# Poincaré Duality

**Def:** A good cover is a open cover of a mfd $M^n = \bigcup \mathcal{U}_\alpha$ s.t. all finite intersections $\mathcal{U}_{\alpha_1} \cap \cdots \cap \mathcal{U}_{\alpha_j}$ are diffeomorphic to $\mathbb{R}^n$.

**Fact:** All compact manifolds admit a finite good cover.

**Prop:** If $M$ admits a finite good cover, then $H^i(M)$ is finite dimensional $\forall i$.

**Pf:** If $M = \mathcal{U}_1 \cup \mathcal{U}_2$ with $\mathcal{U}_i \simeq \mathcal{U}_1 \cap \mathcal{U}_2 \simeq \{pt\}$, then by Mayer-Vietoris

$$\to H^{i-1}(\mathcal{U}_1 \cap \mathcal{U}_2) \xrightarrow{\delta} H^i(M) \xrightarrow{r} H^i(\mathcal{U}_1) \oplus H^i(\mathcal{U}_2) \xrightarrow{s} H^i(\mathcal{U}_1 \cap \mathcal{U}_2) \to$$

$\ker r = \operatorname{im} \delta < \infty$, $\operatorname{im} r = \ker s < \infty$

$H^i(M) = \ker r \oplus \operatorname{im} r$ finite dim

Next consider $M = (\mathcal{U}_1 \cup \mathcal{U}_2) \cup \mathcal{U}_3$ and continue by induction. $\square$

**Thm:** (Poincaré duality) Let $M^n$ be orientable with finite good cover. Then:

$$\underset{\substack{\cap \\ H^k(M)}}{[\eta]} \mapsto \underbrace{\int_M \eta \wedge \cdot}_{\in (H_c^{n-k}(M))^*}$$ is an isomorphism.

**Cor:** If $M^n$ compact orientable, then $H^i(M) \cong (H^{n-i}(M))^*$.
$\therefore \dim H^i(M) = \dim H^{n-i}(M)$.

**Cor:** If $M^n$ orientable with finite good cover, $H^i(M) \cong (H_c^{n-i}(M))^*$

**Cor:** If $M^n$ compact connected orientable, $H^n(M) \cong \mathbb{R}$ and

$$\begin{aligned} H^n(M) &\to \mathbb{R} \\ [\eta] &\mapsto \int_M \eta \end{aligned}$$

is an isomorphism. If $\eta \in \Omega^n(M)$ satisfies $d\eta = 0$ + $\int \eta = 0$, then $\exists \alpha \in \Omega^{n-1}(M)$ s.t. $\eta = d\alpha$.

**Pf of Poincaré duality:** To start, suppose $M = \mathcal{U} \cup V$, $\mathcal{U} \cong V \cong \mathcal{U} \cap V \cong \mathbb{R}^n$

(later: induct on good cover)

1) Mayer-Vietoris seq with cpt support:

verify this seq is exact

$$0 \leftarrow \Omega_c^*(M) \xleftarrow{s} \Omega_c^*(\mathcal{U}) \oplus \Omega_c^*(V) \xleftarrow{\iota} \Omega_c^*(\mathcal{U} \cap V) \leftarrow 0$$

$$\text{sum} \qquad (\omega|_{\mathcal{U}}, -\omega|_V) \leftarrow\!\shortmid \omega$$

2) Long exact seq in cohomology with cpt supp:

$$\leftarrow H_c^{k+1}(M) \xleftarrow{s} H_c^{k+1}(\mathcal{U}) \oplus H_c^{k+1}(V) \xleftarrow{\iota} H_c^{k+1}(\mathcal{U} \cap V) \xleftarrow{\delta} H_c^k(M) \xleftarrow{s} H_c^k(\mathcal{U}) \oplus H_c^k(V) \leftarrow$$

Exercise: verify $\delta[\eta] = [d(\rho_{\mathcal{U}}\eta)]$

Snake $\Rightarrow$ "$\delta = i^{-1} d s^{-1}$" $\Rightarrow \delta[\eta] = [d(\rho_{\mathcal{U}}\eta)]$ where $\rho_{\mathcal{U}}, \rho_V$ partion of unity subordinate to $\mathcal{U}, V$

3) Dual seq:

$$\to H_c^{k+1}(M)^* \to H_c^{k+1}(\mathcal{U})^* \oplus H_c^{k+1}(V)^* \to H_c^{k+1}(\mathcal{U} \cap V) \xrightarrow{\delta^*} H_c^k(M) \to$$

where $f: V \to W$, $f^*: W^* \to V^*$ with $f^*(w^*)(v) = w^*(f(v))$.

