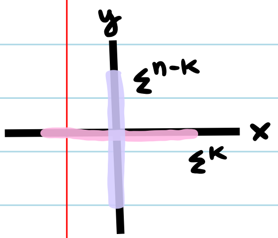
Euler Class and Poincaré Duality

Def: Let Σ_1, Σ_2 be submanifolds of M.
If all $p \in \Sigma_1 \cap \Sigma_2$ satisfy

$$T_pM = T_p\Sigma_1 + T_p\Sigma_2$$

then Σ_1 and Σ_2 are said to intersect transversely.

Prop: If Σ^k, Σ^{n-k} are transverse submfd of M^n of complementary dimension, then $\forall p \in \Sigma^k \cap \Sigma^{n-k}$ $\exists$ coord chart $p \in \mathcal{U} \subseteq M$ st.



$$\Sigma^k \cap \mathcal{U} = \{(x^1, \dots, x^k, y^1, \dots, y^{n-k}) \in \mathcal{U} : y^1 = \dots = y^{n-k} = 0\}$$

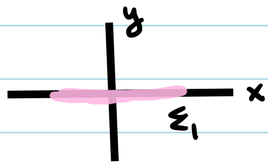
$$\Sigma^{n-k} \cap \mathcal{U} = \{(x^1, \dots, x^k, y^1, \dots, y^{n-k}) \in \mathcal{U} : x^1 = \dots = x^k = 0\}.$$

Pf: Let's just prove the case when: $\dim \Sigma_1 = 1$, $\dim \Sigma_2 = 2$, $\dim M = 2$.

Let $p \in \Sigma_1 \cap \Sigma_2$.

1) Since $\Sigma_1 \subseteq M$ is a submfd, $\exists$ nbhd $\mathcal{U}$ of p and coords s.t.

$$\Sigma_1 \cap \mathcal{U} = \{(x,y) \in \mathcal{U} : y = 0\}.$$



2) Since $\Sigma_2 \subseteq M$ is a submfd, locally

$$\Sigma_2 \cap \mathcal{U} = \{(x,y) \in \mathcal{U} : f(x,y) = 0\} \text{ with } Df|_p : \mathbb{R}^2 \to \mathbb{R} \text{ surjective.}$$

(*)

3) Claim $\frac{\partial f}{\partial x}(p) \neq 0$. Indeed:

$$T_pM = \operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$$

$$T_p\Sigma_1 = \operatorname{span}\left\{\frac{\partial}{\partial x}\right\}$$

$$T_p\Sigma_2 = \ker Df|_p = \left\{ v^1\frac{\partial}{\partial x} + v^2\frac{\partial}{\partial y} : v^1\frac{\partial f}{\partial x} + v^2\frac{\partial f}{\partial y} = 0 \right\}$$

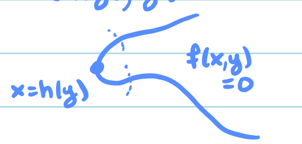
For $T_pM = T_p\Sigma_1 \oplus T_p\Sigma_2$, need $\frac{\partial f}{\partial x}(p) \neq 0$ (else $\frac{\partial f}{\partial y}(p) \neq 0$ (*), $v^2 = 0$)

4) $\frac{\partial f}{\partial x}(p) \neq 0 \Rightarrow$ after shrinking $\mathcal{U}$, $\exists \varepsilon > 0$ s.t.

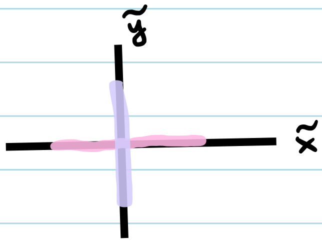
$$\mathcal{U} \cap f^{-1}(0) = \{ (h(y), y) : |y| < \varepsilon \}.$$

Implicit function Thm

$f(h(y), y) = 0$



5) Let: $\tilde{x} = x - h(y)$.
$\tilde{y} = y$



Then: $\{\tilde{x} = 0\} = \Sigma_2$
$\{\tilde{y} = 0\} = \Sigma_1$. $\square$

Lemma: Let $E \to M^n$ be a rank k bundle.
A section $s \in \Gamma(E)$ is transverse to the zero section

$$\Leftrightarrow \text{ all } Ds_\alpha\big|_p : \mathbb{R}^n \to \mathbb{R}^k \text{ are surjective, } p \in \{s_\alpha = 0\}$$

Recall: $M = \bigcup \mathcal{U}_\alpha$ trivializing cover

$s\big|_{\mathcal{U}_\alpha}(x) = (x, s_\alpha(x))$, $s_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^k$.

