Cech Cohomology

Def: A presheaf $\mathcal{F}$ on a mfd X is an assignment:

- $\mathcal{U} \subseteq X$ open $\rightsquigarrow$ $\mathcal{F}(\mathcal{U})$ a set
- $i^V_{\mathcal{U}} : V \hookrightarrow \mathcal{U}$ $\rightsquigarrow$ $\mathcal{F}(i^V_{\mathcal{U}}) : \mathcal{F}(\mathcal{U}) \to \mathcal{F}(V)$ restriction

  inclusion of opens

s.t. a) $\mathcal{F}(i^{\mathcal{U}}_{\mathcal{U}}) = \text{identity}$

b) $\mathcal{F}(i^W_V)\,\mathcal{F}(i^V_{\mathcal{U}}) = \mathcal{F}(i^W_{\mathcal{U}})$.

Notation: $\rho^{\mathcal{U}}_V = \mathcal{F}(i^V_{\mathcal{U}})$, $\rho^{\mathcal{U}}_V : \mathcal{F}(\mathcal{U}) \to \mathcal{F}(V)$ restriction map.

Def: 1) $\mathcal{F}$ is a presheaf of abelian groups if each $\mathcal{F}(\mathcal{U})$ is an abelian group and all $\rho^{\mathcal{U}}_V$ are group homomorphisms.

2) $\mathcal{F}$ is a presheaf of R-modules if each $\mathcal{F}(\mathcal{U})$ is a $R(\mathcal{U})$-module and all $\rho^{\mathcal{U}}_V$ are compatible with the module structure. Here $\mathcal{U}$ is assigned a ring $R(\mathcal{U})$.

Def: A sheaf $\mathcal{F}$ on a mfd X is a presheaf s.t. whenever $\Omega = \bigcup \mathcal{U}_\alpha$ union of opens, then:

a) If $s_\alpha \in \mathcal{F}(\mathcal{U}_\alpha)$ are given s.t.

$$\rho^{\mathcal{U}_\alpha}_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}(s_\alpha) = \rho^{\mathcal{U}_\beta}_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}(s_\beta) \quad \forall\, \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$$

then $\exists\, s \in \mathcal{F}(\Omega)$ s.t. $\rho^{\Omega}_{\mathcal{U}_\alpha}(s) = s_\alpha$.

b) If $t, s \in \mathcal{F}(\Omega)$ are s.t. $\rho^{\Omega}_{\mathcal{U}_\alpha}(s) = \rho^{\Omega}_{\mathcal{U}_\alpha}(t)$ $\forall \alpha$, then $s = t$.

ex) Define $\mathcal{F}(\mathcal{U}) = \mathbb{R}$ $\forall$ open $\mathcal{U} \subseteq X$, $\rho^{\mathcal{U}}_V = \text{id}$.
$\mathcal{F}$ is a presheaf but not a sheaf: let $\mathcal{U}, V \subseteq M$ be open + disjoint.
Take: $s_{\mathcal{U}} = 1 \in \mathcal{F}(\mathcal{U})$. No $s \in \mathcal{F}(\mathcal{U} \cup V)$ restricting to $s_{\mathcal{U}}, s_V$.
$s_V = 0 \in \mathcal{F}(V)$

ex) Constant sheaf with group G: $\mathcal{F}(\mathcal{U}) = \left\{ \begin{array}{c} \text{locally constant functions} \\ \mathcal{U} \to G \end{array} \right\}$

$\rho^{\mathcal{U}}{}_{V}$ = restriction of functions.

ex) Sheaf of differential k-forms $\Omega^k(M)$:
$\mathcal{F}(\mathcal{U}) = \Omega^k(\mathcal{U}) = \{k\text{-forms on } \mathcal{U} \subseteq M\}$
$\rho^{\mathcal{U}}{}_{V}$ = restriction (i.e. pullback)

ex) $E \to M$ vector bundle $\rightsquigarrow$ sheaf on M.
$\mathcal{F}(\mathcal{U}) = \{\text{smooth sections over } \mathcal{U} \subseteq M\}$

ex) Sheaf of nowhere vanishing smooth functions $C^{\infty *}$
$\mathcal{F}(\mathcal{U}) = \{\text{group of smooth nowhere vanishing } f: \mathcal{U} \to \mathbb{R}^*\}$

<u>Def</u>: Suppose $\mathcal{U} \subseteq X$ open set on cplx mfd X.
$\mathcal{O}(\mathcal{U})$ = ring of hol'c functions $f: \mathcal{U} \to \mathbb{C}$.
$\mathcal{O}^*(\mathcal{U})$ = abelian group of non-vanishing hol'c $f: \mathcal{U} \to \mathbb{C}^\times$

ex) I_0 = sheaf of hol'c fun on $\mathbb{C}^2$ vanishing at $(0,0) \in \mathbb{C}^2$.

