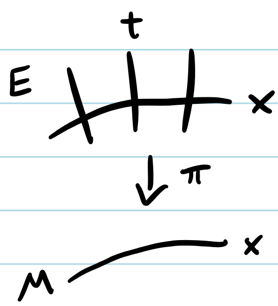
Thom Isomorphism

- $E \to M$ vector bundle rank n over mfd of dim m.
- Total space E is mfd of dim $(n+m)$.
- Suppose E is oriented bundle, M oriented mfd

$$\Omega^k_{cv}(E) = k\text{-forms on } E \text{ with compact support in vertical dir}$$

$$= \left\{ \alpha \in \Omega^k(E) : \alpha\big|_{\pi^{-1}(p)} \text{ has compact support} \right\}$$

$$H^k_{cv}(E) = \frac{\ker \{ d : \Omega^k_{cv}(E) \to \Omega^{k+1}_{cv}(E) \}}{\operatorname{im} \{ d : \Omega^{k-1}_{cv}(E) \to \Omega^k_{cv}(E) \}}$$



__Notation__ for coords on total space E:

Over $\pi^{-1}(\mathcal{U})$, coords $(\underbrace{x^1, \dots, x^m}_{\text{mfd coords}}, \underbrace{t^1, \dots, t^n}_{\text{fiber coords}})$.

__Def__: Integration along the fiber:

$$\pi_* : \Omega^k_{cv}(E) \to \Omega^{k-n}(M), \qquad k \geq n.$$

$$\pi_*\left(\pi^*\varphi \, f(x,t)\, dt^1 \wedge \cdots \wedge dt^n + \sum_{0 \le k < n} \pi^*\psi_k \, f_k(x,t) \underbrace{dt^{i_1} \wedge \cdots \wedge dt^{i_k}}_{\text{less than } n \text{ "dt"}} \right)$$

$$= \varphi \int_{\mathbb{R}^n} f(x,t)\, dt^1 \cdots dt^n .$$

Need to check π_* is well-defined.

Let's check case $n=1$ (1D fibers) and $k=1$ (1-form)

$$\alpha \in \Omega^1_{cv}(E), \quad \alpha = \alpha_t(x,t)\,dt + \alpha_{x^i}(x,t)\,dx^i$$

On overlap $\pi^{-1}(\mathcal{U}) \cap \pi^{-1}(\tilde{\mathcal{U}})$, $\tilde{x} = \psi(x)$ change coords on M

$\tilde{t} = g_{\mu\nu}(x)\, t$ bundle trans fun

$$\alpha = \tilde{\alpha}_{\tilde{t}}(\tilde{x},\tilde{t})\,d\tilde{t} + \tilde{\alpha}_{\tilde{x}^i}(\tilde{x},\tilde{t})\,d\tilde{x}^i$$

$$\pi_*(\alpha) = \int_{\mathbb{R}} \tilde{\alpha}_{\tilde{t}}(\tilde{x},\tilde{t})\,d\tilde{t} \overset{?}{=} \int_{\mathbb{R}} \alpha_t(x,t)\,dt$$

$$\alpha_t = \underbrace{\frac{\partial \tilde{x}}{\partial t}}_{=0} \tilde{\alpha}_{\tilde{x}} + \frac{\partial \tilde{t}}{\partial t} \tilde{\alpha}_{\tilde{t}} = g_{\mu\nu}(x)\, \tilde{\alpha}_{\tilde{t}}(\tilde{x}, \tilde{t}).$$

$$\eta_i = \frac{\partial \tilde{u}^l}{\partial u^i} \tilde{\eta}_l$$

$$\Rightarrow \int_{\mathbb{R}} \alpha_t(x,t)\, dt = \int_{\mathbb{R}} g_{\mu\nu}(x)\, \tilde{\alpha}_{\tilde{t}}(\tilde{x},\tilde{t})\, dt$$

let $\tilde{t} = g_{\mu\nu}(x)\, t$

$$= \int_{\mathbb{R}} \tilde{\alpha}_{\tilde{t}}(\tilde{x}, \tilde{t})\, d\tilde{t} \checkmark$$

change variables of integration

Prop: $\pi_* d = d \pi_*$. **Cor:** $\pi_* : H^{k+n}_{cv}(E) \to H^k(M)$ well-defined.

