Mayer-Vietoris Sequence

Let $C = \bigoplus_{k\in\mathbb{Z}} C^k$ be a direct sum of vector spaces.

C is a differential complex if $\exists$ homomorphisms

$$\cdots \to C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \to \cdots \quad \text{s.t. } d^2 = 0.$$

Cohomology of C: $H^k(C) = \dfrac{\operatorname{Ker}\{d : C^k \to C^{k+1}\}}{\operatorname{Im}\{d : C^{k-1} \to C^k\}}$

Chain map: $f : A \to B$ between differential complexes s.t. $f\, d_A = d_B\, f$.

Exact sequence: Vector spaces V_k with homomorphisms

$$\cdots \to V_{k-1} \xrightarrow{f_{k-1}} V_k \xrightarrow{f_k} V_{k+1} \xrightarrow{f_{k+1}} \cdots \quad \text{s.t. } \operatorname{Ker} f_i = \operatorname{Im} f_{i-1} \ \forall i.$$

Snake Lemma: Given a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where: A, B, C differential complexes
f, g chain maps,

then $\exists$ long exact sequence in cohomology

$$\cdots \to H^k(A) \xrightarrow{f} H^k(B) \xrightarrow{g} H^k(C) \xrightarrow{\delta} H^{k+1}(A) \xrightarrow{f} H^{k+1}(B) \xrightarrow{g} H^{k+1}(C) \xrightarrow{\delta} \cdots$$

where $\delta[c] = [f^{-1} d\, g^{-1}(c)]$.

Proof: Let $c \in C^k$ with $dc = 0$. $\delta[c] = [a]$, where:

$$\begin{array}{ccccccccc}
 & & a & & db & & & & \\
0 & \to & A^{k+1} & \xrightarrow{f} & B^{k+1} & \xrightarrow{g} & C^{k+1} & \to & 0 \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \to & A^{k} & \xrightarrow{f} & B^{k} & \xrightarrow{g} & C^{k} & \to & 0 \\
 & & & & b & & c & &
\end{array}$$

$g(db) = d\, g(b) = dc = 0.$

$f\, da = d f a = dd b = 0$
$\Rightarrow da = 0.$

Must check: $[a] \in H^{k+1}(A)$ is indep of: $c \in [c]$

: choice of b

: choice of a

: long exact sequence is exact.

Omitted.

$\square$

Back to de Rham cohomology

Suppose $M = \mathcal{U} \cup V$ union of open sets. Consider:

$$0 \to \Omega^k(M) \to \Omega^k(\mathcal{U}) \oplus \Omega^k(V) \to \Omega^k(\mathcal{U} \cap V) \to 0$$

$$\omega \qquad\qquad \gamma \qquad\qquad -\omega + \gamma$$

1) All maps are chain maps
2) Sequence is exact.

e.g. if $\omega \in \Omega^k(\mathcal{U} \cap V)$, then $\omega = -(-\rho_V \omega) + \rho_{\mathcal{U}} \omega$

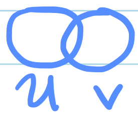
$\uparrow$

$(-\rho_V \omega, \rho_{\mathcal{U}} \omega) \in \Omega^k(\mathcal{U}) \oplus \Omega^k(V)$

where: $\rho_{\mathcal{U}} + \rho_V = 1$

$\operatorname{supp} \rho_{\mathcal{U}} \subseteq \mathcal{U}$, $\operatorname{supp} \rho_V \subseteq V$.

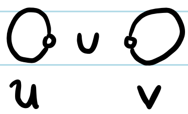
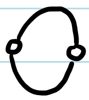
NB: $\rho_V \omega \in \Omega^k(\mathcal{U})$



By the Snake Lemma, $\exists$ long exact seq in cohomology:

$$\cdots \to H^k(M) \to H^k(\mathcal{U}) \oplus H^k(V) \to H^k(\mathcal{U} \cap V) \to H^{k+1}(M) \to \cdots$$

called the Mayer-Vietoris sequence.