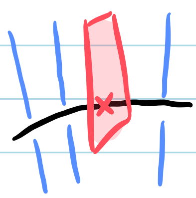
If $(0,0) \notin \mathcal{U}$, $I_0(\mathcal{U}) = \mathcal{O}(\mathcal{U})$-module generated by 1.

If $(0,0) \in \mathcal{U}$, $I_0(\mathcal{U}) = \mathcal{O}(\mathcal{U})$-module generated by x, y.
Any hol'c $f: \mathcal{U} \to \mathbb{C}$ with $f(0,0) = 0$ can be written:

$$f(x,y) = \underset{\mathcal{O}(\mathcal{U})}{\underset{\cap}{g(x,y)}}\, x + \underset{\mathcal{O}(\mathcal{U})}{\underset{\cap}{h(x,y)}}\, y$$

$$\therefore \ \operatorname{rank}(I_0(\mathcal{U})) = \begin{cases} 1 & (0,0) \notin \mathcal{U} \\ 2 & (0,0) \in \mathcal{U} \end{cases}$$

"like vector bundle where rank can jump"


$\mathbb{C}^2$

But also: $I_0(\mathcal{U})$ not a free module
$-y \cdot x + x \cdot y = 0.$

ex) $\mathcal{F}$ presheaf of abelian groups on $\mathbb{C}$:

$$\mathcal{F}(\mathcal{U}) = \mathcal{O}^*(\mathcal{U}) / \exp(\mathcal{O}(\mathcal{U}))$$

Not a sheaf: consider

$\mathcal{U} = \mathbb{C} \setminus \{x > 0\}$
$V = \mathbb{C} \setminus \{0\}$

$s_{\mathcal{U}} = [z]$. Note $s_{\mathcal{U}} = s_V$ on $\mathcal{U} \cap V = \mathbb{C} \setminus \{x=0\}$:
$s_V = [1]$

can write $z = e^{\log z}$ on $\mathbb{C}\setminus\{x=0\} \Rightarrow [1]=[z]$.

But: $\nexists\, s \in \mathcal{F}(\mathcal{U} \cup V)$ s.t. $s|_V = [1]$. Cannot solve $z = e^f$ on $\mathbb{C}\setminus\{0\}$.

$s|_{\mathcal{U}} = [z]$

Cech Cohomology:

Let $X = \bigcup \mathcal{U}_\alpha$, $\underline{\mathcal{U}} = \{\mathcal{U}_\alpha\}_{\alpha\in I}$ open cover of a mfd

Let $\mathcal{F}$ be a presheaf of abelian groups.

$C^p(\underline{\mathcal{U}}, \mathcal{F}) = \prod \mathcal{F}(\mathcal{U}_{\alpha_0 \cdots \alpha_p})$, where $\mathcal{U}_{\alpha_0\cdots\alpha_p} = \mathcal{U}_{\alpha_0} \cap \cdots \cap \mathcal{U}_{\alpha_p}$.

ex) $X = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$

$C^0(\underline{\mathcal{U}}, \mathcal{F}) = \mathcal{F}(\mathcal{U}_1) \oplus \mathcal{F}(\mathcal{U}_2) \oplus \mathcal{F}(\mathcal{U}_3)$

$C^1(\underline{\mathcal{U}}, \mathcal{F}) = \mathcal{F}(\mathcal{U}_1\cap\mathcal{U}_2) \oplus \mathcal{F}(\mathcal{U}_1\cap\mathcal{U}_3) \oplus \mathcal{F}(\mathcal{U}_2\cap\mathcal{U}_3)$

$C^2(\underline{\mathcal{U}}, \mathcal{F}) = \mathcal{F}(\mathcal{U}_1\cap\mathcal{U}_2\cap\mathcal{U}_3)$

Convention: For $\omega \in C^p(\underline{\mathcal{U}}, \mathcal{F})$, assign to each $\mathcal{U}_{\alpha_0\cdots\alpha_p}$ components $\omega_{\alpha_0\cdots\alpha_p} \in \mathcal{F}(\mathcal{U}_{\alpha_0\cdots\alpha_p})$ with $\alpha_0 < \cdots < \alpha_p$. Then allow $\omega_{\alpha_0\cdots\alpha_p}$ with indices of any order by imposing skew-symmetry e.g. $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$.

