# Spectral Sequence of a double complex

**Def:** Let $K = \bigoplus K^{p,q}$ be a direct sum of vector spaces.
$K$ is a double complex if $\exists$ commutative diagram s.t.

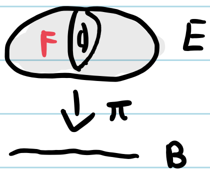
$$\begin{tikzcd}
 & \uparrow & \uparrow & \\
\arrow[r] & K^{p,q+1} \arrow[r] \arrow[u, "d"] & K^{p+1,q+1} \arrow[r] \arrow[u] & {} \\
\arrow[r, "\delta"] & K^{p,q} \arrow[r, "\delta"] \arrow[u, "d"] & K^{p+1,q} \arrow[r, "\delta"] \arrow[u] & {} \\
 & \arrow[u, "d"] & \arrow[u] &
\end{tikzcd}$$

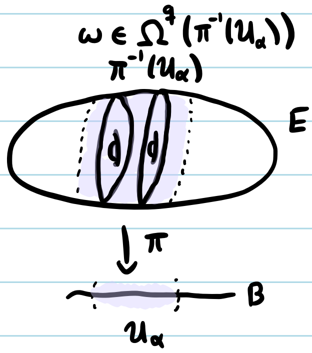
$\delta : K^{p,q} \to K^{p+1,q}$ homomorphisms
$d : K^{p,q} \to K^{p,q+1}$

$\delta d = \delta d$
$\delta^2 = 0$
$d^2 = 0$

ex) $X$ complex manifold
$K^{p,q} = \Omega^{p,q}(X)$ $(p,q)$-forms
$\partial : \Omega^{p,q} \to \Omega^{p+1,q}$
$\bar\partial : \Omega^{p,q} \to \Omega^{p,q+1}$
$\partial^2 = 0$, $\bar\partial^2 = 0$, $\partial\bar\partial = -\bar\partial\partial$
$\delta = \partial$, $d$ "=" $(-1)^p \bar\partial$

ex) Fiber bundle: let $G$ be a topological group acting effectively on a space $F$. A surjection $\pi : E \to B$ is a fiber bundle with structure group $G$ if $\exists$ open cover $B = \bigcup U_\alpha$ with fiber preserving homeomorphisms
$\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$ s.t.
$g_{\alpha\beta}(x) = \varphi_\alpha \circ \varphi_\beta^{-1} \big|_{\{x\}\times F} \in G$.




Double complex:
$K^{p,q} = C^p(\pi^{-1}\underline{U}, \Omega^q)$

$$K^{p,q} = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(\pi^{-1}(U_{\alpha_0 \cdots \alpha_p}))$$

$\delta : K^{p,q} \to K^{p+1,q}$

$$(\delta\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}}, \quad \omega \in \prod \Omega^q(U_{\alpha_0 \cdots \alpha_p}).$$

$d : K^{p,q} \to K^{p,q+1}$ exterior derivative on open sets

$d\delta = \delta d$ (independent operators)

e.g. $\omega \in K^{0,q}$, $\omega_\alpha \in \Omega^q(\pi^{-1}(U))$
$\delta\omega \in K^{1,q}$, $(\delta\omega)_{\alpha\beta} = \omega_\beta - \omega_\alpha \in \Omega^q(\pi^{-1}(U_{\alpha\beta}))$.

Note: From double complex $K = \bigoplus K^{p,q}$, can form single complex

$$C^k = \bigoplus_{p+q=k} K^{p,q}$$

$D : C^k \to C^{k+1}$

$D = \delta + (-1)^p d$

$D^2 = 0.$

ex) $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$

$\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$

$d : \Omega^k \to \Omega^{k+1}$, $d = \partial + \bar{\partial}$

$$\Omega^k = \bigoplus_{p+q=k} \Omega^{p,q}$$

## Spectral Sequence of $K = \bigoplus K^{p,q}$

$\delta : K^{p,q} \to K^{p+1,q}$

$d : K^{p,q} \to K^{p,q+1}$

Bunch of differential complexes

$$\to E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q} \xrightarrow{d_1} E_1^{p+2,q} \xrightarrow{d_1} E_1^{p+3,q} \to \cdots$$