$s\big|_{\mathcal{U}_\alpha} := \varphi_\alpha \circ s$, $\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^k$
$e \mapsto (x, t)$

Pf: s transverse to zero section means: over $\pi^{-1}(\mathcal{U}_\alpha)$, $e = (p, 0)$
$p \in \mathcal{U}_\alpha$

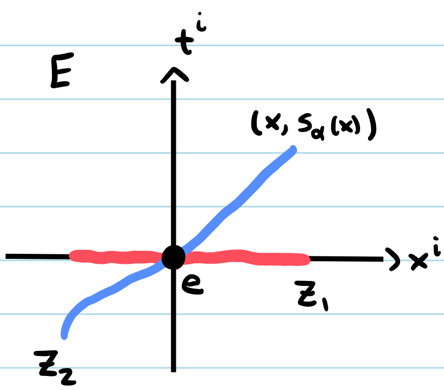
$Z_1 \cap \pi^{-1}(\mathcal{U}_\alpha) = \{ (\overset{x}{x}, \overset{t}{0}) \} \subseteq \pi^{-1}(\mathcal{U}_\alpha)$ zero section
$Z_2 \cap \pi^{-1}(\mathcal{U}_\alpha) = \{ (\underset{x}{x}, \underset{t}{s_\alpha(x)}) \} \subseteq \pi^{-1}(\mathcal{U}_\alpha)$ given section

$T_e E = T_e Z_1 + T_e Z_2$.

$T_e E = \text{span} \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^k} \right\}$

$T_e Z_1 = \text{span} \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$

$T_e Z_2 = \text{span} \left\{ \frac{\partial}{\partial x^i} + \underbrace{\frac{\partial s^\alpha}{\partial x^i} \frac{\partial}{\partial t^\alpha}}_{\underbrace{= Ds\left(\frac{\partial}{\partial x^i}\right)}_{\text{need } Ds \text{ surjective.}}} \right\}_{i=1}^{n}$

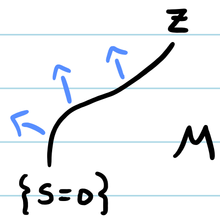


$\square$

$\{s=0\}$ means: $\{s_\alpha(x)=0\}$ over $\mathcal{U}_\alpha$

Prop: Let $E \to M^n$ be a vector bundle of rank k.
Let $s \in \Gamma(E)$ be a transversal section.
Then $\{s=0\} \subseteq M^n$ is a submfd of dim $(n-k)$,
and the normal bundle of $Z := \{s=0\} \subseteq M$ is:

$$E|_Z \to Z.$$



Pf: Let $M = \bigcup \mathcal{U}_\alpha$ be a trivializing cover. Let $S = s^{-1}(0) \subseteq M$.
Local sections: $s_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^k$, $s_\alpha = g_{\alpha\beta}\, s_\beta$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$.

$S \cap \mathcal{U}_\alpha = \{ s_\alpha(x) = 0 \}$

$ds_\alpha : \mathbb{R}^n \to \mathbb{R}^k$ surjective $\Rightarrow$ $S \cap \mathcal{U}_\alpha$ has slice coords
$\Rightarrow$ $S \subseteq M$ submanifold.

$T_pS = \operatorname{Ker} ds_\alpha|_p, \quad p \in S$

$ds_\alpha|_p : T_pM \to \mathbb{R}^k, \quad ds_\alpha|_p : T_pM / \underbrace{\ker ds_\alpha}_{= T_pS} \xrightarrow{\sim} \mathbb{R}^k$ iso

$\Rightarrow ds_\alpha|_p : N_p \xrightarrow{\sim} \mathbb{R}^k.$

Note: $ds_\alpha = g_{\alpha\beta}\, ds_\beta$ on S since $s_\beta(p) = 0 \ \forall p \in S$.