Co-boundary operator: $\delta : C^p(\underline{\mathcal{U}}, \mathcal{F}) \to C^{p+1}(\underline{\mathcal{U}}, \mathcal{F})$

$$(\delta\omega)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i\, \omega_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}}, \quad (*)$$

where restriction of $\omega_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}}$ to $\mathcal{U}_{\alpha_0\cdots\alpha_{p+1}}$ in RHS of $(*)$ suppressed.

ex) $\omega \in C^0(\underline{\mathcal{U}}, \mathcal{F})$, $\omega_\alpha \in \mathcal{F}(\mathcal{U}_\alpha)$

$(\delta\omega)_{\alpha\beta} = \omega_\beta - \omega_\alpha$

ex) $\omega \in C^1(\underline{\mathcal{U}}, \mathcal{F})$, $\quad \omega_{\alpha\beta} \in \mathcal{F}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$

$$(\delta\omega)_{\alpha\beta\gamma} = \omega_{\beta\gamma} - \omega_{\alpha\gamma} + \omega_{\alpha\beta}$$

Prop: $\delta^2 = 0$.

Pf: Let's only check $\delta\delta\omega = 0$ for $\omega \in C^0(\underline{\mathcal{U}}, \mathcal{F})$.

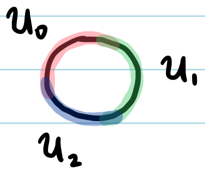
$$\begin{aligned}(\delta^2\omega)_{\alpha\beta\gamma} &= (\delta\omega)_{\beta\gamma} - (\delta\omega)_{\alpha\gamma} + (\delta\omega)_{\alpha\beta} \\ &= (\omega_\gamma - \omega_\beta) - (\omega_\gamma - \omega_\alpha) + (\omega_\beta - \omega_\alpha) \\ &= 0. \quad \square\end{aligned}$$

Def: Cech cohomology of cover $\underline{\mathcal{U}}$ with values in $\mathcal{F}$:

$$H^p(\underline{\mathcal{U}}, \mathcal{F}) = \frac{\operatorname{Ker}\left(C^p(\underline{\mathcal{U}}, \mathcal{F}) \xrightarrow{\delta} C^{p+1}(\underline{\mathcal{U}}, \mathcal{F})\right)}{\operatorname{Im}\left(C^{p-1}(\underline{\mathcal{U}}, \mathcal{F}) \xrightarrow{\delta} C^p(\underline{\mathcal{U}}, \mathcal{F})\right)}.$$

Notation: $H^p(\underline{\mathcal{U}}, \mathbb{R})$ cohomology of constant sheaf $\mathbb{R}$

ex) $S^1 = \mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2$ good cover

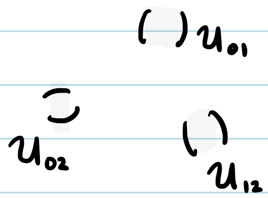


$$\begin{aligned}C^0(\underline{\mathcal{U}}, \mathbb{R}) &= \mathcal{F}(\mathcal{U}_0) \oplus \mathcal{F}(\mathcal{U}_1) \oplus \mathcal{F}(\mathcal{U}_2) \\ &= \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}\end{aligned}$$

$\omega \in C^0(\underline{\mathcal{U}}, \mathbb{R})$, $\quad \omega = (\omega_0, \omega_1, \omega_2) \in \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$

$C^1(\underline{\mathcal{U}}, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$

$\omega \in C^1(\underline{\mathcal{U}}, \mathbb{R})$, $\quad \omega = (\omega_{01}, \omega_{02}, \omega_{12})$,



$C^2(\underline{\mathcal{U}}, \mathbb{R}) = \emptyset$, no triple overlaps.

If $\omega \in C^0(\underline{\mathcal{U}}, \mathbb{R})$ with $\delta\omega = 0$, then $(\delta\omega)_{\alpha\beta} = \omega_\beta - \omega_\alpha = 0 \ \forall \alpha, \beta$

$\Rightarrow \omega_1 = \omega_2 = \omega_3$.

