Vector Bundles

Def: $\pi : E \to M$ is a vector bundle of rank k if:

- $\pi : E \to M$ is a surjective map of manifolds with $\pi^{-1}(p) := E_p$ a vector space $\forall p \in M$.
- $\exists$ open cover

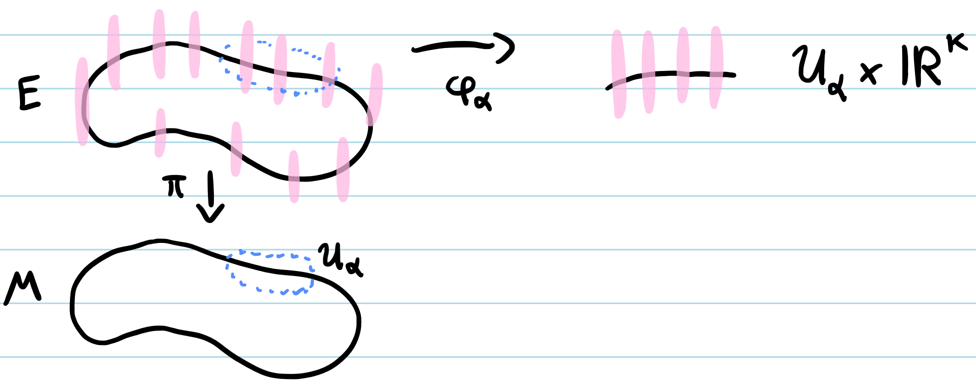
$$M = \bigcup_\alpha \mathcal{U}_\alpha, \quad E = \bigcup_\alpha \pi^{-1}(\mathcal{U}_\alpha)$$

with diffeos

$$\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^k$$

s.t. a) $\pi \circ \varphi_\alpha^{-1}(p, v) = p$

b) $\varphi_\alpha : E_p \to \{p\} \times \mathbb{R}^k$ isomorphism.



Transition functions:

Consider $\varphi_\alpha \circ \varphi_\beta^{-1} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{R}^k \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathbb{R}^k$

$$(x, v) \mapsto (x, g_{\alpha\beta}(x) v).$$

Call $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(k, \mathbb{R})$ transition functions.

Note: $g_{\alpha\beta} \, g_{\beta\gamma} = g_{\alpha\gamma}$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$.

$\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ \varphi_\gamma^{-1} = \varphi_\alpha \circ \varphi_\gamma^{-1}$

Sections: $s : M \to E$ is a section if $s(x) \in \pi^{-1}(x)$ $\forall x \in M$.

Write $s \in \Gamma(E)$.

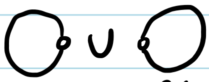
Locally via φ_α: $s\big|_{\mathcal{U}_\alpha} = (x, s_\alpha(x)) = \varphi_\alpha \circ s$

s.t. $s_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^k$

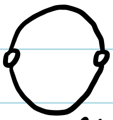
$s_\alpha = g_{\alpha\beta} \, s_\beta$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$.

$\varphi_\alpha \circ s = \varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ s$

ex) Möbius bundle $E \to S^1$ (rank 1)

$S^1 =$ 

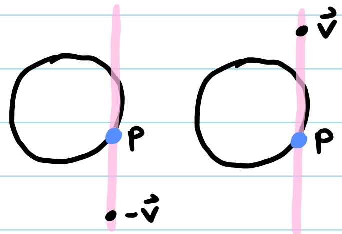
$\mathcal{U}$ $\tilde{\mathcal{U}}$



$\mathcal{U} \cap \tilde{\mathcal{U}} \xrightarrow{g_{12}} \begin{cases} +1 & \text{on } \cap \\ -1 & \text{on } \cup \end{cases}$ $\pm 1 \in GL(1)$

$E = (\mathcal{U} \times \mathbb{R}) \sqcup (\tilde{\mathcal{U}} \times \mathbb{R}) / \sim$, $\pi : E \to S^1$

