de Rham cohomology

$$H^k(M) = \frac{\mathrm{Ker}\,\{d: \Omega^k(M) \to \Omega^{k+1}(M)\}}{\mathrm{Im}\,\{d: \Omega^{k-1}(M) \to \Omega^k(M)\}}$$

k^{th} de Rham cohomology group

ex) $H^0(M) = \{ f \in C^\infty(M) : df = 0 \}$

Exercise: Let M be a connected mfd. Then $H^0(M) = \{\text{constant functions}\} \cong \mathbb{R}$.

Let $p \in M$. Show $\{f(x) = f(p)\} \subseteq M$ is open and closed.

Main Properties: (will all be proved during this course)

A) If $M \simeq N$ homotopic, then $H^k(M) \cong H^k(N)$.

Cor: $H^k(\mathbb{R}^n) = 0 \quad \forall k \geq 1$.

B) Mayer-Vietoris: $M = U \cup V$ union open sets. $\exists$ exact seq

$$\cdots \to H^k(M) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to H^{k+1}(M) \to \cdots$$

C) Künneth formula:

$$H^k(M \times N) = \bigoplus_{p+q=k} H^p(M) \otimes H^q(N)$$

D) Poincaré duality: M compact, orientable, $\dim M = n$.

$$H^k(M) \cong \left(H^{n-k}(M)\right)^*$$

$$[\eta] \in H^k(M) \mapsto f_\eta([\omega]) = \int_M \omega \wedge \eta \quad \text{acting on } [\omega] \in H^{n-k}(M)$$

Cor: If M compact + orientable + connected, $\dim M = n$, then:

$$[\mu] \mapsto \int_M \mu \text{ is isomorphism: } H^n(M) \cong \mathbb{R}.$$

Homotopy:

Def: $\phi, \psi : M \to N$ are homotopic ($\phi \simeq \psi$) if $\exists$ smooth $F: M \times [0,1] \to N$ s.t.
$$F(x,0) = \phi$$
$$F(x,1) = \psi.$$

Def: M and N are homotopic ($M \simeq N$) if $\exists$ smooth maps $f: M \to N$, $g: N \to M$ s.t. $g \circ f \simeq id$, $f \circ g \simeq id$.

ex) $\mathbb{R}^n \simeq \{pt\}$. $f: \mathbb{R}^n \to \{0\}$
$g: \{0\} \hookrightarrow \mathbb{R}^n$

$g \circ f \simeq id$ via $F(x,t) = tx$

Deformation retract: Suppose $i: A \hookrightarrow M$ inclusion and $r: M \to A$ a map s.t. $r \circ i = id_A$. If $\exists F: M \times [0,1] \to M$ s.t. $F(x,0) = x$, $F(x,1) = i \circ r(x)$, then A is a deformation retract of M.

ex) $\mathbb{R}^2 \setminus \{0\} \simeq S^1$ via $r: \mathbb{R}^2 \setminus \{0\} \to S^1$, $r(x) = x/|x|$.
$F(x,t) = (1-t)x + t\, x/|x|$.

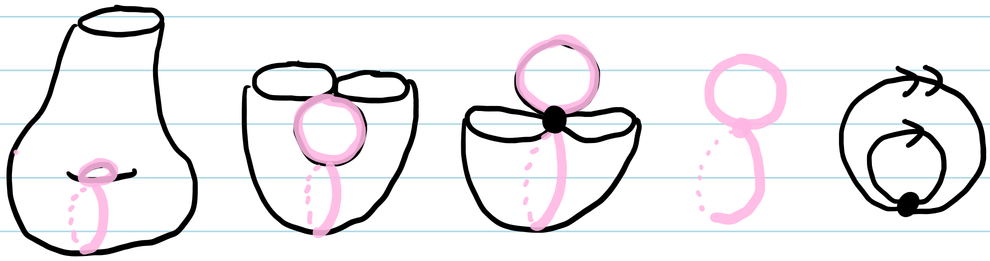
ex) $\mathbb{CP}^n \setminus \{[0,\dots,0,1]\} \simeq \mathbb{CP}^{n-1}$ via $F([z_0,\dots,z_n],t) = [z_0,\dots,z_{n-1},(1-t)z_n]$.
$\mathbb{CP}^{n-1} = \{[z_0,\dots,z_{n-1},0] \in \mathbb{CP}^n\}$

ex) $E \to M$ vector bundle. Then $E \simeq M$ retracts to zero section.