$$\to E_2^{p,q} \xrightarrow{d_2} E_2^{p+2,q-1} \xrightarrow{d_2} E_2^{p+4,q-2} \to \cdots$$

$$\to E_3^{p,q} \xrightarrow{d_3} E_3^{p+3,q-2} \xrightarrow{d_3} E_3^{p+6,q-4} \to \cdots$$

$$\vdots$$

Pattern: $E_k^{p,q} \xrightarrow{d_k} E_k^{p+k,q+1-k}$

s.t. $E_{k+1}^{p,q} = \dfrac{\ker d_k \cap E_k^{p,q}}{\operatorname{im} d_k}$, and

$$E_1^{p,q} = H_d^{p,q} = \frac{\{\sigma \in K^{p,q} : d\sigma = 0\}}{d K^{p,q-1}},$$

$$d_1 = \delta.$$

This means: the first complex $E_1^{p,q}$ is

$$\to H_d^{p,q} \xrightarrow{\delta} H_d^{p+1,q} \xrightarrow{\delta} H_d^{p+2,q} \to \cdots$$

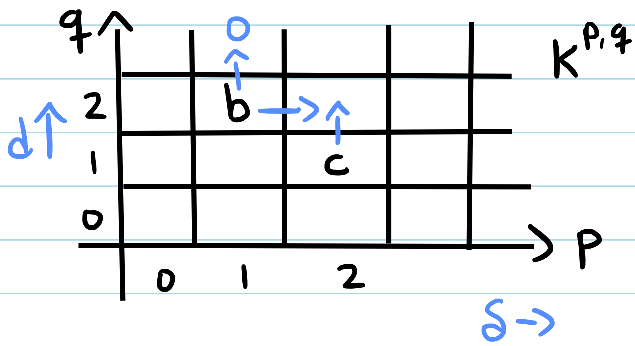
**Level Two:** $E_2^{p,q} = \dfrac{\operatorname{Ker} d_1}{\operatorname{Im} d_1} \cap E_1^{p,q}$

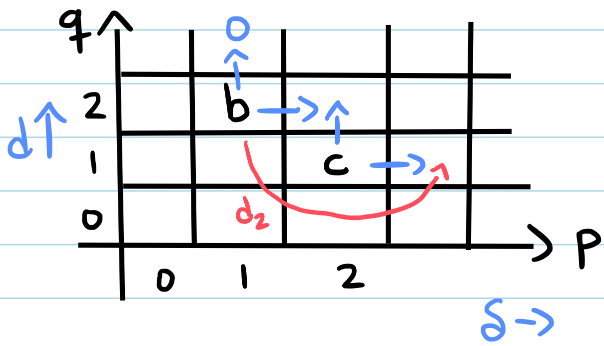
$\Rightarrow E_2^{p,q} = H_\delta^{p,q} H_d.$

What is $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ ?

Let $b \in K^{p,q}$. Say $b$ lives to $E_2^{p,q}$ if: $db = 0$
$[b]_2 \in E_2^{p,q}$ $\quad [\delta b] = 0 \in H_d$
$\delta b = (-1)^{p+1} dc$

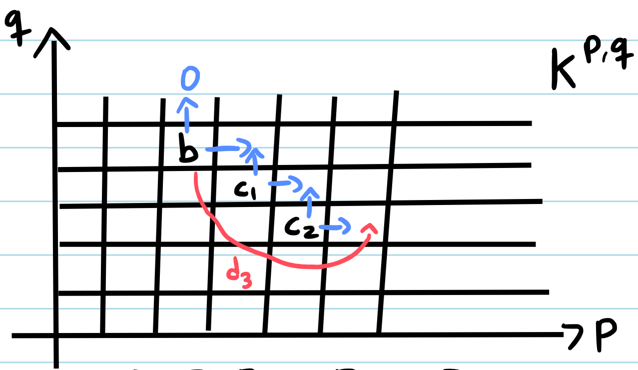
$b \in E_1^{p,q}$
$b \in \ker d_1$




Define: $d_2 [b]_2 = [\delta c]_2$. **Check:** $\delta c \in H_\delta H_d$
$d_2$ indep of choice of $c$.