$(p, v) \sim (p, \tilde{v}) \iff \tilde{v} = g_{12} v$ $(p,v) \mapsto p$



<u>Coords</u> on $E = \pi^{-1}(\mathcal{U}) \cup \pi^{-1}(\tilde{\mathcal{U}})$

- Recall $(\mathcal{U}, \theta)$, $(\tilde{\mathcal{U}}, \tilde{\theta})$ coords on S^1 with

$$\tilde{\theta} = \begin{cases} \theta & \text{on } \cap \\ \theta - 2\pi & \text{on } \cup \end{cases}$$

- Coords over $\pi^{-1}(\mathcal{U})$: (θ, v)
  $\pi^{-1}(\tilde{\mathcal{U}})$: $(\tilde{\theta}, \tilde{v})$

$$\begin{cases} \tilde{\theta} = \theta \\ \tilde{v} = v \end{cases} \quad \text{on } \cap$$

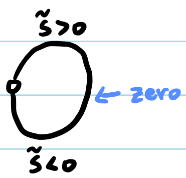
$$\begin{cases} \tilde{\theta} = \theta - 2\pi \\ \tilde{v} = -v \end{cases} \quad \text{on } \cup$$

<u>Note</u>: any section of Möbius bundle must pass through zero.

Let $s : S^1 \to E$, $s|_{\pi^{-1}(\mathcal{U})} = (\theta, s(\theta))$

$s|_{\pi^{-1}(\tilde{\mathcal{U}})} = (\tilde{\theta}, \tilde{s}(\tilde{\theta}))$

If e.g. $s(\theta) > 0 \Rightarrow \tilde{s} = g_{12} s \Rightarrow$ $\tilde{s} > 0$ on $\cap$
$\tilde{s} < 0$ on $\cup$



Coords on total space of a vector bundle:

- $M = \bigcup_\alpha \mathcal{U}_\alpha$ with $(\mathcal{U}_\alpha, x_\alpha)$ coord charts on M
- $E = \bigcup_\alpha \pi^{-1}(\mathcal{U}_\alpha)$ coord charts on E

  $(x^i_\alpha, v^i_\alpha) = \varphi_\alpha(e)$

- On overlaps $(\pi^{-1}(\mathcal{U}_\alpha), (x,v)), (\pi^{-1}(\mathcal{U}_\beta), (\tilde{x}, \tilde{v}))$ with:
  a) $\tilde{x} = f(x)$ change of coords on base M
  b) $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(K, \mathbb{R})$ transition function

The coord change on total space E is:

$$\begin{cases} \tilde{x} = f(x) \\ \tilde{v} = g_{\beta\alpha}(x)\, v \end{cases}$$

or in full: $\tilde{x}^i = f^i(x^1, \dots, x^n)$

$$\tilde{V}^\mu = g_{\beta\alpha}{}^\mu{}_\nu (x)\, V^\nu \quad (*)$$

ex) $TM \to M$ tangent bundle.

If $(\mathcal{U}_\alpha, x), (\mathcal{U}_\beta, \tilde{x})$ coord charts on M, then

$$g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(n, \mathbb{R})$$
$$g_{\beta\alpha} = \frac{\partial \tilde{x}}{\partial x}$$

So $(*)$ becomes: $\tilde{V}^\mu = \dfrac{\partial \tilde{x}^\mu}{\partial x^\nu} V^\nu$.

Sections of TM = Vector fields

$V \in \Gamma(TM)$

$V|_{\pi^{-1}(\mathcal{U})} = (x, V(x))$, $V = \begin{pmatrix} V^1 \\ \vdots \\ V^n \end{pmatrix} \in \mathbb{R}^n$ with $\tilde{V}^\mu = \dfrac{\partial \tilde{x}^\mu}{\partial x^\nu} V^\nu$.

$\Rightarrow$ $V^\mu \dfrac{\partial}{\partial x^\mu}$ well-defined action on $f \in C^\infty(M)$, since

$V^\mu \dfrac{\partial}{\partial x^\mu} f = \tilde{V}^\mu \dfrac{\partial}{\partial \tilde{x}^\mu} f$ by the chain rule.