$\Rightarrow$ Can map $\lambda_0 : N_p \to E_p$
$v \mapsto ds_\alpha|_p(v)$ isomorphism on fibers.

Obtain $\lambda \in \Gamma(\mathrm{Hom}(N,E), S)$ s.t. $\lambda(p) : N_p \xrightarrow{\sim} E_p$. $\therefore N \overset{\sim}{=} E$. ∎ (isomorphic bundles)

Local picture: arrange coords on M : (x,y) s.t. $\left[\dfrac{\partial s_\alpha^k}{\partial y^i}\right]_{k\times k}$ is invertible.

Let: $\lambda_\alpha : \mathcal{U}_\alpha \to GL(k,\mathbb{R})$,
$\lambda_\alpha{}^i{}_j = \dfrac{\partial s_\alpha^i}{\partial y^j}$.

$$\Rightarrow \lambda_\alpha = \underbrace{g_{\alpha\beta}}_{\text{trans fun of } E \to Z}\, \lambda_\beta\, \underbrace{\check{g}_{\alpha\beta}^{-1}}_{\text{trans fun of } N \to Z}, \qquad \lambda|_{\mathcal{U}_\alpha} = (x, \lambda_\alpha)$$

ex) $\mathcal{O}(k) \to \mathbb{C}\mathbb{P}^n$

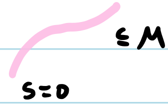
$s \in \Gamma(\mathcal{O}(k))$, s = homogeneous poly of deg k in $[z_0, \dots, z_n]$

$Z = s^{-1}(0)$
= deg k hypersurface. Assume $Z \subseteq \mathbb{C}\mathbb{P}^n$ smooth submfd.

$\Rightarrow N \to Z$ isomorphic to $\mathcal{O}(k)|_Z \to Z$.

Note:
(E, M) oriented
$p \in Z = \{s=0\}$
$T_pM = T_pZ \oplus N_p$
$T_pM = T_pZ \oplus E_p$
oriented
$\Rightarrow T_pZ$ orientable
$\Rightarrow Z$ oriented submfld
$\Rightarrow$ can take PD of $Z \subseteq M$.

Thm: Let $E \to M$ be an oriented bundle of rank K over an oriented M of $\dim M = n$.
Let $s \in \Gamma(E)$ be a transversal section.



$e(E) \in H^K(M)$ is Poincaré dual to $\{s=0\} \subseteq M$.

Pf: Recall $e(E) = [s^* \Phi]$, where $\Phi \in H^K_{cv}(E)$ is the Thom class.

1. Setup Coordinates

Let $p \in s^{-1}(0)$, $p \in \mathcal{U}_\alpha$ a trivialization of E.

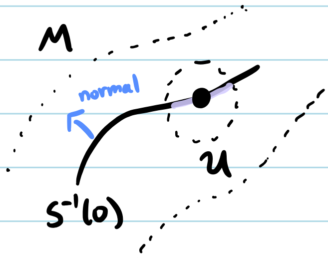
$$s\big|_{\mathcal{U}_\alpha} = (x^1, \dots, x^{n-k}, y^1, \dots, y^k, s_\alpha^1(x,y), \dots, s_\alpha^k(x,y))$$

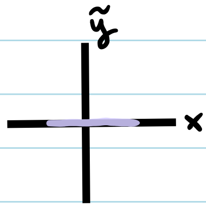
with: $\left[\dfrac{\partial s_\alpha^i}{\partial y^j}\right]_{k \times k}$ invertible. (defn of transverse)

By the implicit function thm, $\exists$ coords $(x, \tilde{y})$, open $\mathcal{U} \subseteq M$, $p \in \mathcal{U}$

$$s^{-1}(0) \cap \mathcal{U} = \{(x, \tilde{y}) \in \mathcal{U} : \tilde{y} = 0\}.$$

$$s_\alpha(x, 0) = 0, \quad \left[\frac{\partial s_\alpha^i}{\partial \tilde{y}^j}\right]_{k \times k} \text{ invertible.}$$





Simplify notation: drop tilde on $\tilde{y}$.