**Level Three:** $b \in K^{p,q}$ lives to $E_3^{p,q}$ if: $db = 0$
$[b]_3 \in E_3^{p,q}$ $\quad \delta b = (-1)^{p+1} d c_1$
$\delta c_1 = (-1)^{p+1} d c_2 \quad (*)$




$d_3 [b]_3 = [\delta c_2]_3$

(*)
Suppose $d_2 [b]_2 = [0]_2$, $b \in E_2^{p,q} = E_1^{p,q} \cap \dfrac{\ker d_1}{\operatorname{im} d_1}$
Then: $\delta b = (-1)^{p+1} dc$
$[\delta c]_2 = 0$
$\Rightarrow [\delta c]_1 = d_1 [a]_1$, $\quad [a]_1 \in H_d$
$\Rightarrow \delta c = \delta a + \delta d \eta$, $\quad da = 0$
$\Rightarrow \delta(c - a) = d\eta$.
Let: $c_1 = c - a$, $\quad c_2 = (-1)^{p+1} \eta$. $\quad (*)$

... Continue like this to define $[b]_r \in E_r^{p,q}$
$d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$

**Def:** Let $K^{p,q}$ be double complex. Say $E_r^{p,q}$ degenerates on page $k$ if
$d_k = d_{k+1} = d_{k+2} = \cdots = 0$.

**Thm:** Let $K^{p,q}$ be a double complex s.t. $E_r^{p,q}$ degenerates on page $k$. Define:

$$E_\infty^{p,q} = E_k^{p,q}.$$
$$C^n = \bigoplus_{p+q=n} K^{p,q}, \quad D = \delta + (-1)^p d$$

$$H_D(C) = \frac{\ker D}{\operatorname{Im} D}, \quad F^p H_D = \bigoplus_{i \geq p} \bigoplus_{q \geq 0} H_D^{i,q} = H_D^{\geq p, \bullet}$$

**Then:**

not canonical

A) $H_D^n(C) \cong \bigoplus_{p+q=n} E_\infty^{p,q}$

canonical

B) $F^p H_D / F^{p+1} H_D \cong E_\infty^{p,q}.$

not canonical

**Note:** B) $\Rightarrow$ A). If $W \subseteq V$ subspace, then $V \cong V/W \oplus W$.

e.g. $H_D^2 \cong \underbrace{\frac{H_D^2}{F^1 H_D^2}}_{H_D^{0,2}} \oplus \underbrace{\frac{F^1 H_D^2}{F^2 H_D^2}}_{H_D^{1,1}} \oplus \underbrace{F^2 H_D^2}_{H_D^{2,0}},$ where: $F^1 H^2 = K^{1,1} \oplus K^{2,0} \cap H_D^2$
$F^2 H^2 = K^{2,0} \cap H_D^2$

$\therefore$ If B), then $H_D^2 \cong E_\infty^{0,2} \oplus E_\infty^{1,1} \oplus E_\infty^{2,0}.$

**Proof of B):** Later.

**Applications:** 1) Kähler geometry
2) $H^k(\underline{\mathcal{U}}, \mathbb{R}) \cong H^k_{dR}(M)$ Cech - de Rham

1) **Kahler geometry:** Let $X$ be a compact Kähler manifold.
The $\partial\bar\partial$-lemma holds in Kähler geometry:

**$\partial\bar\partial$-lemma:** $X$ satisfies the $\partial\bar\partial$-lemma if there holds:

Suppose $\eta \in \Omega^{p,q}(X)$ with $d\eta = 0$. TFAE:

1. $\eta = d\alpha$
2. $\eta = \partial\beta$
3. $\eta = \bar\partial\gamma$
4. $\eta = \partial\bar\partial\chi$.