From $\{\lambda_\alpha\}$ define: $\lambda \in \Gamma(\mathrm{Hom}(E,\check{E}))$

$\lambda|_{\pi^{-1}(\mathcal{U}_\alpha)} = (x, \lambda_\alpha(x))$

$\lambda_\alpha = \check{g}_{\alpha\beta} \lambda_\beta g_{\alpha\beta}^{-1}$

$\mathrm{Hom}(E,\check{E}) = E^* \otimes \check{E}$

$\lambda(p): E_p \to \check{E}_p$ invertible

Def: Let $M = \bigcup_\alpha \mathcal{U}_\alpha$, $E = \bigcup_\alpha \pi^{-1}(\mathcal{U}_\alpha)$, $\check{E} = \bigcup \check{\pi}^{-1}(\mathcal{U}_\alpha)$.

Bundles $E \xrightarrow{\pi} M$, $\check{E} \xrightarrow{\check{\pi}} M$ are isomorphic if $\exists\, \lambda_\alpha : \mathcal{U}_\alpha \to GL(k,\mathbb{R})$ smooth maps s.t.

$$\check{g}_{\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1} \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta.$$

Def: $E = \bigcup_\alpha \pi^{-1}(\mathcal{U}_\alpha)$ is oriented if $\det g_{\alpha\beta} > 0 \quad \forall\, g_{\alpha\beta}$.

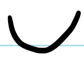
Def: E is orientable if it is isomorphic to an oriented bundle.

ex) Möbius bundle not orientable.

$\mathcal{U}$  $\tilde{\mathcal{U}}$

$\lambda(\theta)$ $\quad$ $\tilde{\lambda}(\tilde{\theta})$

$$\check{g}_{12} = \begin{cases} \lambda \cdot (+1) \cdot \tilde{\lambda}^{-1} & \text{on} \\ \lambda \cdot (-1) \cdot \tilde{\lambda}^{-1} & \text{on} \end{cases}$$



Never gonna make $\check{g}_{12}$ positive on all $\mathcal{U} \cap \tilde{\mathcal{U}}$.

Operations on Vector Bundles

$M = \bigcup \mathcal{U}_\alpha$, $E = \bigcup \pi^{-1}(\mathcal{U}_\alpha)$, $\check{E} = \bigcup \pi^{-1}(\mathcal{U}_\alpha)$. $\pi: E \to M$, $\check{\pi}: \check{E} \to M$ vector bundles.

1. **Direct Sum:** $E \oplus \check{E} \to M$

trivializations: $\varphi_\alpha \oplus \check{\varphi}_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times (\mathbb{R}^k \oplus \mathbb{R}^{\check{k}})$

rank $k + \check{k}$

transition functions: $\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \check{g}_{\alpha\beta} \end{pmatrix}$

2. **Tensor Product:** $E \otimes \check{E} \to M$

trivializations: $\varphi_\alpha \otimes \check{\varphi}_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times (\mathbb{R}^k \otimes \mathbb{R}^{\check{k}})$

rank $k\check{k}$

transition functions: $g_{\alpha\beta} \otimes \check{g}_{\alpha\beta}$

3. <u>Dual Bundle</u>: $E^* \to M$

trivializations: $(\varphi_\alpha^T)^{-1}$

transition functions: $(g_{\alpha\beta}^T)^{-1}$

Point of this: sections $s \in \Gamma(E)$, $\phi \in \Gamma(E^*)$ may be paired together.

$\phi(s) := \phi_\alpha^T s_\alpha$

$s|_{\mathcal{U}_\alpha} = (x, s_\alpha(x))$, $\quad s_\alpha = \begin{pmatrix} s_\alpha^1 \\ \vdots \\ s_\alpha^k \end{pmatrix}$, $\qquad s_\alpha = g_{\alpha\beta}\, s_\beta$

$\phi|_{\mathcal{U}_\alpha} = (x, \phi_\alpha(x))$, $\quad \phi_\alpha^T = ((\phi_\alpha)_1, \dots, (\phi_\alpha)_k)$, $\qquad \phi_\alpha = (g_{\alpha\beta}^T)^{-1} \phi_\beta$

Note: $\phi_\alpha^T s_\alpha = \left((g_{\alpha\beta}^T)^{-1}\phi_\beta\right)^T (g_{\alpha\beta}\, s_\beta) = \phi_\beta^T s_\beta$

$\Rightarrow \phi(s) \in C^\infty(M)$ well-defined scalar function.

