Spectral Sequence of a fiber bundle

- Let $\pi : E \to M$ be a fiber bundle with fiber F.
- $M = \bigcup \mathcal{U}_\alpha$, $\underline{\mathcal{U}} = \{\mathcal{U}_\alpha\}$ good cover
- Double complex: $K^{p,q} = C^p(\pi^{-1}\underline{\mathcal{U}}, \Omega^q)$, $\quad d : K^{p,q} \to K^{p,q+1}$, $\delta : K^{p,q} \to K^{p+1,q}$

  Recall: $C^p(\pi^{-1}\underline{\mathcal{U}}, \Omega^q) = \prod \Omega^q(\pi^{-1}(\mathcal{U}_{\alpha_1 \cdots \alpha_p}))$.

- Spectral Sequence: $E_1^{p,q} = H_d^{p,q}$
  $E_2^{p,q} = H_\delta^{p,q} H_d$

$$E_1^{p,q} = \prod H^q(\pi^{-1}(\mathcal{U}_{\alpha_1 \cdots \alpha_p})) = C^p(\underline{\mathcal{U}}, R^q \pi_* \mathbb{R})$$

$$E_2^{p,q} = H_\delta^p(\underline{\mathcal{U}}, R^q \pi_* \mathbb{R})$$

where: $R^q\pi_* \mathbb{R}$ is the presheaf $\mathcal{U} \mapsto H^q(\pi^{-1}(\mathcal{U}), \mathbb{R})$.

- Total cohomology: $D = \delta + d$, $\quad D : K^i \to K^{i+1}$, $\quad K^i = \bigoplus_{p+q=i} K^{p,q}$,

$$H_D^i(K^\bullet) \cong H_{dR}^i(E).$$

Indeed: $\{\pi^{-1}\underline{\mathcal{U}}\}$ is open cover of E, and argument from last time (using partition of unity, not good cover) shows:

$$\underset{\substack{\| \\ H_D^i(K^\bullet)}}{H_D^i(C^*(\pi^{-1}\underline{\mathcal{U}}, \Omega^*))} \cong \bigoplus_{p+q=K} E_2'^{\,p,q} = H_d^K(E).$$

alternate spectral seq

- If spectral seq degenerates at E_r $(d_r = d_{r+1} = \cdots = 0)$,

  then: $H_D^i(K^\bullet) \cong \bigoplus_{p+q=i} E_r^{p,q}$.

$$\Rightarrow H_{dR}^i(E) \cong \bigoplus_{p+q=i} E_r^{p,q}.$$

<u>NB</u>: This spectral sequence also has a product structure.

$M = \bigcup \mathcal{U}_\alpha$ open cover

$$\underset{\omega}{\cup : C^p(\underline{\mathcal{U}}, \Omega^q)} \otimes \underset{\eta}{C^r(\underline{\mathcal{U}}, \Omega^s)} \to \underset{\omega \cup \eta}{C^{p+r}(\underline{\mathcal{U}}, \Omega^{q+s})}$$

$$(\omega \cup \eta)_{\alpha_0 \cdots \alpha_{p+r}} = (-1)^{qr} \, \omega_{\alpha_0 \cdots \alpha_p} \wedge \eta_{\alpha_p \cdots \alpha_{p+r}}, \quad \text{e.g. } \omega_{\alpha_0 \cdots \alpha_p} \in \Omega^q(\mathcal{U}_{\alpha_0 \cdots \alpha_p}).$$

From now on: just write $\omega\eta$ instead of $\omega \cup \eta$.

<u>Exercise</u>:
- $\delta(\omega\eta) = \delta\omega\,\eta + (-1)^{\deg\omega}\omega\,\delta\eta$
- $D''(\omega\eta) = D''\omega\,\eta + (-1)^{\deg\omega}\omega\, D''\eta$
- $D(\omega\eta) = D\omega\,\eta + (-1)^{\deg\omega}\omega\, D\eta$,

where $\deg\omega = p+q$ for $\omega \in C^p(\underline{\mathcal{U}}, \Omega^q)$,
$D = \delta + D''$, $D'' = (-1)^p d$.

<u>Thm</u>: E_r inherits a product structure, and $d_r : E_r^{p,q} \to E_r^{p+r,\, q-r+1}$ is an antiderivation.