**Cor:** The spectral seq associated to $\Omega^{p,q}$ degenerates on page 1.
$d_1 = d_2 = d_3 = \cdots = 0$.

$$\to H_{\bar\partial}^{p,q} \xrightarrow{\partial} H_{\bar\partial}^{p+1,q} \xrightarrow{\partial} \cdots$$
$$E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q} \xrightarrow{d_1}$$

Let $[\beta] \in H_{\bar\partial}^{p,q}$. Want: $[\partial\beta] = [0] \in H_{\bar\partial}^{p+1,q}$.

Apply $\partial\bar\partial$-lemma to $\partial\beta$ $\rightsquigarrow$ obtain $\partial\beta = \partial\bar\partial\gamma \Rightarrow [\partial\beta] = [0] \in H_{\bar\partial}^{p+1,q}$.
$d(\partial\beta) = 0$

$\Rightarrow d_1 = 0$, $E_2^{p,q} = H_{\bar\partial}^{p,q}$. Let $[\beta]_2 \in E_2^{p,q}$: $\bar\partial\beta = 0$, $\partial\beta = \bar\partial\alpha$. Want: $d_2 = 0$. Want: $[\partial\alpha] = [0] \in H_{\bar\partial}^{p,q}$.

Since $d(\partial\alpha) = \bar\partial\partial\alpha = -\partial\partial\beta = 0$,
can apply $\partial\bar\partial$-lemma to $\partial\alpha$ $\rightsquigarrow$ $\partial\alpha = \partial\bar\partial\gamma \Rightarrow [\partial\alpha] = [0]$.

... Continue on similarly to show all $d_K = 0$.

**Cor:** $X$ compact Kähler. Then: $H_{dR}^{K}(X, \mathbb{C}) \cong \bigoplus_{p+q=K} H_{\bar\partial}^{p,q}(X, \mathbb{C})$.

## 2. Cech-de Rham Theorem

Let $K^{p,q} = C^p(\underline{\mathcal{U}}, \Omega^q) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(\mathcal{U}_{\alpha_0 \cdots \alpha_p})$

where $M = \bigcup \mathcal{U}_\alpha$ and $\underline{\mathcal{U}} = \{\mathcal{U}_\alpha\}$ is a good cover.

First page: $E_1 = H_d$

$$\begin{array}{l|lll}
q \uparrow\ d\uparrow & & \vdots & \\
 & \prod H^2(\mathcal{U}_\alpha) & \prod H^2(\mathcal{U}_{\alpha\beta}) & \cdots \\
 & \prod H^1(\mathcal{U}_\alpha) & \prod H^1(\mathcal{U}_{\alpha\beta}) & \\
 & \prod H^0(\mathcal{U}_\alpha) & \prod H^0(\mathcal{U}_{\alpha\beta}) & \\
\hline
 & \delta \to & & \to p
\end{array}
\qquad
\begin{array}{l|lll}
q \uparrow\ d\uparrow & & & E_1 \\
 & 0 & 0 & 0 \\
 & 0 & 0 & 0 \\
 & 0 & 0 & 0 \\
 & C^0(\mathcal{U}, \mathbb{R}) & C^1(\mathcal{U}, \mathbb{R}) & C^2(\mathcal{U}, \mathbb{R}) \\
\hline
 & \delta \to & & \to p
\end{array}$$

Good cover $\Rightarrow H^q(\mathcal{U}_\alpha) = 0 \quad \forall q \geq 1$.