ex) 

$\mathbb{R}^2 \setminus \{0\}$, cylinder, Möbius strip all retract to S^1.

ex)



retract to bouquet of two circles

Prop: Let $\omega \in \Omega^k(M \times I)$ be s.t. $d\omega = 0$.
Then $[i_1^*\omega] = [i_0^*\omega] \in H^k(M)$, where: $i_t : M \to M \times I$
$p \mapsto (p,t)$.

ex) $\omega = (1+x)dt + t\,dx$, $\omega \in \Omega^1(M \times \mathbb{R})$, $M = \mathbb{R}$.
$d\omega = dx \wedge dt + dt \wedge dx = 0$
$i_1^*\omega = dx$, $i_0^*\omega = 0$.
$i_1^*\omega - i_0^*\omega = d$ (function on M)

Proof: Let $\Theta_t : M \times I \to M \times I$ flow of $\frac{\partial}{\partial t}$.
$\Theta_t(p,s) = (p, s+t)$

$$i_1^*\omega - i_0^*\omega = \int_0^1 \frac{d}{dt} i_t^*\omega \, dt$$

Note: $$\frac{d}{dt} i_t^*\omega = \frac{d}{dt} i_0^* \Theta_t^* \omega = i_0^* \frac{d}{dt} \Theta_t^*\omega$$
$$= \underbrace{i_0^* \Theta_t^*}_{i_t^*} L_{\frac{\partial}{\partial t}} \omega$$ Lie derivative

$$\Rightarrow i_1^*\omega - i_0^*\omega = \int_0^1 i_t^* L_{\partial/\partial t} \omega \, dt$$
$$= \int_0^1 i_t^* \left(d \tfrac{\partial}{\partial t} \lrcorner \omega + \tfrac{\partial}{\partial t} \lrcorner \underbrace{d\omega}_{=0} \right) dt$$ Cartan's formula
$$= d \int_0^1 i_t^* \left(\tfrac{\partial}{\partial t} \lrcorner \omega \right) dt$$
$$= d\,(\text{something}) \quad \square$$

Review of Cartan's formula:

Lie Derivative

$L_V \omega = \left.\frac{d}{dt}\right|_{t=0} \Theta_t^* \omega$, Θ_t flow of V: $\left.\frac{d}{dt}\right|_{t=0} \Theta_t = V$.
$\Theta_t : M \to M$
$\Theta_0 = id_M$
$\Theta_t \circ \Theta_s = \Theta_{t+s}$

Cartan's Magic Formula

$$L_V \omega = V \lrcorner d\omega + d(V \lrcorner \omega)$$

where: $V \lrcorner : \Omega^k(M) \to \Omega^{k-1}(M)$

$$(V \lrcorner \gamma)_{i_1 \cdots i_{k-1}} = V^p \gamma_{p i_1 \cdots i_{k-1}}$$

$$(V \lrcorner \gamma)(W_1, \dots, W_{k-1}) = \gamma(V, W_1, \dots, W_{k-1})$$

Proof of Cartan's Formula: Let's only prove it for 1-forms.

Let $\omega \in \Omega^1(M)$.

$$L_V \omega = \frac{d}{dt}\bigg|_{t=0} \theta_t^* \omega$$

$$= \frac{d}{dt}\bigg|_{t=0} \left(\frac{\partial \theta_t^\alpha}{\partial x^i} \omega_\alpha(\theta_t(x)) \, dx^i \right)$$

$$= (\partial_i V^\alpha \omega_\alpha + \delta^\alpha{}_i \, \partial_p \omega_\alpha V^p) \, dx^i$$

$$= (\partial_i V^\alpha \omega_\alpha + V^p \partial_p \omega_i) \, dx^i$$

$$V \lrcorner d\omega = V^p (d\omega)_{pi} \, dx^i$$
$$= V^p (\partial_p \omega_i - \partial_i \omega_p) \, dx^i$$

$$d(V \lrcorner \omega) = d(V^p \omega_p)$$
$$= \partial_i (V^p \omega_p) \, dx^i$$
$$= (\partial_i V^p \omega_p + V^p \partial_i \omega_p) \, dx^i$$

$$\Rightarrow V \lrcorner d\omega + d(V \lrcorner \omega) = (V^p \partial_p \omega_i + \partial_i V^p \omega_p) \, dx^i \quad \checkmark$$

cancellation

$\square$

uses prev prop: $[i_1^*\Phi] = [i_0^*\Phi]$

Thm: Let $F: M \to N$, $G: M \to N$ be homotopic smooth maps. Let $\omega \in \Omega^k(N)$ with $d\omega = 0$.