ex) $S^1 =$  , $\mathcal{U} \cap V =$  $\simeq \{pt\} \sqcup \{pt\}$

$\mathcal{U} \simeq \{pt\}$, $V \simeq \{pt\}$

$$0 \to H^0(S^1) \to H^0(\mathcal{U}) \oplus H^0(V) \to H^0(\mathcal{U} \cap V) \to$$
$$\to H^1(S^1) \to H^1(\mathcal{U}) \oplus H^1(V) \to H^1(\mathcal{U} \cap V) \to 0$$

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}^2 \to H^1(S^1) \to 0$$

dimker = 0	dimker = 1	dimker = 1	dimker = 1
dimIm = 1	dimIm = 1	dimIm = 1	dimIm = 0

Recall: $T: V \to W$ linear map,

$\dim V = \dim \operatorname{Ker} T + \dim \operatorname{Im} T$.

$\Rightarrow \dim H^1(S^1) = 1.$

Consider $d\theta \in \Omega^1(S^1)$. Note $\int_{S^1} d\theta = 2\pi$.

$\Rightarrow [d\theta] \neq 0 \in H^1(S^1).$

(if $[\alpha]=0$, then $\int_{S^1} \alpha = \int_{S^1} df = 0$)

$\Rightarrow [d\theta]$ is a generator of $H^1(S^1)$.

Note: We will eventually prove Poincaré duality:

$$H^k(M) \cong (H^{n-k}(M))^* \quad \text{for } M \text{ compact oriented.}$$

$\Rightarrow \dim H^k(M) = \dim H^{n-k}(M)$

$\Rightarrow \dim H^n(M) = 1$, $\dim M = n$, M compact connected oriented.

We will use this freely in the following examples.

Def: $b_i = \dim H^i(M)$, i^{th} Betti number

Def: $\chi(M) = \sum_{i=0}^{n} (-1)^i b_i$.

ex) M^2 compact connected oriented surface.

$\Rightarrow \chi(M) = 1 - b_1 + 1 = 2 - \dim H^1(M^2)$

Prop: $M = \mathcal{U} \cup V$ union of open sets.

$\Rightarrow \chi(M) = \chi(\mathcal{U}) + \chi(V) - \chi(\mathcal{U} \cap V)$.

Pf: Given an exact seq of vector spaces

$$0 \to V_1 \to V_2 \to \cdots \to V_k \to 0,$$

then $0 = \sum_{i=1}^{k} (-1)^k \dim V_i$.

(use $\dim V = \dim \ker + \dim \mathrm{Im}$)

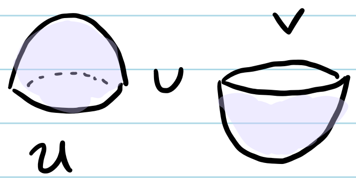
Apply this to Mayer-Vietoris:

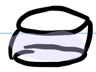
$$0 \to H^0(M) \to H^0(\mathcal{U}) \oplus H^0(V) \to H^0(\mathcal{U}\cap V) \to H^1(M) \to \cdots \to 0.$$

$$\Rightarrow 0 = \sum_{i=0}^{n} (-1)^i \dim H^i(M) - \sum (-1)^i \dim H^i(\mathcal{U}) + \sum (-1)^i \dim H^i(\mathcal{U}\cap V)$$
$$- \sum (-1)^i \dim H^i(V)$$

$$\Rightarrow 0 = \chi(M) - \chi(\mathcal{U}) - \chi(V) + \chi(\mathcal{U}\cap V).$$

$\square$

ex) $S^2 =$  ,

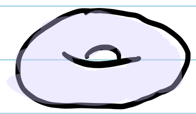
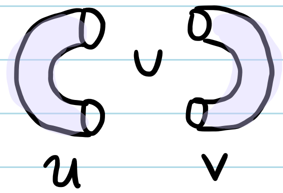
$\mathcal{U} \simeq \{pt\}$
$V \simeq \{pt\}$
$\mathcal{U}\cap V =$  $\simeq$  $= S^1$

$$\chi(S^2) = \chi(\mathcal{U}) + \chi(V) - \chi(\mathcal{U}\cap V)$$
$$= 1 + 1 - \underset{\chi(S^1) = b_0 - b_1}{\underset{\shortparallel}{0}}$$