$d_r(xy) = d_r(x)y + (-1)^{p+q} x\, d_r(y)$
$x \in E_r^{p,q}$

<u>Thm</u>: (Künneth Formula)
Let M, N be manifolds. Assume N has finite-dimensional cohomology. (e.g. N is compact)

$$\text{Then: } H^k(M \times N) \cong \bigoplus_{p+q=k} H^p(M) \otimes H^q(N).$$

<u>Pf</u>: $\pi : M \times N \to M$ product fiber bundle over M.
$M = \bigcup \mathcal{U}_\alpha$ good cover

$$K^{p,q} = \prod \Omega^q(\mathcal{U}_{\alpha_0 \cdots \alpha_p} \times N)$$

$$E_1^{p,q} = \prod H^q(\mathcal{U}_{\alpha_0 \cdots \alpha_p} \times N).$$

Since $H^q(\mathcal{U}_{\alpha_0 \cdots \alpha_p} \times N) \cong H^q(N)$, $\omega \in E_1^{p,q}$ is of the form $\omega = \{\omega_A\}$

$\omega_A = f_A{}^a [\eta_a]$, $f_A{}^a \in \mathbb{R}$, $A = \alpha_0 \cdots \alpha_p$, $[\eta_a]$ generators of $H^*(N)$.

Since $\delta\omega = \delta f^a [\eta_a]$, $d_1 = \delta$, $E_2 = \ker d_1 / \operatorname{im} d_1$, we see:

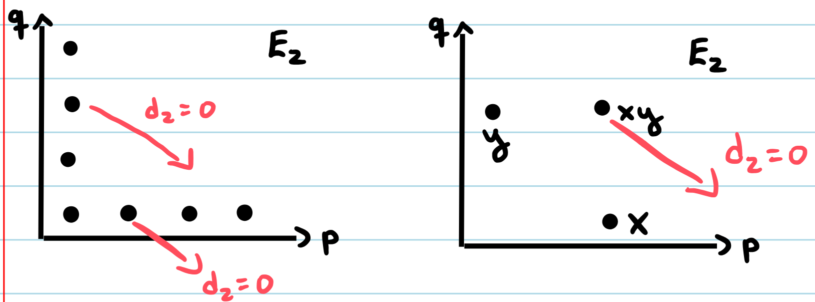
Čech-de Rham Thm (as algebras)

$$E_2^{p,q} = H^p(\underline{\mathcal{U}}, \mathbb{R}) \otimes H^q(N) \cong H^p(M) \otimes H^q(N).$$

<u>Claim</u>: $d_2 = d_3 = \cdots = 0$. $\therefore$ spectral seq deg at E_2, so

$$H_{dR}(M \times N) = H_D^k = \bigoplus_{p+q=k} E_2^{p,q} = \bigoplus_{p+q=k} H^p(M) \otimes H^q(N).$$

<u>Claim 0.5</u>: For $[\omega] \in E_2^{0,q}$, $d_k[\omega] = [0]$. Assuming this:



Given $x \in E_2^{p,0}$
$y \in E_2^{0,q}$

$$d_2(xy) = d_2 x\, y + (-1)^q x\, d_2 y$$
$$= 0 + 0 = 0.$$

Claim 0.5 $\Rightarrow$ claim since $H^p(M) \otimes H^q(N) = E_2^{p,q}$ as algebra.

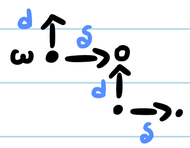
Proving claim 0.5: let $[\omega]_2 \in E_2^{0,q}$ be represented by $\omega \in K^{0,q}$ s.t. $\omega = \{\omega_\alpha\}$,

$\omega_\alpha \in \Omega^q(N)$, $\omega_\alpha = f_\alpha^{\,c}\, \eta_c$, where $[\eta_c]$ is a basis for $H^q(N)$.

$$(\delta\omega)_{\alpha\beta} = (f_\beta^{\,c} - f_\alpha^{\,c})\,\eta_c.$$

Since $[\delta\omega]_d = 0$, (defn E_2) $\Rightarrow [0] = (f_\alpha^{\,c} - f_\beta^{\,c})[\eta_c]$
$\Rightarrow f_\alpha^{\,c} = f_\beta^{\,c} \quad \forall c.$

$$\Rightarrow \delta\omega = 0 \Rightarrow \begin{cases} d_2[\omega]_2 = [0]_2 \\ d_3[\omega]_3 = [0]_3 \\ \vdots \\ d_k[\omega]_k = [0]_k \quad \forall k \geq 2. \end{cases}$$



$\square$

How does this generalize from products $E = M \times N$ to fiber bundles $\pi: E \to M$?