Second Page: $E_2 = H_\delta H_d$

$$
\begin{array}{c|ccc}
q & & & \\
 & 0 & 0 & 0 \\
 & 0 & 0 & 0 \\
 & H^0(\underline{\mathcal{U}}, \mathbb{R}) & H^1(\underline{\mathcal{U}}, \mathbb{R}) & H^2(\underline{\mathcal{U}}, \mathbb{R}) \\
\hline
 & & & \to p
\end{array}
$$

$$
\begin{tikzcd}
0 \arrow[drr, "d_2"] & 0 & 0 \\
0 & 0 & 0 \\
H^0(\underline{\mathcal{U}}, \mathbb{R}) & H^1(\underline{\mathcal{U}}, \mathbb{R}) & H^2(\underline{\mathcal{U}}, \mathbb{R})
\end{tikzcd}
$$

$d_2 = 0$ for degree reasons.

Also: $d_2 = d_3 = d_4 = \cdots = 0$ spectral seq degenerates at page 2.

$$\therefore H_D^k(C^*(\mathcal{U}, \Omega^*)) = \bigoplus_{p+q=k} E_2^{p,q} = H^k(\underline{\mathcal{U}}, \mathbb{R}).$$

Redo: switch role of $d$ and $\delta$: $E_1' = H_\delta$ Alternate spectral seq assoc $K^{p,q}$

$E_2' = H_d H_\delta$

$\vdots$

First Page: $E' = H_\delta$

$$
\begin{array}{c|cc}
q & & \\
 & H_\delta^0(\underline{\mathcal{U}}, \Omega^1) & H_\delta^1(\underline{\mathcal{U}}, \Omega^1) \\
 & H_\delta^0(\underline{\mathcal{U}}, \Omega^0) & H_\delta^1(\underline{\mathcal{U}}, \Omega^0) \\
\hline
 & & \to p
\end{array}
$$

$b \in C^p(\underline{\mathcal{U}}, \Omega^q)$

$[b] \in H_\delta^p(\underline{\mathcal{U}}, \Omega^q)$

$\delta b = 0$

1. If $b \in H_\delta^0(\underline{\mathcal{U}}, \Omega^q)$, $b_\alpha \in \Omega^q(\mathcal{U}_\alpha)$

   $b_\alpha = b_\beta$ $(\delta b = 0)$

   $\Rightarrow b \in \Omega^q(M)$

$$
\begin{array}{c|ccc}
q & & & \\
 & \Omega^2(M) & 0 & 0 \\
 & \Omega^1(M) & 0 & 0 \\
 & \Omega^0(M) & 0 & 0 \\
\hline
 & & & \to p
\end{array}
$$

2. If $b \in H_\delta^1(\underline{\mathcal{U}}, \Omega^q)$,

   $b_{\alpha\beta} \in \Omega^q(\mathcal{U}_{\alpha\beta})$,

claim: $b = \delta\gamma$, $\gamma \in C^0(\underline{\mathcal{U}}, \Omega^q)$.

(use partition of unity)

Check: $b = \delta\gamma$, $\gamma_\alpha = \sum_\mu \rho_\mu b_{\mu\alpha}$, where $\rho_\mu$ partition of unity wrt $\underline{\mathcal{U}} = \{\mathcal{U}_\mu\}$

$$(\delta\gamma)_{\alpha\beta} = \gamma_\beta - \gamma_\alpha$$
$$= \sum \rho_\mu (b_{\mu\beta} - b_{\mu\alpha})$$

Know $\delta b = 0$ $\quad \alpha\mu\beta$

$b_{\mu\beta} - b_{\alpha\beta} + b_{\alpha\mu} = 0$

$$\Rightarrow (\delta\gamma)_{\alpha\beta} = \sum_\mu \rho_\mu b_{\alpha\beta} = b_{\alpha\beta}.$$

Similarly: $H^p_\delta(\underline{\mathcal{U}}, \Omega^q) = 0 \quad \forall p \geq 1.$

$$E'_2 = H_d H_\delta$$

$$\begin{array}{c|ccc}
q & & & \\
 & H^2_d(M) & 0 & 0 \\
 & H^1_d(M) & 0 & 0 \\
 & H^0_d(M) & 0 & 0 \\
\hline
 & & & & p
\end{array}$$