$\Rightarrow 2 - \dim H^1 = 2$

$\Rightarrow \dim H^1(S^2) = 0.$

$$\Rightarrow \begin{cases} H^0(S^2) = \mathbb{R} \\ H^1(S^2) = 0 \\ H^2(S^2) = \mathbb{R} \end{cases}$$

ex) $T^2 =$  $=$  ,

$\mathcal{U} \simeq \bigcirc = S^1$
$V \simeq \bigcirc = S^1$
$\mathcal{U}\cap V \simeq \bigcirc \sqcup \bigcirc = S^1 \sqcup S^1$

$$\chi(T^2) = \chi(\mathcal{U}) + \chi(V) - \chi(\mathcal{U}\cap V)$$
$$= 0 + 0 - 0$$

$\Rightarrow 2 - b_1 = 0 \quad \Rightarrow b_1 = 2$

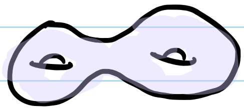
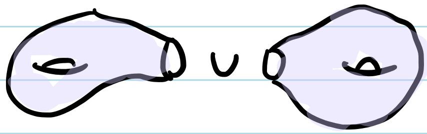
$$\Rightarrow \begin{cases} H^0(T^2) = \mathbb{R} \\ H^1(T^2) = \mathbb{R}^2 \\ H^2(T^2) = \mathbb{R} \end{cases}$$

Generators of $H^1(T^2)$? Write $T^2 = \underset{d\theta^1}{S^1} \times \underset{d\theta^2}{S^1}$

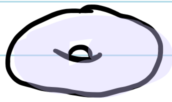
$H^1(T^2) = \text{span}\{[d\theta^1], [d\theta^2]\}$.

These cannot be linearly dependent: if $d\theta^1 = d\theta^2 + df$, $f \in C^\infty(T^2)$,

$$2\pi = \int_{S^1\times\{pt\}} d\theta^1 = \int_{S^1\times\{pt\}} d\theta^2 + \int_{S^1\times\{pt\}} df = 0 + 0. \qquad \Rightarrow\Leftarrow$$

ex) Σ_g, genus $g=2$ =  = 

$\mathcal{U}$ $\qquad$ V

 =  $\cup$ 

T^2 $\qquad$ $\mathcal{U}$ $\qquad$ Cap $\qquad$ Cap $\simeq \{pt\}$

$$\chi(T^2) = \chi(\mathcal{U}) + \chi(\{pt\}) - \chi(S^1)$$

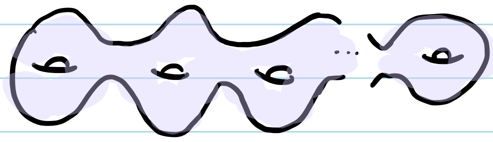
$$0 = \chi(\mathcal{U}) + 1 \quad \Rightarrow \chi(\mathcal{U}) = -1.$$

$$\chi(\Sigma_g) = \chi(\mathcal{U}) + \chi(V) - \chi(\mathcal{U} \cap V)$$

$$= -1 - 1 - 0 \qquad = \chi(S^1)$$

$\Rightarrow \dim H^1(\Sigma_g) = 4.$

General Pattern: $\chi(\Sigma_g) = 2 - 2g$, genus g surface.

 $= T^2 \# T^2 \# \cdots \# T^2$

Exercise: $H^k(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$

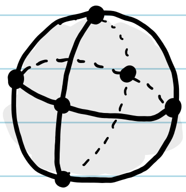
Exercise: $H^k(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & k = 0, 2, 4, \dots, 2n \quad \text{even} \\ 0 & \text{otherwise} \end{cases}$

One more remark on $\chi(M^2)$: Given a triangulation of a compact surface M^2, then:

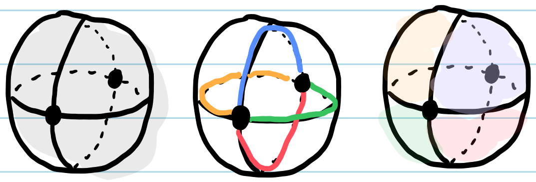
$$\chi(M^2) = V - E + F.$$

Triangulation of M^2: collection of polygons, each contained in a single coordinate chart, s.t.