4) Regular Mayer-Vietoris:

Mayer-Vietoris

$$\to H^k(M) \xrightarrow{\text{restrict}} \underset{\omega}{H^k(\mathcal{U})} \oplus \underset{\tau}{H^k(V)} \xrightarrow{\text{difference}} \underset{-\omega+\tau}{H^k(\mathcal{U} \cap V)} \xrightarrow{\delta} H^{k+1}(M) \to$$

verify

Snake $\Rightarrow \delta[\eta] = [d(\rho_{\mathcal{U}}\eta)]$

$\eta \in H^k(\mathcal{U} \cap V)$

$\eta = -(\rho_V \eta) + (\rho_{\mathcal{U}}\eta)$

$(-\rho_V\eta, \rho_{\mathcal{U}}\eta) \in H^k(\mathcal{U}) \oplus H^k(V)$

5) Big square:

$$\begin{array}{ccccccccc}
\to & H^k(\mathcal{U} \cup V) & \xrightarrow{\text{restrict}} & H^k(\mathcal{U}) \oplus H^k(V) & \xrightarrow{\text{difference}} & H^k(\mathcal{U} \cap V) & \xrightarrow{\delta} & H^{k+1}(\mathcal{U} \cup V) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & H_c^{n-k}(\mathcal{U} \cup V)^* & \to & H_c^{n-k}(\mathcal{U})^* \oplus H_c^{n-k}(V)^* & \to & H_c^{n-k}(\mathcal{U} \cap V)^* & \xrightarrow{\delta^*} & H_c^{n-k-1}(\mathcal{U} \cup V)^* & \to
\end{array}$$

Vertical maps are e.g. $H^k(\mathcal{U}) \to H_c^{n-k}(\mathcal{U})^*$

$$[\eta] \mapsto \int_{\mathcal{U}} \eta \wedge \cdot$$

Exercise: check diagram commutes up to sign.

Let's only check the square:

$$\begin{array}{ccc} \overset{\omega}{H^k(\mathcal{U}\cap V)} & \xrightarrow{\delta} & H^{k+1}(\mathcal{U}\cup V) \\ \downarrow & & \downarrow \\ H_c^{n-k}(\mathcal{U}\cap V)^* & \xrightarrow{\delta^*} & H_c^{n-k-1}(\mathcal{U}\cup V)^* \end{array}$$

Let $[\omega] \in H^k(\mathcal{U}\cap V)$
$[\Upsilon] \in H_c^{n-k-1}(\mathcal{U}\cup V)$.

$\omega$ $\quad (\delta\omega)^*(\Upsilon) = \int_M \delta\omega \wedge \Upsilon = \int_M d(\rho_{\mathcal{U}}\omega)\wedge\Upsilon$

$\omega$ $\quad \delta^*\omega^*(\Upsilon) = \int_M \omega \wedge \delta\Upsilon = \int_M \omega\wedge d(\rho_{\mathcal{U}}\Upsilon)$

$$(\delta\omega)^*(\Upsilon) = \int_M d\rho_{\mathcal{U}}\wedge\omega\wedge\Upsilon \qquad d\omega = 0$$

$$= \int_M (-1)^k \omega\wedge d\rho_{\mathcal{U}}\wedge\Upsilon$$

$$= \int_M (-1)^k \omega\wedge d(\rho_{\mathcal{U}}\Upsilon) \qquad d\Upsilon = 0$$

$$= (-1)^k \delta^*\omega^*(\Upsilon) \checkmark$$

**6)** Five Lemma: Given a commutative diagram

$$\begin{array}{ccccccccc} V_1 & \xrightarrow{f_1} & V_2 & \xrightarrow{f_2} & V_3 & \xrightarrow{f_3} & V_4 & \xrightarrow{f_4} & V_5 \\ \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \downarrow\delta & & \downarrow\varepsilon \\ W_1 & \xrightarrow[\tilde{f}_1]{} & W_2 & \xrightarrow[\tilde{f}_2]{} & W_3 & \xrightarrow[\tilde{f}_3]{} & W_4 & \xrightarrow[\tilde{f}_4]{} & W_5 \end{array}$$

