Top Forms and Integration

Def: A smooth mfd of dim $= n$ is oriented if

$$M = \bigcup_\alpha \mathcal{U}_\alpha \quad \text{s.t.}$$

change of coords $\tilde{x}^i = f^i(x)$ on overlaps $\mathcal{U} \cap \tilde{\mathcal{U}}$ satisfy:

$$\det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) > 0.$$

nxn matrix

ex) Check $S^n = \mathcal{U} \cup \tilde{\mathcal{U}}$ (Stereographic proj) is oriented.

Relevance to top forms: Let $\mu \in \Omega^n(M)$, $\dim M = n$.

On $(\mathcal{U}, x)$: $\mu \overset{\text{loc}}{=} f(x)\, dx^1 \wedge \cdots \wedge dx^n$

On $(\tilde{\mathcal{U}}, \tilde{x})$: $\mu \overset{\text{loc}}{=} \tilde{f}(\tilde{x})\, d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$

$$\Rightarrow \tilde{f} = \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) f. \qquad (*)$$

check for $n=2$:
$$\mu = f(x)\, dx^1 \wedge dx^2$$
$$= \mu_{12}\, dx^1 \wedge dx^2$$

Know: $\tilde{\mu}_{12} = \frac{\partial x^p}{\partial \tilde{x}^1} \frac{\partial x^q}{\partial \tilde{x}^2} \mu_{pq}$

$$\Rightarrow \tilde{\mu}_{12} = \frac{\partial x^1}{\partial \tilde{x}^1} \frac{\partial x^2}{\partial \tilde{x}^2} \mu_{12} + \frac{\partial x^2}{\partial \tilde{x}^1} \frac{\partial x^1}{\partial \tilde{x}^2} \mu_{21}$$

$$\Rightarrow \tilde{\mu}_{12} = \det\begin{pmatrix} \frac{\partial x^1}{\partial \tilde{x}^1} & \frac{\partial x^1}{\partial \tilde{x}^2} \\ \frac{\partial x^2}{\partial \tilde{x}^1} & \frac{\partial x^2}{\partial \tilde{x}^2} \end{pmatrix} \mu_{12}.$$

Significance: Objects you can integrate = $\mu \in \Omega^n(M)$ over oriented mfd M with $\dim M = n$.

1) Let: $D \subseteq \mathcal{U} \subseteq M$, $(\mathcal{U}, x)$ coord chart, M oriented
: $\mu \in \Omega^n(M)$, cpt support in D.
: $\mu = f(x)\, dx^1 \wedge \cdots \wedge dx^n$

$$\int_D \mu := \int_{\mathbb{R}^n} f(x)\, dx^1 \cdots dx^n$$

Well-defined: if $D \subseteq \tilde{\mathcal{U}}$, $(\tilde{\mathcal{U}}, \tilde{x})$ another chart,
$\alpha = \tilde{f}(\tilde{x})\, d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$, then

$$\int_{\mathbb{R}^n} \tilde{f}(\tilde{x})\, d\tilde{x}^1 \cdots d\tilde{x}^n = \int_{\mathbb{R}^n} \tilde{f}(x) \det\left(\frac{\partial \tilde{x}}{\partial x}\right) dx^1 \cdots dx^n$$

change of var formula

$$= \int_{\mathbb{R}^n} f(x)\, dx^1 \cdots dx^n \quad ✓$$

(*)

2) Let: M oriented, $\dim M = n$
: $\mu \in \Omega^n(M)$ with cpt support

$$\int_M \mu := \sum_\alpha \int_M \rho_\alpha \mu$$

where $\{\rho_\alpha\}$ is a partion of unity subordinate to $\{\mathcal{U}_\alpha\}$:

a) $\rho_\alpha : \mathcal{U}_\alpha \to \mathbb{R}_{\geq 0}$, $\operatorname{supp}(\rho_\alpha) \subset\subset \mathcal{U}_\alpha$
b) $\sum_\alpha \rho_\alpha = 1$

Well-defined: if take $\{\tilde{\rho}_\alpha\}$ alternate partition of unity,

$$\sum_\alpha \int_M \tilde{\rho}_\alpha \mu = \sum_{\alpha,\beta} \int_M \tilde{\rho}_\alpha \rho_\beta \mu = \sum_\beta \int_M \rho_\beta \mu \quad ✓$$

ex) $\int_{S^1} d\theta = \int_{S^1\setminus\{pt\}} d\theta = \int_0^{2\pi} d\theta = 2\pi$

(Recall coords: θ on $\mathcal{U} = \{e^{i\theta} : 0 < \theta < 2\pi\}$
$\tilde{\theta}$ on $\tilde{\mathcal{U}} = \{e^{i\tilde{\theta}} : -\pi < \theta < \pi\}$
$\tilde{\theta} = \begin{cases} \theta \\ \theta - 2\pi \end{cases} \Rightarrow S^1 = \mathcal{U} \cup \tilde{\mathcal{U}}$ oriented)

ex) $\int_{T^2} d\theta^1 \wedge d\theta^2 = (2\pi)(2\pi)$, $T^2 = S^1 \times S^1$

ex) $\mathbb{CP}^1 = \mathcal{U}_0 \cup \mathcal{U}_1$, $[Z_0, Z_1] \in \mathbb{CP}^1$, over $\mathcal{U}_0$, $z = Z_1/Z_0$
over $\mathcal{U}_1$, $\tilde{z} = Z_0/Z_1$

$$\omega = \begin{cases} i\partial\bar{\partial} \log(1+|z|^2) & \text{over } \mathcal{U}_0 \\ i\partial\bar{\partial} \log(1+|\tilde{z}|^2) & \text{over } \mathcal{U}_1 \end{cases}$$

check: $\omega \in \Omega^{1,1}(\mathbb{P}^1)$ well-defined.