Then: $[F^*\omega] = [G^*\omega]$.

Pf: Let $H: M \times I \to M$ be a homotopy from F to G.

$$[F^*\omega] = [(H \circ i_0)^*\omega] = [i_0^* H^*\omega] = [i_1^* H^*\omega] = [(H \circ i_1)^*\omega] = [G^*\omega]. \quad \square$$

Note: Given $\varphi: M \to N$, can define

$$\varphi^*: H^k(N) \to H^k(M) \text{ by: } \varphi^*[\eta] = [\varphi^*\eta].$$

$d\varphi^*\eta = \varphi^* d\eta = 0$

Thm: $M \simeq N \Rightarrow H^k(M) \cong H^k(N)$.

Pf: Take $f: M \to N$, $g: N \to M$ with $g \circ f \simeq id$, $f \circ g \simeq id$.

$$\Rightarrow (f \circ g)^*[\eta] = [\eta] \in H^k(N)$$
$$(g \circ f)^*[\alpha] = [\alpha] \in H^k(M)$$

Note: $(g \circ f)^* = f^* g^*$

$$\Rightarrow g^* f^* [\eta] = [\eta]$$
$$f^* g^* [\alpha] = [\alpha]$$

inverses

$\Rightarrow f^*: H^k(N) \to H^k(M)$ is isomorphism. $\square$

Cor: $H^k(\mathbb{R}^n) = 0 \quad \forall k \geq 1$.

Cor: If $\omega \in \Omega^k(\mathbb{R}^n)$, $k \geq 1$, and $d\omega = 0$, then $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}(\mathbb{R}^n)$.

Cor: Let M be a smooth manifold. $\forall p \in M$, $\exists$ nbhd $\mathcal{U}$ on which every closed form is exact.

ex) $H^k(T^2 \setminus \{p\}) = H^k\Big($  $\Big) \cong H^k\Big($  $\Big)$

but not really because  is not a mfd.

Cohomology with compact support

$\Omega_c^k(M) = C_0^\infty(M) \otimes \Omega^k(M)$,

$C_0^\infty(M) = \{ f \in C^\infty(M) : \exists K \subseteq M \text{ compact s.t. } f|_{M \setminus K} \equiv 0 \}$.

$$H_c^k(M) = \frac{\operatorname{Ker}\{ d: \Omega_c^k(M) \to \Omega_c^{k+1}(M) \}}{\operatorname{Im}\{ d: \Omega_c^{k-1}(M) \to \Omega_c^k(M) \}}$$

NB: $H_c^k(M)$ rather different than $H^k(M)$ for non-compact M.
(They are the same for M compact)

ex) $H_c^0(\{pt\}) = \mathbb{R}$

ex) $H_c^0(\mathbb{R}^n) = 0$ $\quad$ $f \equiv$ const + f cpt supp $\Rightarrow f \equiv 0$.

Thm: $H_c^{k+1}(M \times \mathbb{R}) \underset{e_*}{\overset{\pi_*}{\rightleftarrows}} H_c^k(M)$ isomorphic.

Here: $\pi_* : \Omega_c^{k+1}(M \times \mathbb{R}) \to \Omega_c^k(M)$ integration along the fiber

$$\pi_*\left(u(x,t)\pi^*\varphi_{k+1} + f(x,t)\, \pi^*\varphi_k \wedge dt \right) = \left(\int_{-\infty}^{\infty} f(x,t)\, dt \right) \varphi_k$$

where: $u, f \in C_0^\infty(M \times \mathbb{R})$
$\varphi_k \in \Omega^k(M)$
$\pi : M \times \mathbb{R} \to M$ projection

$e_* : \Omega_c^k(M) \to \Omega_c^{k+1}(M \times \mathbb{R})$
$e_*(\varphi) = \pi^*\varphi \wedge \rho(t)\, dt$ $\quad$ where $\rho(t)$ cpt support with $\int \rho\, dt = 1$.