$d_2$

$d'_2 : E'^{\,p,q}_2 \to E'^{\,p-1,q+2}_2$

$d'_2 = 0.$

Also: $d'_3 = d'_4 = \cdots = 0$

$\therefore$ Spectral seq deg at $E'_2$.

$$H_D^k(C^*(\underline{\mathcal{U}}, \Omega^*)) = \bigoplus_{p+q=k} E'^{\,p,q}_2 = H^k_d(M)$$

Combine both results for $E$ and $E'$:

$$H^k(\underline{\mathcal{U}}, \mathbb{R}) = H^k_{dR}(M).$$

de Rham coho $\cong$ Čech coho of good cover

<u>Remark</u>: Same argument can be used to show: Dolbeault isomorphism.

$$H^k(\underline{\mathcal{U}}, \Omega^p_X) \cong H^{p,q}_{\bar\partial}$$ on a cplx mfd $X$,

where: $\Omega^p_X =$ sheaf of hol'c $\Omega^{p,0}$ - forms

$\alpha \in \Omega^p_X \Rightarrow \alpha = \alpha_{i_1 \cdots i_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, $\bar\partial\alpha = 0$

$$H^{p,q}_{\bar\partial} = \frac{\operatorname{Ker}\bar\partial \cap \Omega^{p,q}}{\operatorname{im}\bar\partial}, \qquad \Omega^{p,0} \xrightarrow{\bar\partial} \Omega^{p,1} \xrightarrow{\bar\partial} \Omega^{p,2} \to \cdots$$

Dolbeault complex

**Proof of Main Thm:** Let $K^{p,q}$ be double complex s.t. $E_r^{p,q}$ degenerates at page $k$: $d_k = d_{k+1} = \cdots = 0$.

Then: $F^p H_D^{p+q} / F^{p+1} H_D^{p+q} \cong E_\infty^{p,q}$.

We only check the following simple case for concreteness:
Assume $d_2 = d_3 = d_4 = \cdots = 0$. Show $E_2^{1,1} \cong F^1 H_D^2 / F^2 H_D^2$.

where: $F^1 H^2 = K^{1,1} \oplus K^{2,0} \cap H_D^2$
$F^2 H^2 = K^{2,0} \cap H_D^2$
$E_2^{1,1} = H_{D'}^{1,1} H_{D''}$, $\quad D' = \delta \qquad D' : K^{p,q} \to K^{p+1,q}$
$D'' = (-1)^p d \qquad D'' : K^{p,q} \to K^{p,q+1}$
$D = D' + D'' \qquad D : C^k \to C^{k+1}$.

<u>Recall</u>: $[\omega]_2 \in E_2^{p,q}$ means: $\omega \in K^{p,q}$ and can find $\gamma \in K^{p+1,q-1}$ s.t.
$d_1 [\omega]_1 = [0]_1$
$$\begin{cases} D''\omega = 0 \\ D'\omega = -D''\gamma \end{cases}$$

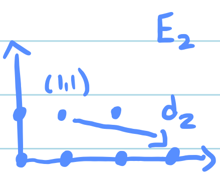
(*) earlier

$d_2 [\omega]_2 = [0]_2$ means: $\omega \in K^{p,q}$ and can find $\gamma_1 \in K^{p+1,q-1}$, $\gamma_2 \in K^{p+2,q-2}$ s.t.
$[\omega]_3 \in E_3^{p,q}$
$$\begin{cases} D''\omega = 0 \\ D'\omega = -D''\gamma_1 \\ D'\gamma_1 = -D''\gamma_2 . \end{cases}$$

$[\omega]_2 = [0]_2$ means: $\omega \in K^{p,q}$ and can find $\eta \in K^{p-1,q}$, $\alpha \in K^{p,q-1}$ s.t.
$\omega = D'\eta + D''\alpha$, with $D''\eta = 0$.

In general: $d_2 = 0$ does not imply $d_3 = d_4 = \cdots = 0$.

