U(1) bundles

Def: A rank 1 complex vector bundle $E \to M$ is a U(1) bundle if all transition functions are of the form
$g_{\alpha\beta} = e^{i\varphi_{\alpha\beta}}$.

Two ways to arrive at U(1) bundle:
1) SO(2) - bundle
2) Complex line bundle

1) Suppose $E \to M$ is a real rank 2 bundle with transition functions of the form:

$$g_{\alpha\beta} = \begin{pmatrix} \cos\varphi_{\alpha\beta} & -\sin\varphi_{\alpha\beta} \\ \sin\varphi_{\alpha\beta} & \cos\varphi_{\alpha\beta} \end{pmatrix}$$

Identify $\mathbb{R}^2 \leftrightarrow \mathbb{C}$ and $g_{\alpha\beta} \leftrightarrow e^{i\varphi_{\alpha\beta}}$.

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow e^{i\varphi}(x+iy).$$

$\therefore$ $E \to M$ can be understood as U(1) bundle, with coord change:
$(\pi^{-1}(\mathcal{U}_\alpha), \tilde{x}, \tilde{w})$, $(\pi^{-1}(\mathcal{U}_\beta), x, w)$

$\tilde{x} = f(x)$ $\qquad$ x coord on M
$\tilde{w} = e^{i\varphi_{\alpha\beta}(x)} w$ $\qquad$ $w \in \mathbb{C}$ coord on fiber

2) Let $L \to M$ be a complex line bundle.
$M = \bigcup_\alpha \mathcal{U}_\alpha$ trivializing cover, $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{C}^*$.

Def: A metric h on L is a collection $\{h_\alpha\}$

$$h_\alpha : \mathcal{U}_\alpha \to (0,\infty) \quad \text{s.t.} \quad h_\alpha = \frac{1}{|g_{\alpha\beta}|^2} h_\beta.$$

Point of this: given sections $u, v \in \Gamma(L)$, then

$\langle u, v\rangle_h := u_\alpha h_\alpha \bar{v}_\alpha$ is well-defn, i.e. $u_\alpha h_\alpha \bar{v}_\alpha = u_\beta h_\beta \bar{v}_\beta$.
$u_\alpha = g_{\alpha\beta} u_\beta$

Every line bundle admits metrics:
e.g. $h_\alpha = \exp\left(-\sum_\mu \rho_\mu \log |g_{\alpha\mu}|^2\right)$ where $\{\rho_\alpha\}$ is partition of unity wrt $\{\mathcal{U}_\alpha\}$.

Given a metric on L, can construct isomorphism to $U(1)$ bundle:
$$(g_{new})_{\alpha\beta} = \lambda_\alpha (g_{old})_{\alpha\beta} \lambda_\beta^{-1}$$

Take $\lambda_\alpha = \sqrt{h_\alpha}$. Then $\sqrt{h_\alpha}\, g_{\alpha\beta} \sqrt{h_\beta}^{-1} = \dfrac{g_{\alpha\beta}}{|g_{\alpha\beta}|} \in \mathcal{U}(1)$. ✓

$\therefore$ $L \to M$ cplx line bundle $\rightsquigarrow$ can assume $g_{\alpha\beta} = e^{i\varphi_{\alpha\beta}}$.

Connections on U(1)-bundles:

- $L \to M$, $g_{\alpha\beta} = e^{i\varphi_{\alpha\beta}} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to U(1)$.

Def: A connection on L is a collection $\{A_\alpha\}$

$A_\alpha \in \Omega^1(\mathcal{U}_\alpha, i\mathbb{R})$ s.t. $A_\alpha = A_\beta - i\, d\varphi_{\alpha\beta}$.

Every line bundle admits a connection, e.g.
$A_\alpha = \sum_\mu \rho_\mu\, d\varphi_{\alpha\mu}$.

Point of this: $A \overset{loc}{=} (A_\mu)_i\, dx^i$

covariant derivative
$\nabla_K s_\alpha = \partial_K s_\alpha + (A_\alpha)_K s_\alpha$
$\nabla_K s_\alpha = e^{i\varphi_{\alpha\beta}} \nabla_K s_\beta$
$\therefore \nabla s \in \Omega^1(L)$.