$V_i, W_i$ vector spaces

with: $V_1 \to V_2 \to V_3 \to V_4 \to V_5$ exact sequences
$W_1 \to W_2 \to W_3 \to W_4 \to W_5$

: $\alpha, \beta, \delta, \varepsilon$ isomorphisms.

**Then:** middle one $\gamma$ is an isomorphism. **Proof:** no

7) Apply the Five Lemma to the big square: Know

Show $H^0(\mathbb{R}^n) \to H^n_c(\mathbb{R}^n)^*$

$$H^k(\mathcal{U}) \to H^{n-k}_c(\mathcal{U})^*$$
$$H^k(V) \to H^{n-k}_c(V)^*$$
$$H^k(\mathcal{U}\cap V) \to H^{n-k}_c(\mathcal{U}\cap V)^*$$

all iso since $\mathcal{U} \cong V \cong \mathcal{U}\cap V \cong \mathbb{R}^n$

use direct calculation

$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k=0 \\ 0 \end{cases}$

$H^k_c(\mathbb{R}^n) = \begin{cases} 0 \\ \mathbb{R} & k=n \end{cases}$

$a \mapsto \int_{\mathbb{R}^n} a \cdot$

$a \in \mathbb{R}$, $\int_{\mathbb{R}^n} a \,\cdot \in (H^n_c)^*$

isomorphism

$\Rightarrow$ Poincaré duality applies to $\mathcal{U}\cup V = M$.

8) Consider $M = (\mathcal{U}_1 \cup \mathcal{U}_2)\cup \mathcal{U}_3$, repeat until reach good cover $M = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \dots \cup \mathcal{U}_N$. $\square$

Note: $(\mathcal{U}_1\cup\mathcal{U}_2)\cup\mathcal{U}_3 \cong \sqcup\, \mathbb{R}^n$

**Degree of a map**: $M, N$ oriented compact connected both of dimension $n$.

$f: M \to N$ smooth map.

Define: $\deg(f) = \int_M f^*\omega$, $\omega \in \Omega^n(N)$ s.t. $\int_N \omega = 1$.

Well-defined:

$\omega_1, \omega_2 \in \Omega^n(N)$ with $\int_N \omega_1 = \int_N \omega_2 \Rightarrow \int_N (\omega_1 - \omega_2) = 0$

$+ \ d(\omega_1-\omega_2) = 0$

Poincaré duality $H^n \cong \mathbb{R}$

$\Rightarrow \omega_1 - \omega_2 = d\alpha$

$\Rightarrow \int_M f^*\omega_1 - \int_M f^*\omega_2 = \int_M d f^*\alpha = 0.$

**Properties:**

a) $f, g: M \to N$ homotopic $\Rightarrow \deg(f) = \deg(g)$

b) $f: M\to N$ not surjective $\Rightarrow \deg(f) = 0$

c) If $f: M \to N$ surjective and $q \in N$ regular value, then

$$\deg(f) = \sum_{f^{-1}(q)} \pm 1.$$

**Def:** $f: M^m \to N^n$. Say $f(p)$ is a regular value if in local coords,

$$\left[\frac{\partial f^\mu}{\partial x^i}(p)\right]_{n\times m} \quad \text{is a surjective matrix } (\text{rank} = n).$$

**Fact:** Sard's Thm: $f: M^m \to N^n$. The set of critical values has measure zero.

critical values $= \{ f(p) : f(p) \text{ not regular value} \} \subseteq N$.