4. <u>Homomorphism Bundle</u>: $\mathrm{Hom}(E, \check{E}) = E^* \otimes \check{E}$

5. <u>Pullback Bundle</u>: $f : N \to M$, $f^*E \to N$, $\quad N = \bigcup_\alpha f^{-1}(\mathcal{U}_\alpha)$

$f^*E = \{(n, e) : f(n) = \pi(e)\} \xrightarrow{p} N$

$$\begin{tikzcd} f^*E \arrow[d] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

trivializations: $f^*\varphi_\alpha : p^{-1}(f^{-1}(\mathcal{U}_\alpha)) \to f^{-1}(\mathcal{U}_\alpha) \times \mathbb{R}^k$

$f^*\varphi_\alpha(f^{-1}(x), e) = \varphi_\alpha(e)$

transition functions: $f^* g_{\alpha\beta} = g_{\alpha\beta} \circ f$

<u>Remark</u>: A complex vector bundle of rank k has trivialisations

$\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{C}^k$

and transition functions

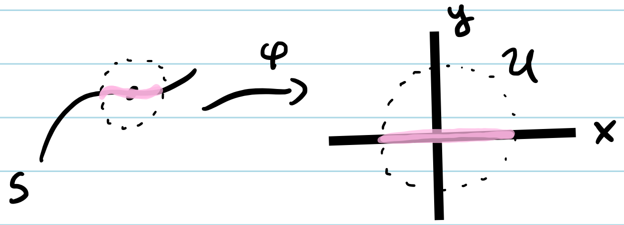
$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(k, \mathbb{C})$.

<u>Note</u>: If M cplx mfd + $g_{\alpha\beta}$ hol'c, say $E \to M$ hol'c bundle.

The normal bundle of a submanifold

Def: $S \subseteq M$ is a submfd if $\forall p \in S$, $\exists\, \mathcal{U} \subseteq M$ coord chart with coords (x^i, y^i) s.t.

$$S \cap \mathcal{U} = \{ (x^1, \dots, x^k, y^1, \dots, y^{n-k}) \in \mathcal{U} : y^i = 0 \ \forall i \}.$$



slice coords

In these coords, $T_p M = \mathrm{span} \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{n-k}} \right\}$

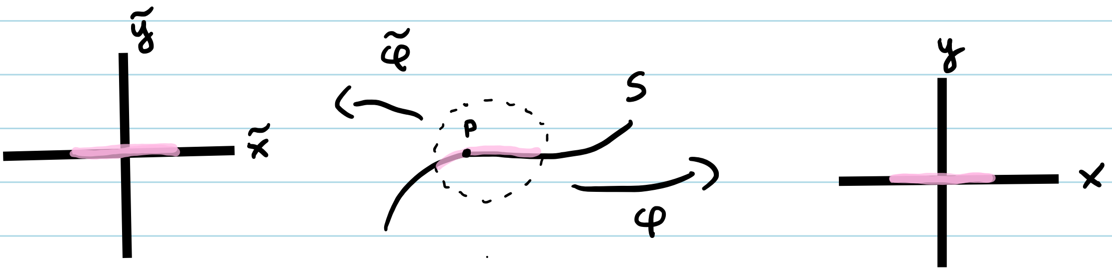
$$T_p S = \mathrm{span} \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k} \right\}.$$

$\exists$ exact sequence

$$0 \to TS \to TM|_S \to N \to 0, \qquad N = TM|_S / TS.$$

Transition functions of $TS \to S$ and $N \to S$:

On overlap $p \in \mathcal{U} \cap \tilde{\mathcal{U}}$ with slice coords (x,y), $(\tilde{x}, \tilde{y})$:



$$\begin{pmatrix} \tilde{x}(x,y) \\ \tilde{y}(x,y) \end{pmatrix} = \tilde{\varphi} \circ \varphi^{-1}(x,y).$$

If $p \in S$, then $\varphi(p) = (x(p), 0)$.
$\tilde{\varphi}(p) = (\tilde{x}(p), 0)$

$\Rightarrow \tilde{y}(x(p), 0) \equiv 0.$

Transition function of $TM|_S$ $= \begin{pmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{y}}{\partial x} \\ \frac{\partial \tilde{x}}{\partial y} & \frac{\partial \tilde{y}}{\partial y} \end{pmatrix}_{y=0} = \begin{pmatrix} \frac{\partial \tilde{x}}{\partial x} & 0 \\ \frac{\partial \tilde{x}}{\partial y} & \frac{\partial \tilde{y}}{\partial y} \end{pmatrix}_{y=0}$

Transition function of $TS \to S$: $\frac{\partial \tilde{x}}{\partial x}$ Transition function of $N \to S$: $\frac{\partial \tilde{y}}{\partial y}$

Cocycle cond: $TM|_S \to S$, $\check{g}_{\alpha\beta}$
$TS \to S$, $g_{\alpha\beta}$
$N \to S$, $h_{\alpha\beta}$ } trans fun on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \subseteq S$

$\check{g}_{\alpha\beta} = \check{g}_{\alpha\gamma}\check{g}_{\gamma\beta}$ Know TM satisfies cocycle cond.

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ * & h_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} g_{\alpha\gamma} & 0 \\ * & h_{\alpha\gamma} \end{pmatrix}\begin{pmatrix} g_{\gamma\beta} & 0 \\ * & h_{\gamma\beta} \end{pmatrix} = \begin{pmatrix} g_{\alpha\gamma}g_{\gamma\beta} & 0 \\ * & h_{\alpha\gamma}h_{\gamma\beta} \end{pmatrix}$$

$\Rightarrow$ $g_{\alpha\beta} = g_{\alpha\gamma}g_{\gamma\beta}$ $\Rightarrow$ $TS \to S$ valid bundles.
$h_{\alpha\beta} = h_{\alpha\gamma}h_{\gamma\beta}$ $N \to S$

ex) $\mathbb{CP}^1 \subseteq \mathbb{CP}^2$
$\parallel$
$S = \{[z_0, z_1, 0] \in \mathbb{CP}^2\}$. Compute normal bundle.

Coords on $\mathcal{U}_0$: $(x, y) = \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right)$, $S \cap \mathcal{U}_0 = \{y = 0\}$

Coords on $\mathcal{U}_1$: $(\tilde{x}, \tilde{y}) = \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}\right)$, $S \cap \mathcal{U}_1 = \{\tilde{y} = 0\}$

$\tilde{y} = x^{-1}y$

$\frac{\partial \tilde{y}}{\partial y} = x^{-1}$ $\Rightarrow$ transition function $h_{10}: \mathcal{U}_0 \cap \mathcal{U}_1 \to \mathbb{C}^*$
$h_{10} = x^{-1}$.

Defn: The complex line bundle $\mathcal{O}(1) \to \mathbb{CP}^n$ is defined by:
$g_{ij} = \frac{z_j}{z_i}$ on $\mathcal{U}_i \cap \mathcal{U}_j$.
$\in \mathbb{C}^*$

$\Rightarrow$ Normal bundle of $\mathbb{CP}^1 \subseteq \mathbb{CP}^2$ is $\mathcal{O}(1) \to \mathbb{CP}^1$.
rank 2 real bundle

(*) **Prop**: If $S = \{P_d(x) = 0\} \subseteq \mathbb{CP}^n$ is a submfd cut out by a homogeneous poly $P_d(x)$ of deg d, the normal bundle of S is $\mathcal{O}(d)|_S$. ($\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$)

Def: Let X be complex mfd. A smooth analytic hypersurface $D \subseteq X$ is a submfd s.t. $\forall p \in D$, $\exists$ hol'c chart $\mathcal{U}$ with $p \in \mathcal{U}$ s.t.