$$\int_{\mathbb{P}^1} \omega = ?$$

$$\int_{\mathbb{P}^1} \omega = \int_{\mathbb{P}^1 \setminus [0,1]} \omega = \int_{\mathcal{U}_0} i\partial\bar{\partial} \log(1+|z|^2)$$

$$= \int_{\mathbb{C}} i\partial\left(z(1+|z|^2)^{-1} d\bar{z} \right) = \int_{\mathbb{C}} \frac{1}{(1+|z|^2)^2}\, i\, dz \wedge d\bar{z}$$

$$= \int_0^{2\pi}\int_0^{\infty} \frac{1}{(1+r^2)^2}\, 2r\, dr \wedge d\theta$$

$z = re^{i\theta}$
$dz = e^{i\theta}dr + ire^{i\theta}d\theta$
$d\bar{z} = e^{-i\theta}dr - ire^{-i\theta}d\theta$

$= 2\pi$.

Stokes's Theorem: M oriented, $\dim M = n$

$$\int_M d\omega = 0 \qquad \forall\, \omega \in \Omega_c^{n-1}(M).$$

Note: $\Omega_c^k(M) =$ k-forms with compact support

Note: M compact oriented $\Rightarrow \int_M d\omega = 0 \quad \forall \omega \in \Omega^{n-1}(M)$.

Note: There is a notion of mfd with boundary. In that case,

$$\int_M d\omega = \int_{\partial M} \omega \qquad \forall\, \omega \in \Omega_c^{n-1}(M).$$

Proof of Stokes's (no boundary)

$$\int_M d\omega = \int_M d\Big(\big(\sum_\alpha \rho_\alpha\big)\omega\Big)$$

ρ_α partion of unity subordinate to oriented coord cover $M = \bigcup_\alpha \mathcal{U}_\alpha$.

$$= \sum_\alpha \int_M d(\underbrace{\rho_\alpha \omega}_{\text{supp} \subseteq \mathcal{U}_\alpha})$$

Will show: $\int_{\mathbb{R}^n} d\eta = 0 \quad \forall \eta \in \Omega_c^{n-1}(\mathbb{R}^n)$. e.g. $\eta = \rho_\alpha \omega$

Indeed: $$\int_{\mathbb{R}^n} d\eta = \int d\left(\frac{1}{(n-1)!}\eta_{i_1 \cdots i_{n-1}} dx^{i_1}\wedge\cdots\wedge dx^{i_{n-1}}\right)$$

$$= \int \left(\partial_1 \eta_{23\cdots n} - \partial_2 \eta_{13\cdots n} + \partial_3 \eta_{124\cdots n} - \cdots\right) dx^1\wedge\cdots\wedge dx^n$$

$$= 0 - 0 + 0 - \cdots \quad \text{since e.g.}$$

$$\int\!\!\int\left(\int_{-\infty}^{\infty} \frac{\partial}{\partial x^1}\eta_{23\cdots n}\, dx^1\right) dx^2\wedge\cdots\wedge dx^n$$

$$= \int\!\!\int\left(\Big|_{-\infty}^{\infty} \eta_{23\cdots n}\right) dx^2\wedge\cdots\wedge dx^n = 0$$ since $\eta \equiv 0$ outside a compact set. $\square$

Change of Variables:

$f: M \to N$ diffeo

$\omega \in \Omega^n_c(N)$ top-form

$$\int_M f^*\omega = \int_{f^*(N)} f^*\omega = \pm \int_N \omega .$$

$\pm$ depending on whether f is orientation preserving.

Recall pull back:

$$\omega = \frac{1}{k!} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

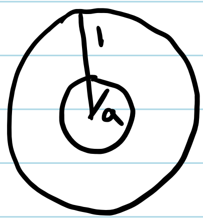
$$f^*\omega(y) = \frac{1}{k!} \omega_{i_1 \cdots i_k}(f(y)) \underbrace{d x^{i_1} \circ f}_{= df^{i_1} = \frac{\partial f^{i_1}}{\partial y^p} dy^p} \wedge \cdots \wedge dx^{i_k} \circ f$$

If $\varphi \in C^\infty(N)$, $f^*\varphi(y) = \varphi(f(y))$.

ex) $\partial B_1 = \{r = 1\} \subseteq \mathbb{R}^{n+1}$, $r = |x|$. ∂B_a = n-sphere radius $a > 0$.

$\omega_1 \in \Omega^n(\partial B_1)$ arbitrary.

Consider $S_a : \{r=1\} \to \{r = a\}$ orientation preserving
$x \mapsto ax$



$$\omega_a = a^n S_{a^{-1}}^* \omega_1 \in \Omega^n(\partial B_a).$$

If $\int_{\partial B_1} \omega_1 = M$, then $\int_{\partial B_a} \omega_a = ?$

$$\int_{\{r=a\}} \omega_a = \int_{S_{a^{-1}}^*\{r=1\}} a^n S_{a^{-1}}^* \omega_1 = a^n \int_{\{r=1\}} \omega_1 = a^n M.$$

$S_{a^{-1}}^*\{r=1\} = \{S_{a^{-1}}^* r = 1\} = \{a^{-1} r = 1\}$