For $[\omega] \in E_2^{1,1}$, $d_3 = d_4 = \cdots = 0$ automatically for degree reasons. So only $d_2 = 0$ has new content.




Let $\omega_{11} \in K^{1,1}$ s.t. $[\omega_{11}]_2 \in E_2^{1,1}$ and $d_2 [\omega_{11}]_2 = [0]_2$.

Then: $D''\omega_{11} = 0$
$D'\omega_{11} = -D''\phi_{20}$, $\quad \phi_{20} \in K^{2,0}$
$D'\phi_{20} = 0$ cannot be $D''$(something) by type considerations.

Map: $E_2^{1,1} \to F^1H^2 / F^2H^2$
$[\omega_{11}] \mapsto [\omega_{11} + \phi_{20}]$. $\qquad$ Want: this is an isomorphism.

<u>Step 0</u>: well-defined.
$$D(\omega_{11} + \phi_{20}) = D'\omega_{11} + \underbrace{D'\phi_{20}}_{=0} + D''\phi_{20} = 0 \Rightarrow \omega_{11} + \phi_{20} \in F^1 H^2.$$

Indep of choice of $\Phi_{20}$ : suppose $D'\omega_{11} = -D''\Phi_{20}$
$D'\omega_{11} = -D''\tilde{\Phi}_{20}$

then $[\omega_{11} + \Phi_{20}] = [\omega_{11} + \tilde{\Phi}_{20}]$ in $F'H^2/F'H'$ because :

$D''(\Phi_{20} - \tilde{\Phi}_{20}) = 0$

$D'(\Phi_{20} - \tilde{\Phi}_{20}) = 0$ by (*) $\Rightarrow D(\Phi_{20} - \tilde{\Phi}_{20}) = 0$

$[\Phi_{20} - \tilde{\Phi}_{20}] \in F^2H^2$.

$$\therefore \underbrace{[\omega_{11} + \Phi_{20}]}_{\in F'H^2} = \underbrace{[\omega_{11} + \tilde{\Phi}_{20}]}_{\in F'H^2} + \underbrace{[\Phi_{20} - \tilde{\Phi}_{20}]}_{\in F^2H^2}$$

<u>Step 1</u>: injective. Let $[\omega_{11}]_2 \in E_2^{1,1}$.

If $\underbrace{[\omega_{11} + \Phi_{20}]}_{\in F'H^2/F^2H^2} = 0 \Rightarrow \omega_{11} + \Phi_{20} = D(\alpha_{10} + \alpha_{01}) + \eta_{20}$

$= D'\alpha_{10} + D''\alpha_{10} + D'\alpha_{01} + D''\alpha_{01} + \eta_{20}$

By type consideration : $\omega_{11} = D''\alpha_{10} + D'\alpha_{01}$, $D''\alpha_{01} = 0$.

$\therefore [\omega_{11}]_1 = [D'\alpha_{01}]_1 \in E_1^{1,1}$

$\therefore [\omega_{11}]_2 = [0]_2 \in E_2^{1,1}$.

$[D'\alpha_{01}] = d_1 [\alpha_{01}]$, $[\alpha_{01}] \in H_{D''}$

<u>Step 2</u>: surjective. Let $\alpha \in F'H^2/F^2H^2$.

$\alpha = \alpha_{11} + \alpha_{20}$ with $D\alpha = 0$, so: $\underbrace{D'\alpha_{20}}_{=0} + \underbrace{(D'\alpha_{11} + D''\alpha_{20})}_{=0} + \underbrace{D''\alpha_{11}}_{=0} = 0$

$\Rightarrow [\alpha_{11}] \in E_2^{1,1}$ since $D'\alpha_{11} = -D''\alpha_{20}$
$D''\alpha_{11} = 0$.

$\underbrace{[\alpha_{11}]}_{\in E_2^{11}} \mapsto \underbrace{[\alpha_{11} + \alpha_{20}]}_{\in F'H^2}$.

$\Rightarrow E_2^{1,1} \cong F'H^2/F^2H^2$.

$\square$