Proof of Sard: No

**Proof of properties of degree:**

a) If $f \simeq g$, then $f^*\eta = g^*\eta + d(\cdots) \quad \forall \eta \in \Omega^k(N),\ d\eta = 0$.

b) Let $q \in N$ not in $f(M)$.
$M$ cpt $\Rightarrow f(M)$ cpt $\Rightarrow \exists\, \mathcal{U} \subseteq N$ open nbhd of $q$ not hit by $f$.
Let $\eta \in \Omega^n_c(\mathcal{U}) \Rightarrow f^*\eta = 0 \Rightarrow \deg(f) = 0$.

$\eta_i(f(x))$

not in $\text{supp}(\eta) \subseteq \mathcal{U}$

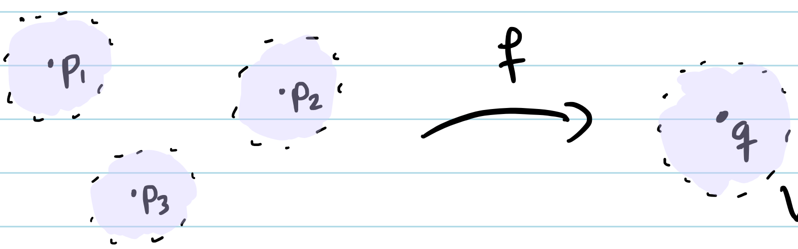
c) Recall: If $f: M \to N$ diffeo, then

$$\int_{f(M)} \omega = \pm \int_M f^*\omega. \quad (*)$$

Sard's Thm $\Rightarrow \exists q \in N$ regular value. Let $f^{-1}(q) = \{p_1, \dots, p_k\}$.

$$\Rightarrow \left[\frac{\partial f^\mu}{\partial x^i}(p_i)\right] \quad \text{invertible matrix.}$$

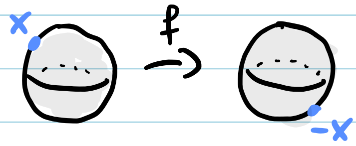
IFT $\Rightarrow \exists V$ nbhd of $q$ s.t. $f^{-1}(V) = \mathcal{U}_1 \cup \dots \cup \mathcal{U}_k$ with $\underset{p_i}{\overset{\Psi}{\mathcal{U}_i}}$ disjoint and $f: \mathcal{U}_i \to V$ diffeo.



Let $\eta \in \Omega^n_c(V)$ with $\int_N \eta = 1$. By $(*)$, $\int_{U_i} f^*\eta = \pm 1$.

$$\Rightarrow \int_M f^*\eta = \sum_{i=1}^{K} \int_{U_i} f^*\eta = \sum_{i=1}^{K} \pm 1. \qquad \square$$

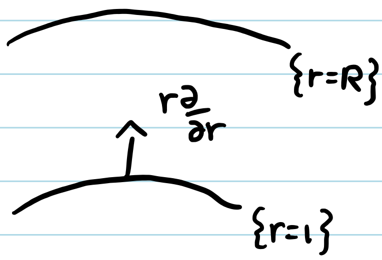
ex) $f: S^n \to S^n$
$\quad x \mapsto -x$ antipodal map




1. Need $\omega \in \Omega^n(S^n)$ generator of $H^n$.

Motivation from polar coords:
$S^n = \{r=1\} \subseteq \mathbb{R}^{n+1}$
$r^2 = |x|^2$

$(*)$ $dr = \frac{1}{r} \sum_{i=1}^{n+1} x_i\, dx_i \qquad 2r\,dr = 2 \sum x_i\, dx_i$




Polar coords formula: " $dx_1 \cdots dx_{n+1} = r^n\, dr\, d\vec{\theta}$ "
Differential forms: Want to find $\omega \in \Omega^n(S^n)$ s.t.
$$dx_1 \wedge \cdots \wedge dx_{n+1} = r^n\, dr \wedge p^*\omega \quad \text{on } \mathbb{R}^{n+1}, \text{ where}$$
$$p: \mathbb{R}^{n+1} \to S^n, \quad x \mapsto x/r$$