Pf: Let's check when $\operatorname{rk} E = 1$ and act on forms like: $\omega = \pi^* \varphi\, f(x,t)\, dt$.

$$d \pi_* \omega = d\left(\left(\int_{\mathbb{R}} f(x,\cdot)\right) \varphi\right)$$

$$= \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x^i}(x,\cdot)\right) dx^i \wedge \varphi + \left(\int_{\mathbb{R}} f(x,\cdot)\right) d\varphi$$

$$\pi_* d\omega = \pi_*\left(\pi^* d\varphi\, f(x,t)\, dt + (-1)^k \pi^* \varphi \frac{\partial f}{\partial x^i} dx^i \wedge dt \right)$$

$$= \left(\int_{\mathbb{R}} f(x,\cdot)\right) d\varphi + (-1)^k \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x^i}(x,\cdot)\right) \varphi \wedge dx^i . \checkmark$$

Prop: (A) $\pi_*(\pi^* \tau \wedge \omega) = \tau \wedge \pi_* \omega$, $\quad \tau \in \Omega^k(M)$, $\omega \in \Omega^l_{cv}(E)$.

(B) $\displaystyle\int_E \pi^* \tau \wedge \omega = \int_M \tau \wedge \pi_* \omega.$

Pf: If $\omega = \pi^* \varphi\, f(x,t)\, dt^{i_1} \wedge \cdots \wedge dt^{i_k}$, $k < \underset{n}{\operatorname{rk} E}$, then (A) is $0 = 0$.
If $\omega = \pi^* \varphi\, f(x,t)\, dt^1 \wedge \cdots \wedge dt^n$, then

$$\pi_*(\pi^* \tau \wedge \omega) = \tau \wedge \varphi \int_{\mathbb{R}^n} f(x,\cdot) = \tau \wedge \pi_* \omega.$$

Partition unity wrt $\mathcal{U}_\alpha$, $\pi^{-1}(\mathcal{U}_\alpha) = \mathcal{U}_\alpha \times \mathbb{R}^n$

$$\text{(B)}: \int_E \pi^* \tau \wedge \omega = \sum_\alpha \int_{\mathcal{U}_\alpha \times \mathbb{R}^n} \pi^* \tau \wedge (\rho_\alpha \omega) = \sum_\alpha \int_{\mathcal{U}_\alpha} \pi_*(\pi^* \tau \wedge (\rho_\alpha \omega))$$

$$= \int_M \tau \wedge \pi_* \omega. \quad \square \quad \text{(A)}$$

Thm: (Thom Isomorphism)

Let $E \to M$ be orientable vector bundle of rank n.
Let M have finite good cover
Then:

$$H^{k+n}_{cv}(E) \cong H^k(M) \quad \text{with iso } \pi_*: H^{k+n}_{cv}(E) \to H^k(M).$$

Pf: First, suppose $M = \mathcal{U} \cup V$, $\pi^{-1}(\mathcal{U}) \cong \mathcal{U} \times \mathbb{R}^n$, $\pi^{-1}(V) \cong V \times \mathbb{R}^n$, $\mathcal{U} \cong V \cong \mathcal{U} \cap V \cong \mathbb{R}^m$

$$0 \to \Omega^k_{cv}(E) \to \Omega^k_{cv}(E|_{\mathcal{U}}) \oplus \Omega^k_{cv}(E|_V) \to \Omega^k_{cv}(E|_{\mathcal{U}\cap V}) \to 0$$

Long exact seq:

$$\begin{tikzcd}
\to H^{n+k}_{cv}(E) \arrow[r] \arrow[d, "\pi_*"] & H^{n+k}_{cv}(E|_{\mathcal{U}}) \oplus H^{n+k}_{cv}(E|_V) \arrow[r] \arrow[d, "\pi_*"] & H^{n+k}_{cv}(E|_{\mathcal{U}\cap V}) \arrow[r, "d^*"] \arrow[d, "\pi_*"] & H^{n+k+1}_{cv}(E) \to \arrow[d, "\pi_*"] \\
\to H^k(M) \arrow[r] & H^k(\mathcal{U}) \oplus H^k(V) \arrow[r] & H^k(\mathcal{U}\cap V) \arrow[r, "d^*"] & H^{k+1}(M) \to
\end{tikzcd}$$

Check commutative: use $d^*(\omega) = d(\rho_{\mathcal{U}}\omega)$ + $d\pi_* = \pi_* d$

$\rho_{\mathcal{U}} + \rho_V = 1$
$d\rho_{\mathcal{U}} = -d\rho_V$

Note: $E|_{\mathcal{U}} \to \mathcal{U}$, $E|_V \to V$, $E|_{\mathcal{U}\cap V} \to \mathcal{U}\cap V$ are all trivial bundles: $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^n$

due to: **Fact:** A vector bundle over a contractible mfd is trivial.