Physical origins:
Magnetic field $\vec{B}$ on $\mathbb{R}^3$
$\vec{B} \rightsquigarrow B_1 dx^{23} + B_2 dx^{31} + B_3 dx^{12}$
$\nabla \cdot \vec{B} = 0 \rightsquigarrow dB = 0$
$\vec{B} = \nabla \times \vec{A} \rightsquigarrow B = dA$
$A = A_i\, dx^i$
$\vec{A} \mapsto \vec{A} + \nabla f \rightsquigarrow A_\alpha = A_\beta - i d\varphi_{\alpha\beta}$
"gauge transformation"

Def: The curvature of a connection A on $L \to M$ is:
$$F = dA. \quad F \in \Omega^2(M).$$

Def: The Euler class of $L \to M$ is:
$$e(L) = \frac{i}{2\pi}[F] \in H^2(M).$$

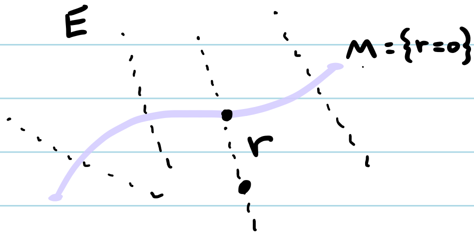
(also denoted: $c_1(L) = \frac{i}{2\pi}[F]$)

Euler class does not depend on choice of connection:
Let $A, \tilde{A}$ two connections.

$$A_\alpha - \check{A}_\alpha = A_\beta - \check{A}_\beta \Rightarrow A - \check{A} \in \Omega^1(M)$$

$$\frac{i}{2\pi}(F - \check{F}) = \frac{1}{2\pi} d(iA - i\check{A}) \Rightarrow \left[\frac{iF}{2\pi}\right] = \left[\frac{i\check{F}}{2\pi}\right] \in H^2(M).$$

Angular form on total space



$E^\circ = E \setminus \{\text{zero section}\}$

Coords on E:

$\tilde{x}^i = f^i(x)$, x^i coords on M

$\tilde{w} = e^{i\varphi_{\alpha\beta}} w$, $w \in \mathbb{C}$ along fiber

Coords on E°: write $w = re^{i\theta}$. $\Rightarrow \tilde{r}e^{i\tilde{\theta}} = e^{i\varphi_{\alpha\beta}} r e^{i\theta}$

$\Rightarrow \tilde{r} = r$

$\Rightarrow \tilde{\theta} = \theta + \varphi_{\alpha\beta}(x) + 2\pi k$, $k \in \mathbb{Z}$.

Define on E°:

$$\psi = \frac{1}{2\pi}(d\theta - i\pi^* A), \quad \psi \in \Omega^1(E^\circ).$$

Check:

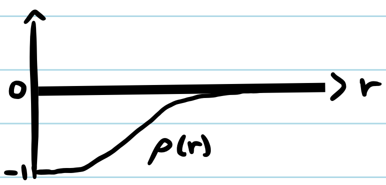
$$d\tilde{\theta} - i\pi^*\tilde{A} = d(\theta + \varphi_{\alpha\beta} + 2\pi k) - i\pi^* A - d\varphi_{\alpha\beta}$$
$$= d\theta - i\pi^* A. \checkmark$$

Note: $d\psi = -\frac{i}{2\pi}\pi^* dA = -\frac{i}{2\pi}\pi^* F.$ $\quad d\psi = -\pi^* e$

Thom class

$$\Phi = [d(\rho(r)\psi)] \in H^2_{cv}(E),$$

where:



$\rho \equiv -1$ near $r = 0$

$\rho \equiv 0$ $\forall r \geq 1$.

Note: $d(\rho(r)\psi) = d\rho \wedge \psi - \frac{i}{2\pi}\rho\, \pi^* F$, $\quad d\rho = \rho'(r)dr$

$d\rho \equiv 0$ near zero section

$\therefore d(\rho(r)\psi) \in \Omega^2(E)$ even though $\psi \in \Omega^1(E^\circ)$.

By uniqueness of Thom class, need to check

$$\int_F \Phi = 1 \qquad \forall\ F = \pi^{-1}(p) \text{ fiber.}$$

$$\int_F \Phi = \int_0^\infty \int_0^{2\pi} d\rho(r) \wedge \frac{d\theta}{2\pi}$$

$d(\rho(r)\psi) = d\rho \wedge \frac{d\theta}{2\pi} - \underbrace{d\rho \wedge \frac{i}{2\pi}\pi^* A - \frac{i}{2\pi}\rho\, \pi^* F}_{\text{restrict to zero on } \pi^{-1}(p)}$

$$= \int_0^\infty \rho'(r)\, dr = \rho(\infty) - \rho(0) = 1.$$

Euler class $\overset{?}{=} j^*\Phi$, $\quad j : M \hookrightarrow E$ zero section

$$j^*\Phi = [d(\rho(0)\, j^*\psi)] = [\tfrac{i}{2\pi} dA] = e.$$

Hol'c line bundle:

- Let $E \to M$ be hol'c line bundle over cplx mfd M.
  $M = \bigcup_\alpha \mathcal{U}_\alpha$, $\quad g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{C}^*$ hol'c: $\bar\partial g_{\alpha\beta} = 0$.
- Let h be a metric on E:
  $$h_\alpha = \frac{1}{|g_{\alpha\beta}|^2} h_\beta$$
- Let $L \to M$ be the associated $U(1)$-bundle with transition functions
  $$e^{i\varphi_{\alpha\beta}} = \frac{g_{\alpha\beta}}{|g_{\alpha\beta}|} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to U(1).$$

Claim: $A = \frac{1}{2}\partial \log h - \frac{1}{2}\bar\partial \log h$ is a connection on L.
(written in unitary frame wrt h)
$\bar A = -A$

check: Know

$$\frac{1}{2}\partial \log h_\alpha = \frac{1}{2}\partial \log h_\beta - \frac{1}{2}\partial \log |g_{\alpha\beta}|^2$$

$$= \frac{1}{2}\partial \log h_\beta - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{g_{\alpha\beta}} \qquad \text{since } \partial \bar g_{\alpha\beta} = \overline{\bar\partial g_{\alpha\beta}} = 0.$$

$$\Rightarrow A_\alpha = A_\beta - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{g_{\alpha\beta}} + \frac{1}{2}\frac{\bar\partial \bar g_{\alpha\beta}}{\bar g_{\alpha\beta}} \qquad (a)$$

Need: $\frac{1}{2}\frac{\partial g_{\alpha\beta}}{g_{\alpha\beta}} = i\,\partial\varphi_{\alpha\beta}$ (b)

$g_{\alpha\beta} = |g_{\alpha\beta}|\, e^{i\varphi_{\alpha\beta}}$

$\partial \log g_{\alpha\beta} = \frac{1}{2}\partial \log |g_{\alpha\beta}|^2 + i\,\partial\varphi_{\alpha\beta}$

$= \frac{1}{2}\frac{\partial g_{\alpha\beta}}{g_{\alpha\beta}} + i\,\partial\varphi_{\alpha\beta}.$

(a) + (b):

$A_\alpha = A_\beta - i\,d\varphi_{\alpha\beta}$

Cor: Let $E \to M$ be a hol'c line bundle.
Let h be a metric on E. Then $\therefore$ $e = \left[\frac{-i}{2\pi}\partial\bar\partial \log h\right]$

check: $F = dA$

$F = d\left(\frac{1}{2}\partial \log h - \frac{1}{2}\bar\partial \log h\right) = -\partial\bar\partial \log h$

$iF = -i\,\partial\bar\partial \log h$

ex) $\mathcal{O}(1) \to \mathbb{P}^n$ hol'c line bundle

$g_{\alpha\beta} = \frac{z_\beta}{z_\alpha}.$

$h_{FS} = (1+\sum|w^i|^2)^{-1}$, $w^i = \frac{z_i}{z_\alpha}$, $\tilde{w}^i = \frac{z_i}{z_\beta}$

$h_\alpha = (1+\sum|w^i|^2)^{-1}$

$\frac{h_\beta}{|g_{\alpha\beta}|^2} = \left(1+\sum_i\left|\frac{z_i}{z_\beta}\right|^2\right)^{-1}\frac{|z_\alpha|^2}{|z_\beta|^2} = h_\alpha$

$e = \left[\frac{-i}{2\pi}\partial\bar\partial \log h\right] = \left[\frac{i}{2\pi}\partial\bar\partial \log(1+\sum|w^i|^2)\right] = [\omega_{FS}]$

ex) $\mathcal{O}(\ell) \to \mathbb{P}^n$, $\mathcal{O}(\ell) = \mathcal{O}(1)^{\otimes \ell}$, $g_{\alpha\beta} = \left(\frac{z_\beta}{z_\alpha}\right)^\ell$, $\ell \in \mathbb{Z}$

$h = h_{FS}^{\ell}$ valid metric.

Note: $\mathcal{O}(-1) = \mathcal{O}(1)^{\otimes -1}$
with convention:
$L^{-1} = L^*$ dual

$e = \ell\,[\omega_{FS}]$ since $i\,\partial\bar\partial \log h^\ell = \ell\, i\,\partial\bar\partial \log h.$