Solution turns out to be
$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

$$dr \wedge p^*\omega = dr \wedge \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i}{r}\, d(r^{-1}x_1) \wedge \cdots \wedge \widehat{d(r^{-1}x_i)} \wedge \cdots \wedge d(r^{-1}x_{n+1})$$
$$= r^{-n-1}\, dr \wedge \sum_{i=1}^{n+1} (-1)^{i-1} x_i\, dx_1 \wedge \cdots \wedge (\widehat{dx_i}) \wedge \cdots \wedge dx_{n+1}.$$

Stokes's Thm $\Rightarrow \int_{S^n} \omega = \int_{\partial B_1(0)} \omega = \int_{B_1(0)} d\omega = (n+1)\int_{B_1(0)} dx_1 \cdots dx_{n+1} = (n+1)\,\mathrm{Vol}(B_1)$

$\therefore [\omega] \neq 0$

2. Compute degree of $f$.

$f^*\omega = (-1)^{n+1}\omega \qquad x_i \leftrightarrow -x_i$

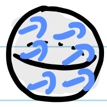
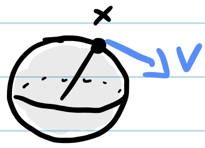
$$\Rightarrow \int f^*\omega = (-1)^{n+1} \int \omega$$

$$\Rightarrow \deg(f) = (-1)^{n+1}.$$

**Cor:** $S^n$ with $n$ even does not admit a nowhere vanishing vector field.

**Pf:** Let $V$ be a nowhere vanishing vector field.

$V: S^n \to \mathbb{R}^{n+1}$
$V(x) \cdot x = 0$

 $\subseteq \mathbb{R}^{n+1}$ 


Let $U(x) = \dfrac{V(x)}{|V(x)|}$, $|U(x)| = 1$, $|x| = 1$, $U(x) \cdot x = 0$.

$F(x,t) = \cos t \; x + \sin t \; U(x)$ satisfies $|F(x,t)|^2 = 1$, so $F: S^n \times [0,\pi] \to S^n$.

$\therefore$ $F$ homotopy from $x \mapsto x$ to $x \mapsto -x$.
$\Rightarrow\Leftarrow$ since $\deg(x \mapsto x) = 1$
$\deg(x \mapsto -x) = (-1)^{n+1} = -1$. $\square$

# Poincaré Dual of a Submanifold

- $M$ oriented mfd dim $n$
- $S \subseteq M$ oriented submfd dim $k$

$$\omega \mapsto \int_S \omega, \quad \omega \in \Omega_c^k(M)$$

defines a linear functional on $H_c^k(M)$. (Stokes's Thm)

Poincaré duality $\Rightarrow \exists! \, [\eta_S] \in H^{n-k}(M)$ s.t.

$$\int_M \omega \wedge \eta_S = \int_S \omega \quad \forall [\omega] \in H_c^k(M).$$

ex) $p \in M$, point in compact connected mfd.
$[\eta_p] \in H^n(M)$ is represented by any $\eta_p \in \Omega^n(M)$ with $\int \eta = 1$.
Indeed: if $[f] \in H^0(M) \Rightarrow f \equiv c$ const.

$$\int_M f \, \eta_p = c = \int_{\{p\}} f$$

ex) $p \in \mathbb{R}^n$.

$[\eta_p] \in H^n(\mathbb{R}^n)$ is represented by any top form $\eta$.

If $[f] \in H^0_c(\mathbb{R}^n) \Rightarrow f \equiv \text{const} + \text{cpt supp} \Rightarrow f \equiv 0$

$$\int_{\mathbb{R}^n} f \, \eta_p = 0 = \int_{\{p\}} f .$$

ex) All of $M$.

$[\eta_M] \in H^0(M)$, take $\eta_M \equiv 1$.

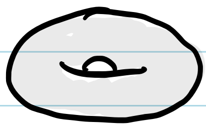
$$\int_M \omega \wedge \eta_M = \int_M \omega \qquad \forall \omega \in H^n_c(M)$$

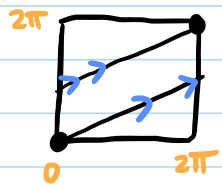
ex) $T^2 = S^1 \times S^1$, $\quad S^1 = \{ e^{i\theta} : 0 \le \theta \le 2\pi \}$

$L \subseteq T^2$, $\quad L = \text{Image}(\gamma)$

$\gamma : [0, 4\pi] \to T^2$

$\gamma(t) = (e^{it}, e^{it/2})$






<u>Know</u>: $H^1(T^2) = \text{span}\{ d\theta^1, d\theta^2 \}$

$[\eta_L] \in H^1(T^2) \Rightarrow \eta_L = A \, d\theta^1 + B \, d\theta^2$.