$s^{-1}(0) \cap \mathcal{U} = \{y = 0\}$

$s : \mathcal{U} \to E$

$s(x,y) = (x, y, s_\alpha(x,y)), \quad s_\alpha(x, 0) = 0 \quad \forall x$

2. Notice $s^* \Phi \in \Omega^k(\mathcal{U})$ has cpt support in y-coord.

Global: $s^* \Phi \in \Omega^k(\text{Tubular nbhd } s^{-1}(0))$ has cpt support in vertical dir.

Indeed: $\Phi \in \Omega_{cv}(E)$, choose s.t. support is very close to the zero section.

In $(x, y, t^1, \dots, t^k)$ coords over $\mathcal{U}$, $\Phi \equiv 0$ if $|t| \geq \varepsilon$.

base | fiber

Want: $s^* \Phi \equiv 0$ if $|y| \geq R\varepsilon$, $R \gg 1$.

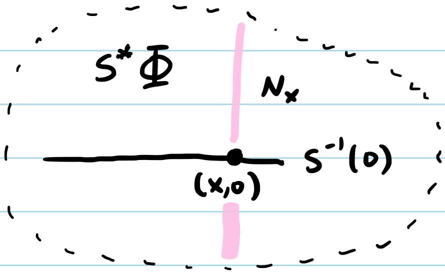
Want: if $\phi(x, y, t)$ function s.t. $\phi \equiv 0$ if $|t| \geq \varepsilon$, then $\phi(x, y, s_\alpha(x,y)) \equiv 0$ if $|y| \geq R\varepsilon$.

True if: $|s_\alpha(x,y)| \geq \varepsilon \quad \forall |y| \geq R\varepsilon$. Prove using $\left[\dfrac{\partial s_\alpha}{\partial y^j}\right]_{k \times k}$ non-deg.

$S^k = \{(x,0)\in \mathcal{U}\}$
(x,y) coord on $\mathcal{U}^n$
$\psi: N_S \to \mathbb{R}^k \times \mathbb{R}^{n-k}$
$\psi\left(v^i \frac{\partial}{\partial y^i}\Big|_{(x,0)}\right)$
$=(x,v)$
$\therefore \psi: N_S \xrightarrow{\cong} \mathcal{U}$
$\therefore N_{(x,0)} \cong \{(x,y)\in \mathcal{U}\}$

3. Objective: show $\int_{N_x} s^*\Phi = 1 \quad \forall x \in \mathcal{U}$,

where: $N \to s^{-1}(0)$ is the normal bundle
: N_x = fiber over x.



$\mathcal{U} \subseteq M$

Fix x. Then:
$N_x = \{(x,y)\in \mathcal{U}\}$

$$\int_{N_x} s^*\Phi = \int_{\{|y|\le R\varepsilon\}} F_1^*\Phi, \qquad F_1: N_x \to E, \quad (x,y) \mapsto (x,y,s_\alpha(x,y))$$

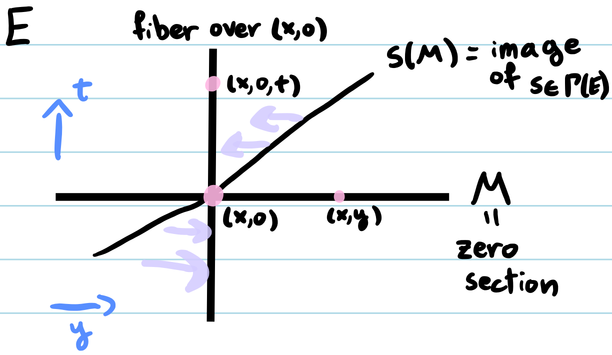
$F_1 = s|_{N_x}$

$$= \int_{\{|y|\le R\varepsilon\}} F_0^*\Phi \qquad F_0: N_x \to E, \quad (x,y) \mapsto (x,0,y)$$

Since: $F_0 \simeq F_1 \Rightarrow [F_1^*\Phi] = [F_0^*\Phi]$ $\qquad F_t = (x, ty, ts_\alpha + (1-t)y)$

$$= \int_{E_{(x,0)}} \Phi$$ integrate Φ over fiber $\pi^{-1}(x,0)$

$$= 1$$ Defn of Thom class.