$H^0(\underline{\mathcal{U}}, \mathbb{R}) = \{ \omega \in C^0(\underline{\mathcal{U}}, \mathbb{R}) : \omega = (t, t, t) \text{ for } t \in \mathbb{R} \}$

$\dim H^0(\underline{\mathcal{U}}, \mathbb{R}) = 1.$

If $\omega \in C^1(\underline{\mathcal{U}}, \mathbb{R})$, then $\delta\omega = 0$. $\Rightarrow H^1 = \mathbb{R}^3 / \mathrm{Im}\, \delta$.
If $\omega \in C^1(\underline{\mathcal{U}}, \mathbb{R})$ is s.t. $\omega = \delta \zeta$, then $\omega_{\alpha\beta} = \zeta_\beta - \zeta_\alpha$.

$$\begin{aligned} \omega_{01} &= \zeta_1 - \zeta_0 \\ \omega_{02} &= \zeta_2 - \zeta_0 \\ \omega_{12} &= \zeta_2 - \zeta_1 \end{aligned} \qquad \delta = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \qquad \delta \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} \omega_{01} \\ \omega_{02} \\ \omega_{12} \end{pmatrix}$$

$\dim \mathrm{Im}\, \delta = 2$
$\dim H^1(\underline{\mathcal{U}}, \mathbb{R}) = 3 - 2 = 1$

Note: $\omega \in C^1(\underline{\mathcal{U}}, \mathbb{R})$ given by $(\omega_{01}, \omega_{02}, \omega_{12}) = (0,0,1)$
is not exact.

<u>Prop</u>: $M = \bigcup \mathcal{U}_\alpha$ good cover.
If $H^1(\underline{\mathcal{U}}, \mathbb{R}) = 0$, then every closed 1-form is exact.

<u>Pf</u>: Let $\eta \in \Omega^1(M)$ be s.t. $d\eta = 0$.

Since $\mathcal{U}_\alpha \cong \mathbb{R}^n$, $\eta\big|_{\mathcal{U}_\alpha} = df_\alpha$ for $f_\alpha : \mathcal{U}_\alpha \to \mathbb{R}$.

Remark: $f_\alpha - f_\beta \equiv$ const on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, since: $d(f_\alpha - f_\beta) = \eta - \eta = 0$.

$\Rightarrow$ obtain $\omega \in C^1(\underline{\mathcal{U}}, \mathbb{R})$ by $\omega_{\alpha\beta} = f_\alpha - f_\beta \equiv$ const.

Note: $(\delta\omega)_{\alpha\beta\gamma} = \omega_{\beta\gamma} - \omega_{\alpha\gamma} + \omega_{\alpha\beta} = 0$.

$\Rightarrow [\omega] \in H^1(\underline{\mathcal{U}}, \mathbb{R})$.
$\Rightarrow [\omega] = [0]$
$\Rightarrow \exists\, c \in C^0(\underline{\mathcal{U}}, \mathbb{R})$ s.t. $\omega_{\alpha\beta} = c_\beta - c_\alpha = (\delta c)_{\alpha\beta}$

Then $f_\alpha + c_\alpha : \mathcal{U}_\alpha \to \mathbb{R}$ glues to a global function $F : M \to \mathbb{R}$ since
$f_\alpha + c_\alpha = f_\beta + c_\beta$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$.
$f_\alpha - f_\beta = \omega_{\alpha\beta} = c_\beta - c_\alpha$

$\therefore dF = \eta$.
$\square$

Prop: $H^1(\underline{\mathcal{U}}, C^{\infty *})$ = isomorphism classes of complex line bundles, where:

$C^{\infty *}(\mathcal{U})$ = multiplicative group of smooth non-vanishing functions $f: \mathcal{U} \to \mathbb{C}^*$.

Pf:
$g \in C^1(\underline{\mathcal{U}}, C^{\infty *})$ with $\delta g = 0$ is given by:

$g_{\alpha\beta}: \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{C}^*$ smooth s.t. $(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma}\, g_{\alpha\gamma}^{-1}\, g_{\alpha\beta} = 1$ multiplicative group notation

$\Rightarrow g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma}$.

$\therefore$ $g \in C^1(\underline{\mathcal{U}}, C^{\infty *})$ with $\delta g = 0$ $\longleftrightarrow$ line bundle with triv $M = \bigcup \mathcal{U}_\alpha$ and transition functions $g_{\alpha\beta}: \mathcal{U}_{\alpha\beta} \to \mathbb{C}^*$.

Recall: L isomorphic to $\check{L}$
$\Leftrightarrow \exists\, \lambda_\alpha: \mathcal{U}_\alpha \to GL(1, \mathbb{C})$ s.t. $\check{g}_{\alpha\beta} = \lambda_\alpha\, g_{\alpha\beta}\, \lambda_\beta^{-1}$ on $\mathcal{U}_{\alpha\beta}$
$\Leftrightarrow g_{\alpha\beta}\, \check{g}_{\alpha\beta}^{-1} = \lambda_\beta\, \lambda_\alpha^{-1}$

$\Leftrightarrow [g] = [\check{g}] \in H^1(\underline{\mathcal{U}}, C^{\infty *})$. $\square$