Recall: $H^{n+k}_{cv}(M \times \mathbb{R}^n) \xrightarrow{\pi_*} H^k(M)$ is isomorphism.
(Same proof as cohomology with cpt support)

5-lemma $\Rightarrow$ $H^{n+k}_{cv}(E) \xrightarrow{\pi_*} H^k(M)$ is isomorphism.

Next: consider the case $M = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$ and proceed by iteration. $\square$

Def: $E \to M$ oriented, $\operatorname{rk} E = n$, finite good cover.

Thom class: $\Phi \in H^n_{cv}(E)$, $\Phi = \pi_*^{-1}(1)$ $\qquad \pi_*: H^n_{cv}(E) \to H^0(M)$

$\Phi \qquad 1$

Note: $\pi_*(\pi^*\varphi \wedge \Phi) = \varphi \wedge \pi_* \Phi = \varphi$ $\qquad \pi_* \Phi = 1.$

Prop: Thom class $\Phi \in H^n_{cv}(E)$ uniquely characterized by property:
$$\int_F \Phi = 1 \quad \text{for all fibers } F = \pi^{-1}(p).$$

Pf: ($\Rightarrow$) Thom class satisfies $\pi_* \Phi = \pi_* \pi_*^{-1}(1) = 1 \Rightarrow \int_F \Phi = 1, \ \forall F = \pi^{-1}(p).$

($\Leftarrow$) Suppose $\Phi' \in H^n_{cv}(E)$ solves $\int_F \Phi' = 1 \ \ \forall$ fibers F. $\Rightarrow \pi_* \Phi' = 1.$

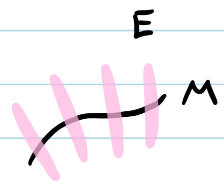
Thom iso: $H^n_{cv}(E) \cong H^0(M) \cong \mathbb{R} \Rightarrow \Phi' = c\,\Phi$
$\Rightarrow \pi_* \Phi' = c\,\pi_* \Phi$
$\Rightarrow c = 1.$

$\square$

Thm: $E \to M$ oriented bundle over compact oriented mfd.
$Z \subseteq E$ zero section, $\dim M = m$, $\operatorname{rk} E = n$.
$\Phi \in H^n_{cv}(E)$ Thom class

$\Phi = \eta_Z$ = Poincaré dual of zero section.
$$\int_E \omega \wedge \Phi = \int_M \omega \qquad \forall \omega \in \Omega^m_c(E) \text{ with } d\omega = 0.$$

Recall: Zero section in coords:
$\pi^{-1}(\mathcal{U})$, (x^i, t^i) coords on E.
$\pi^{-1}(\mathcal{U}) \cap Z = \{ t^i = 0 \ \ \forall i \}$



Pf: Take $\omega \in \Omega^m_c(E)$, $d\omega = 0$.
Since $E \simeq M$ by deformation retraction, $H^m(M) = H^m(E)$.

$\therefore \ \omega = \pi^*\varphi + d\gamma, \quad [\omega] = [\pi^*\varphi], \quad \varphi \in \Omega^m(M).$

$$\int_E \omega \wedge \Phi = \int_E \pi^*\varphi \wedge \Phi + \int_E d\gamma \wedge \Phi$$

$$= \int_E d(\gamma \wedge \Phi) = 0, \quad \Phi \text{ cpt supp}$$

(B)
$$= \int_M \varphi \wedge \pi_* \Phi$$
$$= \int_M \varphi. \quad \square \qquad \pi_* \Phi = 1.$$

Euler Class: $E \to M$ oriented over cpt mfd M, $\mathrm{rk}\, E = n$.

$$e(E) = j^* \Phi, \quad e(E) \in H^n(M)$$

where: $j: M \hookrightarrow E$ inclusion into zero section

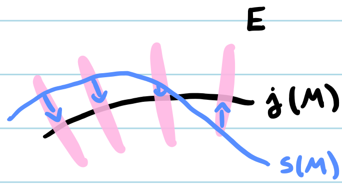
: Φ is Thom class.

Note:
$$\int_M e(E) = \int_M j^* \Phi = \int_E \Phi \wedge \Phi = \int_E \eta_Z \wedge \eta_Z$$

will interpret this as self-intersection number of zero section.

Note: $e(E) = [s^* \Phi]$ for any section $s \in \Gamma(E)$.

Indeed: $j: M \to E$ are homotopic.
$s: M \to E$



$\Rightarrow [j^* \Phi] = [s^* \Phi]$.