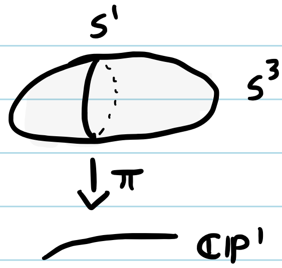
<u>Leray's Thm</u>:

Let $\pi: E \to M$ be a fiber bundle with fiber F.
Let $M = \bigcup \mathcal{U}_\alpha$ be a good cover.
Suppose M is simply-connected.

Then: $E_2^{p,q} = H^p(M) \otimes H^q(F)$ as a graded algebra.

<u>Pf</u>: Bott-Tu, Thm 14.18. $\square$

ex) $$\begin{array}{ccc} S^1 \to & S^3 & \\ & \downarrow & \\ & \mathbb{CP}^1 \cong S^2 & \end{array}$$ Hopf fibration.



$\pi : \underset{\mathbb{C}^2}{\overset{}{S^3}} \to \mathbb{CP}^1$, $\quad \pi(z_1, z_2) = [z_1, z_2]$. ($S^3 \subset \mathbb{C}^2$)

$$\pi^{-1}([z_1, z_2]) \cong S^1.$$

Since $\mathbb{CP}^1 \cong S^2$, Leray's Thm applies.

$\therefore E_2^{p,q} = H^p(S^2) \otimes H^q(S^1)$.

<u>But</u>: $H^k(S^3) \neq \bigoplus_{p+q=k} H^p(S^2) \otimes H^q(S^1)$

Must go deeper.

$$E_2^{p,q}: \quad \begin{array}{c|cccc} q & & & & \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & \mathbb{R} & 0 & \mathbb{R} & 0 \\ & \mathbb{R} & 0 & \mathbb{R} & 0 \\ \hline & & & & \to p \end{array}$$

Note: $d_3 : E_3^{p,q} \to E_3^{p+3,q-2}$ moves down 2 steps, so $d_3 = 0$.
Also $d_4 = d_5 = \cdots = 0$.

$\Rightarrow$ Spectral seq degenerates at E_3.

$\Rightarrow H^k(S^3) = \bigoplus_{p+q} E_3^{p,q}$.

$$E_3^{p,q}: \quad \begin{array}{c|cccc} q & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \mathbb{R} & 0 \\ 0 & \mathbb{R} & 0 & 0 & 0 \\ \hline & 0 & 1 & 2 & \to p \end{array}$$

$H^3(S^3) = h^{1,2}$

$$E_2^{p,q}: \quad \begin{array}{c|cccc} q & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & \mathbb{R} & 0 & \mathbb{R} & 0 \\ & \mathbb{R} & 0 & \mathbb{R} & 0 \\ \hline & & & & \to p \end{array}$$

d_2 isomorphism

Thm: Every simply connected manifold is orientable.

Pf: 1) Equip $TM \to M$ with a metric g.

We can then arrange the transition functions s.t. $g_{\alpha\beta} \in O(n)$.

$\Rightarrow$ Obtain S^{n-1} fiber bundle with same $g_{\alpha\beta} \in O(n) \circlearrowright S^{n-1}$.

$$\begin{array}{ccc} S^{n-1} \to & S(TM), & M = \bigcup \mathcal{U}_\alpha, \\ & \downarrow \pi & \\ & M & \end{array}$$

$$\begin{aligned} \pi^{-1}(\mathcal{U}_\alpha) &\cong \mathcal{U}_\alpha \times S^{n-1} \ni (x,t) \\ \pi^{-1}(\mathcal{U}_\beta) &\cong \mathcal{U}_\beta \times S^{n-1} \ni (\tilde{x},\tilde{t}) \\ \tilde{x} &= \psi(x) \\ \tilde{t} &= g_{\alpha\beta}(x)\, t \end{aligned}$$

2) $E_2^{p,q} = H^p(M) \otimes H^q(S^{n-1})$ (By Leray's Thm: uses simply-connected)

σ here

$$\begin{array}{c|ccc} q & & & \\ n-1 & \mathbb{R} & 0 & * \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & * \\ 0 & \mathbb{R} & 0 & * \\ \hline & & & \to p \end{array} \qquad E_2$$

Find $\sigma \neq 0$, $\sigma \in E_1^{0,n-1}$, $d_1 \sigma = 0$

$\sigma \in C^0(\mathcal{U}, R^{n-1}\pi_* \mathbb{R})$, $d_1 = \delta$

$\sigma \in \dfrac{\operatorname{Ker} d_1}{\operatorname{im} d_1} \cap E_1^{0,n-1} \cong \mathbb{R}$

Obtain $[\sigma_\alpha] \in H^{n-1}(\pi^{-1}(\mathcal{U}_\alpha)) = H^{n-1}(\mathcal{U}_\alpha \times S^{n-1})$ s.t.