Cor: $H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & \text{otherwise} \end{cases}$

e.g. $H_c^2(\mathbb{R}^2) = H_c^1(\mathbb{R}) = H_c^0(\{pt\}) = \mathbb{R}$
$H_c^1(\mathbb{R}^2) = H_c^0(\mathbb{R}) = 0$
$H_c^0(\mathbb{R}^2) = 0$.

Pf of Theorem: (outline)

1) Show $d\pi_* = \pi_* d$ so $\pi_* : H_c^{k+1}(M\times\mathbb{R}) \to H_c^k(M)$ well-defn.
(omitted)

2) Notice $e_* : H_c^k(M) \to H_c^{k+1}(M\times\mathbb{R})$ and
$\pi_* \circ e_* (\eta) = \eta \quad \forall \eta \in \Omega_c^k(M)$. (easy)

3) To finish construction of inverse, need:

$e_* \circ \pi_* (\eta) = \eta + d(\text{Something}) \quad \forall \eta \in \Omega_c^{k+1}(M\times\mathbb{R})$
with $d\eta = 0$.

Suppose for example $\eta = f(x,t)\,\pi^*\varphi \wedge dt$, (also need to check $\eta = u(x,t)\,\pi^*\varphi$)

where: $f \in C_0^\infty(M\times\mathbb{R})$, $\varphi \in \Omega^k(M)$, $d\eta = 0$.

Compare η with:
$$e_* \circ \pi_* (\eta) = \left(\int_{-\infty}^{\infty} f(x,\cdot)\right) \pi^*\varphi \wedge \rho(t)\,dt$$

Consider:

write: φ instead of $\pi^*\varphi$

$$d\left[\left(\int_{-\infty}^{t} f(x,u)\,du\right)\varphi - \left(\int_{-\infty}^{\infty} f(x,u)\,du\right)\left(\int_{-\infty}^{t}\rho(v)\,dv\right)\varphi\right]$$

$d(**)$ //

$$= f(x,t)\,dt\wedge\varphi + \int_{-\infty}^{t}\frac{\partial f}{\partial x^i}(x,u)\,du\,dx^i\wedge\varphi + \left(\int_{-\infty}^{t} f\right)d\varphi$$
$$-\left(\int_{-\infty}^{\infty}\frac{\partial f}{\partial x^i}(x,u)\,du\right)dx^i\wedge\left(\int_{-\infty}^{t}\rho\right)\varphi$$
$$-\left(\int_{-\infty}^{\infty} f(x,u)\,du\right)\rho(t)\,dt\wedge\varphi - \left(\int_{-\infty}^{\infty} f(x,u)\,du\right)\left(\int_{-\infty}^{t}\rho\right)d\varphi .$$

Use $d\eta = 0$:

$\Rightarrow 0 = d(f\,\varphi \wedge dt) \Rightarrow \frac{\partial f}{\partial x^i} dx^i \wedge \varphi \wedge dt \overset{(*)}{=} - f\, d\varphi \wedge dt$

(*1)
$\Rightarrow \int_{-\infty}^{t} \frac{\partial f}{\partial x^i}(x,u)\,du \; dx^i \wedge \varphi = - \int_{-\infty}^{t} f(x,u)\,du\, d\varphi$, also

(*2)
$\Rightarrow \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x^i}(x,u)\,du\right)\left(\int_{-\infty}^{t} \rho\right) dx^i \wedge \varphi = -\left(\int_{-\infty}^{\infty} f(x,u)\,du\right)\left(\int_{-\infty}^{t} \rho\right) d\varphi.$

Sub (*1), (*2) into $d(**)$

$\therefore d(**) = f(x,t)\,dt \wedge \varphi - \left(\int_{-\infty}^{\infty} f(x,u)\,du\right) \rho(t)\,dt \wedge \varphi.$

$\Rightarrow d(**) = (-1)^k (\eta - e_* \circ \pi_*(\eta))$

$\Rightarrow [e_* \circ \pi_*(\eta)] = [\eta].$

Exercise: Show $[e_* \circ \pi_*(\eta)] = [\eta]$ for $\eta = u(x,t)\,\pi^*\varphi$,
with $\varphi \in \Omega^{k+1}(M)$,
$u \in C_0^\infty(M \times \mathbb{R})$,
$d\eta = 0$.

$\square$