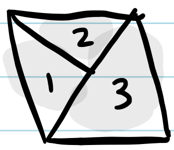
1) Every $p \in M$ inside at least 1 polygon
2) Two polygons are disjoint or their intersection is an edge or common vertex
3) Each edge is the edge of exactly 2 polygons.

ex)  use octants as triangles

$\chi(S^2) = 6 - 12 + 8 = 2$

ex)  use quarters as 2-gons

$\chi(S^2) = 2 - 4 + 4 = 2$

ex)  not valid

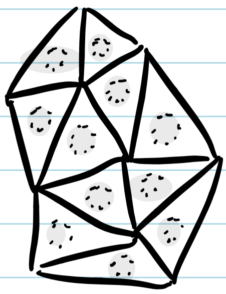
ex) use halves as 1-gons

$\chi = 1 - 1 + 2$

Proof of $\chi = V - E + F$:

Write $M = \mathcal{U} \cup M \setminus \{p_1, \dots, p_F\}$

where: take one point p_i in each polygon. (# polygons = F)
: $\mathcal{U}$ = disjoint union of balls centred at p_i.



$$\mathcal{U} \simeq \bigsqcup_F \{pt\}$$

$$\mathcal{U} \cap (M \setminus \{p_1, \dots, p_F\}) \simeq \bigsqcup_F S^1$$

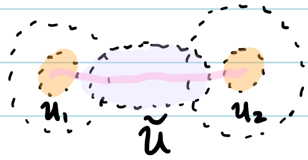
$$\chi(M) = \chi(\mathcal{U}) + \chi(M \setminus \{p_1, \dots, p_F\}) - \chi(\mathcal{U} \cap (M \setminus \{p_1, \dots, p_F\}))$$

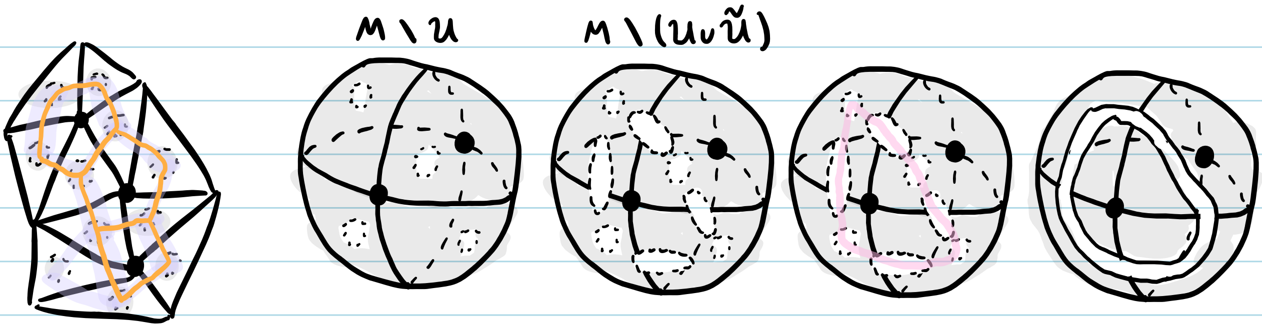
$$\chi(M) = F + \chi(M \setminus \{p_1, \dots, p_F\}). \qquad (*)$$

Next: $M \setminus \{p_1, \dots, p_F\} = \tilde{\mathcal{U}} \cup M \setminus \{\text{arcs}\}$

where: $\tilde{\mathcal{U}}$ = disjoint union of contractible opens, one for each edge, linking balls of old $\mathcal{U}$ with 2x radius.

arcs = paths in $2\mathcal{U} \cup \tilde{\mathcal{U}}$ joining the p_i.





$\tilde{U} = \bigsqcup_{E}$

$\tilde{U} \cap (M \setminus \{\text{arcs}\}) = \bigsqcup_{E}$

$M \setminus \{\text{arcs}\} \simeq \bigsqcup_{V}$

$\simeq \bigsqcup_{2E}$

$\Rightarrow \chi(M \setminus \{p_1, \dots, p_F\}) = E + V - 2E$

$\Rightarrow \chi(M) = F - E + V.$

(*) $\square$