$[\sigma_\alpha] = [\sigma_\beta]$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \times S^{n-1}$.

$[\delta\sigma] = 0 \in H_d^{n-1}$.

3) Find new transition functions $\tilde{g}_{\alpha\beta}$ with $\det \tilde{g}_{\alpha\beta} = 1$.

Have $[\sigma_\alpha] \in H^{n-1}(\mathcal{U}_\alpha \times S^{n-1}) \cong H^{n-1}(S^{n-1}) = \operatorname{span}[\omega]$, where:

$$\omega = \sum_{i=1}^{n} (-1)^i x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \in \Omega^{n-1}(S^{n-1}),$$

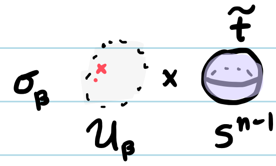
$$\int_{S^{n-1}} \omega = \int_{\partial B_1(0)} \omega = \int_{B_1(0)} d\omega = (n+1)\operatorname{Vol}(B_1). \Rightarrow [\omega] \neq 0.$$

$\therefore\ \sigma_\alpha = \underset{\substack{\cap \\ \mathbb{R}}}{\lambda_\alpha} \omega + d(\eta_\alpha)$ in (x,t) coords on $\mathcal{U}_\alpha \times S^{n-1}$.

Now on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \times S^{n-1}$,

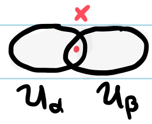
Need to write $\sigma_\alpha, \sigma_\beta$ in same (x,t) coords

$$\sigma_\alpha = \lambda_\alpha \sum (-1)^i t_i \, dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_n + d\eta_\alpha$$
$$\sigma_\beta = \lambda_\beta \sum (-1)^i \tilde{t}_i \, d\tilde{t}_1 \wedge \cdots \wedge \widehat{d\tilde{t}_i} \wedge \cdots \wedge d\tilde{t}_n + d\eta_\beta$$



where: $\tilde{t} = g_{\alpha\beta}(x) t$, $t \in S^{n-1}$, $g_{\alpha\beta} \in O(n)$.

Fix $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $\pi^{-1}(x) = S^{n-1}$, $g : S^{n-1} \to S^{n-1}$ induced by $g_{\alpha\beta}(x)$.



$$g^*\sigma_\beta \Big|_{\pi^{-1}(x)} = \lambda_\beta \, g^* \left(\sum (-1)^i \tilde{t}_i \, d\tilde{t}_1 \wedge \cdots \wedge \widehat{d\tilde{t}_i} \wedge \cdots \wedge d\tilde{t}_n \right) + d\, g^*\eta_\beta \Big|_{\pi^{-1}(x)}$$

$t \mapsto (x,t)$

$i : S^{n-1} \to \mathcal{U}_{\alpha\beta} \times S^{n-1}$

$[\sigma_\alpha] = [\sigma_\beta] \Rightarrow [i^*\sigma_\alpha] = [i^*\sigma_\beta] \Rightarrow \lambda_\alpha \omega = \lambda_\beta g^*\omega + d(\cdots)$ on S^{n-1}.

$$\Rightarrow \lambda_\alpha \int_{S^{n-1}} \omega = \lambda_\beta \int_{S^{n-1}} g^*\omega = \lambda_\beta \det g_{\alpha\beta} \int_{S^{n-1}} \omega, \quad \text{since}$$

$$\int_{\partial B_1} g^*\omega = \int_{B_1} d g^*\omega = \int_{B_1} g^* \left((n+1) dt^1 \wedge \cdots \wedge dt^n \right) = \det g_{\alpha\beta} \int_{B_1(0)} d\omega .$$

$$\therefore \lambda_\alpha = \underbrace{(\det g_{\alpha\beta})}_{\in \{\pm 1\}} \lambda_\beta$$

Note: $\lambda_\alpha \neq 0$ since $\sigma \neq 0$.

$\therefore \operatorname{sgn}(\lambda_\alpha) \det g_{\alpha\beta} \operatorname{sgn}(\lambda_\beta) = 1$.