$$\mathcal{U} \cap D = \{ f = 0 \} \cap \mathcal{U}, \quad \text{where } f \text{ hol'c function with } df \text{ surj}, \; f : \mathcal{U} \to \mathbb{C},$$

and s.t. on overlaps $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$,

$$\frac{f_\alpha}{f_\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{C}^*.$$

Def: Given $D \subseteq M$, $\exists$ line bundle $\mathcal{O}(D) \to M$ defined by

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}, \quad g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{C}^* \text{ hol'c.}$$

ex) $D = \{ \sum_{i=0}^n z_i^3 = 0 \} \subseteq \mathbb{P}^n$

$(\mathcal{U}_0, w)$, $w_i = \frac{z_i}{z_0}$, $\mathcal{U}_0 \cap D = \{ 1 + \sum_{i=1}^n w_i^3 = 0 \}$

$(\mathcal{U}_1, \tilde{w})$, $\tilde{w}_i = \frac{z_i}{z_1}$, $\mathcal{U}_1 \cap D = \{ 1 + \sum_{i=1}^3 \tilde{w}_i^3 = 0 \}$

$g_{01} : \mathcal{U}_0 \cap \mathcal{U}_1 \to \mathbb{C}^*$, $g_{01} = \frac{1 + \sum w_i^3}{1 + \sum \tilde{w}_i^3} = \frac{z_1^3}{z_0^3}$

$\Rightarrow \mathcal{O}(D) = \mathcal{O}(3) = \mathcal{O}(1)^{\otimes 3}$

ex) $D = \{ P_d = 0 \} \subseteq \mathbb{P}^n$, P_d homog poly deg d.

same calc $\Rightarrow \mathcal{O}(D) = \mathcal{O}(d)$.

Prop: M complex mfd
$D \subseteq M$ smooth analytic hypersurface

$$N_D \cong \mathcal{O}(D).$$

Cor: Prop (*).

Pf: $M = \bigcup \mathcal{U}_\alpha$, $D \cap \mathcal{U}_\alpha = \{ f_\alpha = 0 \}$

1) Notice df_α nowhere zero section of $N_D^* \to D \cap \mathcal{U}_\alpha$.

This because: Since $f_\alpha \equiv 0$ on D, $T_pD = \ker df_\alpha|_p$.

$$N_p = T_pM / T_pD \Rightarrow df_\alpha(V) \neq 0 \quad \forall V \in N_p.$$

2) On $D \cap \mathcal{U}_\alpha \cap \mathcal{U}_\beta$:

$$df_\alpha = d\left(\frac{f_\alpha}{f_\beta} f_\beta\right) = dg_{\alpha\beta} f_\beta + g_{\alpha\beta} df_\beta, \quad g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$$

$df_\alpha = g_{\alpha\beta} df_\beta$ on $D \cap \mathcal{U}_\alpha \cap \mathcal{U}_\beta$

$\Rightarrow df_\alpha \in \Gamma(\mathcal{O}(D) \otimes N_D^*)$ nowhere vanishing section.

3) $\mathcal{O}(D) \otimes N_D^*$ trivial $\Rightarrow N_D \cong \mathcal{O}(D)$.

$\square$

<u>Note</u>: $L \to M$ complex line bundle admitting a section $s \in \Gamma(L)$ which is nowhere vanishing. Then: $L \cong \mathbb{C}$ trivial bundle $g_{\alpha\beta} = 1$

Isomorphic: need $\lambda_\alpha : \mathcal{U}_\alpha \to \mathbb{C}^*$ s.t. $1 = \lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}$.
Nowhere zero section: have $s_\alpha : \mathcal{U}_\alpha \to \mathbb{C}^*$ s.t. $s_\alpha = g_{\alpha\beta} s_\beta$.

Take $\lambda_\alpha = s_\alpha^{-1}$. Then $\lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1} = s_\alpha^{-1} g_{\alpha\beta} s_\beta = s_\alpha^{-1} s_\alpha = 1$ ✓