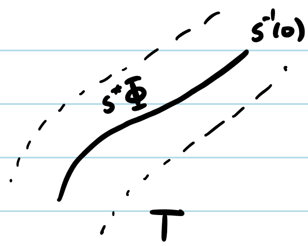


$$\int_y F_0^*\Phi = \int_y F_0^*\left(f(x,y,t)\,dt^1\wedge\cdots\wedge dt^k + \sum_{i<k} \pi^*\psi_i\, f_i\, dt^{\partial_1}\wedge\cdots\wedge dt^{\partial_i}\right)$$

$$= \int_y f(x,0,y)\,dy^1\wedge\cdots\wedge dy^k$$

$$= \int_{E_{(x,0)}} \Phi$$

4. Altogether: $T \subseteq M$ tubular nbhd of $s^{-1}(0)$.
: $T \cong \operatorname{tot}(N \to s^{-1}(0))$
: $s^*\Phi \in \Omega^k_{cv}(\operatorname{tot}(N \to s^{-1}(0)))$
with $\int_{N_x} s^*\Phi = 1 \quad \forall x$.



: $s^*\Phi$ is Thom class of $N \to s^{-1}(0)$
: $s^*\Phi$ is Poincaré dual to zero section of N.

$$\int_M \omega \wedge s^*\Phi = \int_{s^{-1}(0)} \omega \quad \forall \omega \in \Omega_c^{n-k}(M). \quad \square$$

Cor: Let $L \to M$ be a hol'c line bundle over a complex mfd.
Let $e = c_1(L) = \left[\frac{-i}{2\pi} \partial\bar{\partial} \log h \right] \in H^2(M, \mathbb{R})$

Let $s \in \Gamma(L)$ be a transversal section.

Then: $c_1(L)$ is Poincaré dual to $Z = s^{-1}(0)$.

ex) $Z = \left\{ \sum_{i=0}^{n} z_i^d = 0 \right\} \subseteq \mathbb{CP}^n$, $d \in \{1, 2, 3, \dots\}$

Exercise: can define a transversal section $s \in \Gamma(\mathcal{O}(d))$ s.t. $Z = s^{-1}(0)$ by: $s_\alpha : \mathcal{U}_\alpha \to \mathbb{C}$

$$s_\alpha = \frac{1}{z_\alpha^d} \sum_{i=0}^{n} z_i^d$$

$$\mathcal{U}_\alpha = \{ z_\alpha \neq 0 \}$$

$$g_{\alpha\beta} = \frac{z_\beta}{z_\alpha}, \quad g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{C}^*.$$

Recall: $e(\mathcal{O}(d)) = d\,[\omega_{FS}]$.

$\therefore \eta_Z = d\,[\omega_{FS}]$ Poincaré dual of Z.

ex) $Z = \{ z_0^4 + z_1^4 + z_2^4 = 0 \} \subseteq \mathbb{CP}^2$, $\quad \eta_Z = 4[\omega_{FS}]$.

over $\mathcal{U}_0 = \{ z_0 \neq 0 \}$:

$s_0 : \mathcal{U}_0 \to \mathbb{C}$
$s_0(w) = 1 + (w^1)^4 + (w^2)^4$, where: $w^i = \frac{z_i}{z_0}$ local coords on $\mathcal{U}_0$.

$$Z \cap \mathcal{U}_0 = \{ 1 + (w^1)^4 + (w^2)^4 = 0 \} \subseteq \mathbb{C}^2$$

$$ds_0\big|_{(w^1, w^2)} = (4(w^1)^3, 4(w^2)^3)$$ surjective, since $(w^1, w^2) = (0,0) \notin Z$.