Test $\omega \in H^1(T^2)$, $\omega = a \, d\theta^1 + b \, d\theta^2$.

$$\int_L \omega = \int_{T^2} \omega \wedge \eta_L \iff \int_0^{4\pi} \gamma^*(a\,d\theta^1 + b\,d\theta^2) = \int_0^{2\pi}\int_0^{2\pi} (a\,d\theta^1 + b\,d\theta^2) \wedge (A\,d\theta^1 + B\,d\theta^2)$$

$$4\pi a + 2\pi b = (2\pi)^2 (aB - bA) \qquad \forall a, b$$

$$\Rightarrow B = \frac{1}{\pi}, \quad A = -\frac{1}{2\pi} \quad \Rightarrow \eta_L = \frac{1}{\pi} d\theta^1 - \frac{1}{2\pi} d\theta^2 .$$

ex) $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$ given by $S = \{ [z_0, z_1, 0] \in \mathbb{C}P^2 \}$.

$[\eta_S] \in H^2(\mathbb{C}P^2)$. Know: $H^2(\mathbb{C}P^2) \cong \mathbb{R}$, and $[\omega_{FS}] \in \Omega^2(\mathbb{C}P^2)$.

$$\omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log (1 + |z^1|^2 + |z^2|^2) \text{ over } \mathcal{U}_0 = \{ z_0 \neq 0 \}, \quad z^i = z_i / z_0 .$$

Since $\int_S \omega_{FS} = \int_{\mathbb{C}P^1} \frac{i}{2\pi} \partial\bar\partial \log(1+|z|^2) = 1$, cannot have $\omega_{FS} = d\alpha$.

$\therefore [\omega_{FS}]$ generates $H^2(\mathbb{C}P^2)$.

$\therefore [\eta_S] = \lambda[\omega_{FS}]$.

$$\int_{\mathbb{C}P^2} \omega_{FS} \wedge \eta_S = \int_S \omega_{FS} \Rightarrow \lambda \int_{\mathbb{C}P^2} \omega_{FS}^2 = 1.$$

Compute $\int_{\mathbb{C}P^n} \omega_{FS}^n$:

1) $\omega_{FS} = \frac{i}{2\pi} g_{j\bar{k}}\, dz^j \wedge d\bar{z}^k$, $\quad g_{j\bar{k}} = \dfrac{(1+|z|^2)\delta_{jk} - z^j \bar{z}^k}{(1+|z|^2)^2}$

2) $\dfrac{\omega_{FS}^n}{n!} = \dfrac{1}{(2\pi)^n} (\det g_{j\bar{k}})\, i\, dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge i\, dz^n \wedge d\bar{z}^n$

3) $\det g_{j\bar{k}} = (1+|z|^2)^{-(n+1)}$

4) $\frac{i}{2} dz \wedge d\bar{z} = dx \wedge dy$, $\quad z = x+iy$

5) $\int_{\mathbb{C}P^n} \omega_{FS}^n = \int_{\mathcal{U}_0} \omega_{FS}^n$ remove set measure zero

$$= \int_{\mathbb{C}^n} \frac{n!}{\pi^n} (1+r^2)^{-(n+1)} r^{2n-1}\, dr\, d\vec{\theta}$$

$$= \left(\frac{n!}{\pi^n}\right)\left(\frac{1}{2n}\right) \mathrm{Vol}(S^{2n-1}) = 1.$$

$$\mathrm{Vol}(S^{2n-1}) = \frac{2\pi^n}{(n-1)!}$$

$\therefore \lambda = 1$ and $[\eta_{\mathbb{C}P^1}] = [\omega_{FS}] \in H^2(\mathbb{C}P^2)$, $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$.