$\therefore \tilde{g}_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1}$ are new transition functions for isomorphic bundle with $\det \tilde{g}_{\alpha\beta} = +1$, where $h_\alpha : \mathcal{U}_\alpha \to GL(n)$

$$h_\alpha = \begin{pmatrix} \operatorname{sgn}(\lambda_\alpha) & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}_{n \times n} .$$

$\square$

More on direct image sheaf $R^q(\pi_* \mathbb{R})$.

$\pi: E \to M$ fiber bundle with fiber F, for contractible $\mathcal{U} \subseteq M$

$$\mathcal{U} \mapsto H^q(\pi^{-1}(\mathcal{U}), \mathbb{R}) \cong H^q(F, \mathbb{R})$$

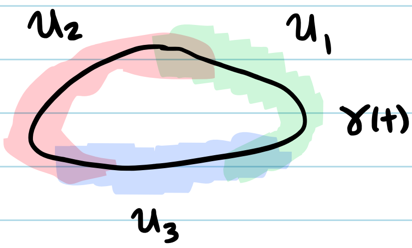
bundle over contractible $\mathcal{U}$ is trivial

$$\mathcal{U} \mapsto \mathbb{R}^{h^q(F)}.$$

<u>Def</u>: Sheaf $\mathcal{F}$ is a locally constant presheaf on a good cover $\underline{\mathcal{U}}$ if all $\mathcal{F}(\mathcal{U}_{\alpha_1 \cdots \alpha_p})$ are isomorphic and restriction maps are isomorphisms.

$\therefore$ $R^q(\pi_* \mathbb{R})$ is a locally constant presheaf over any good cover of $M = \bigcup \mathcal{U}_\alpha$.

Monodromy: take a loop γ on base M.
Cover $\gamma(t)$ by a chain of opens $\mathcal{U}_i$



Write $\mathcal{F} = R^q(\pi_* \mathbb{R})$

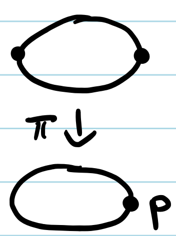
$$\mathcal{F}(\mathcal{U}_1) \cong \mathcal{F}(\mathcal{U}_1 \cap \mathcal{U}_2)$$
$$\cong \mathcal{F}(\mathcal{U}_2)$$
$$\vdots$$

and so on until return to $\mathcal{U}_1$.

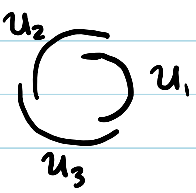
Obtain $\mathcal{F}(\mathcal{U}_1) \to \mathcal{F}(\mathcal{U}_1)$

$$\mathbb{R}^{h^q(F)} \xrightarrow{T_\gamma} \mathbb{R}^{h^q(F)}$$

ex) $\pi: S^1 \to S^1$ Fiber bundle with $F = \pi^{-1}(p) = \{2 \text{ pts}\}$
$z \mapsto z^2$



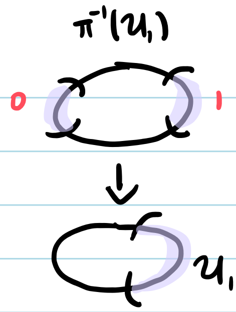
Good cover on base:

$S^1 =$ 

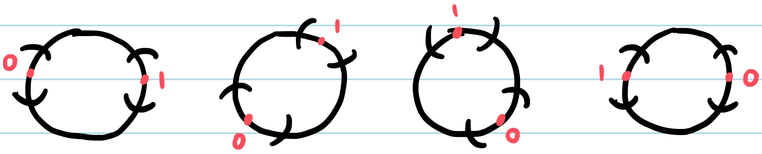
$$S^1 = \{e^{i\theta} : 0 \le \theta \le 2\pi\}$$

$$\pi^{-1}(e^{i\theta}) = \left\{ e^{i\theta/2}, e^{i(\frac{\theta}{2} + \pi)} \right\}$$

$\eta \in H^0(\pi^{-1}(\mathcal{U}_\alpha))$ is a function on 2 pts.



Consider e.g. $\eta \in H^0(\pi^{-1}(\mathcal{U}_1))$ with $\eta(e^0) = 1$
$\eta(e^{i\pi}) = 0$



$\eta \in H^0(\pi^{-1}(\mathcal{U}_1)) \to \eta \in H^0(\pi^{-1}(\mathcal{U}_2)) \to \eta \in H^0(\pi^{-1}(\mathcal{U}_3)) \to \eta \in H^0(\pi^{-1}(\mathcal{U}_1))$

$\eta = (1,0) \overset{T}{\longmapsto} (0,1)$.

