int_M e(TM) = \chi \Rightarrow \int_M iF = 2\pi\chi$$

since $e(L) = \frac{i}{2\pi} F$ for $U(1)$-bundle

$$\therefore \int_M \frac{R}{2} dvol_g = 2\pi\chi.$$