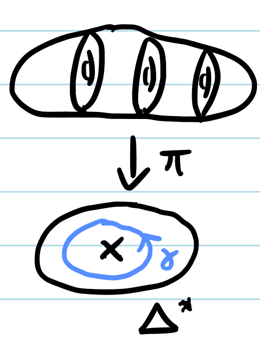
$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$, $\eta = (a,b)$ means: $\eta(e^0) = a$
$\eta(e^{i\pi}) = b$.

ex) $\pi : \mathcal{X} \to \Delta^*$, $(z,t) \mapsto t$, $\mathcal{X} = (\mathbb{C} \times \Delta^*)/\sim$, $(z,t) \sim (z+1, t)$
$(z,t) \sim (z + \frac{1}{2\pi i}\log t, t)$

$\pi^{-1}(t) = \mathbb{C}/\Lambda_t$, Λ_t = lattice spanned by $\langle 1, \frac{1}{2\pi i}\log t \rangle$.

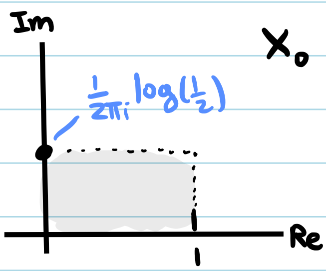
Note: Λ_t indep of choice of branch of log.

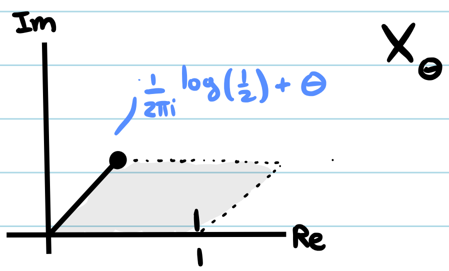
Take loop $\gamma(\theta) = \frac{1}{2} e^{2\pi i \theta}$, $0 \le \theta \le 1$.



Denote: $X_\theta = \pi^{-1}(\gamma(\theta)) = \mathbb{C}/\Lambda_\theta$

$$\Lambda_\theta = \langle 1, \frac{1}{2\pi i}\log(\tfrac{1}{2}) + \theta \rangle$$



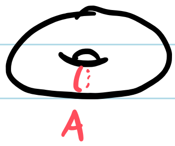


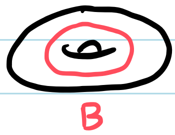
Take $\omega \in H^1(X_0)$. Go around $\gamma(t)$. Monodromy of $R^1(\pi_* \mathbb{R})$?

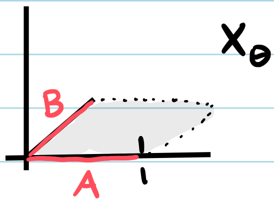
$H^1(X_0) \cong H^1(\pi^{-1}(\mathcal{U}))$ contractible $\mathcal{U}$, $\gamma(0) \in \mathcal{U}$

$$\omega \mapsto a_\theta \eta_{A_\theta} + b_\theta \eta_{B_\theta}, \qquad a_\theta = \int_{B_\theta} \omega, \quad b_\theta = \int_{A_\theta} \omega$$

where: η_A Poincaré dual to A
η_B Poincaré dual to B

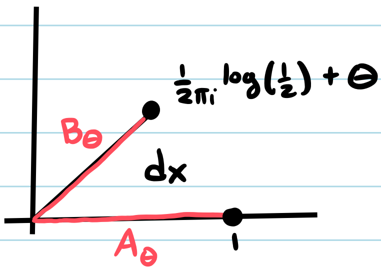






Take e.g. $\omega = [dx] \in H^1(X_0)$, $(a(\theta), b(\theta)) = \left(\int_{B_\theta} \omega, \int_{A_\theta} \omega \right)$

$(a(0), b(0)) = (0,1)$
$(a(\theta), b(\theta)) = (\theta, 1)$



Let $\theta \to 1$

$(0,1) \mapsto (1,1)$.

Take $[\omega] = \frac{2\pi}{\log 2} [dy] \in H^1(X_0)$

$(a(0), b(0)) = (1,0)$
$(a(\theta), b(\theta)) = (1,0)$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix}$$

Thm: Let $\pi : E \to M$ be a fiber bundle.
Let M be simply-connected.
Then $R^q(\pi_* \mathbb{R})$ is the constant sheaf. $\Rightarrow$ Monodromy always the identity map

Pf: Bott-Tu [Thm 13.2] + [Thm 13.4]. $\square$