Differential Topology: Homework 1

Due Date: September 20

- **Problem 1:** Let $\alpha \in \Omega^1(\mathbb{R}^2)$ be a smooth 1-form defined on all of $\mathbb{R}^2$. Further suppose that
$$\alpha = f(\theta)\, d\theta \quad \text{on } \mathbb{R}^2 \backslash \{0\},$$
using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Show that $\alpha \equiv 0$.

- **Problem 2:** Consider the 1-form defined on $\mathbb{R}^2\backslash\{0\}$ given by
$$\omega = -\frac{y\, dx}{x^2+y^2} + \frac{x\, dy}{x^2+y^2}.$$
Show that ω is closed but not exact.

- **Problem 3:** Let ω be the 1-form from the previous problem, defined now on
$$U = \mathbb{R}^2 \backslash \{x = 0\}.$$
Write down an explicit function $f : U \to \mathbb{R}$ such that $\omega = df$. Hint: consider $\arctan(u)$.

- **Problem 4:** Consider $\mathbb{P}^2 = \mathbb{C}^3/\sim$ and denote points by $[Z_0, Z_1, Z_2] \in \mathbb{P}^2$. Consider the coordinate charts
$$\mathbb{P}^2 = U_0 \cup U_1 \cup U_2$$
where $U_i = \{Z_i \neq 0\}$. On $U_0 \cap U_1$, let
$$(z^1, z^2) = \left(\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0}\right), \quad (\tilde{z}^1, \tilde{z}^2) = \left(\frac{Z_0}{Z_1}, \frac{Z_2}{Z_1}\right).$$
Compute the change of coordinates $z = f(\tilde{z})$ and verify that the change of coordinates is holomorphic.

- **Problem 5:** The Fubini-Study metric ω on $\mathbb{P}^2$ is given by
$$\omega = i\partial\bar{\partial}\log(1 + |z^1|^2 + |z^2|^2)$$
over each coordinate chart (U_i, z). Show that $\omega \in \Omega^{1,1}(\mathbb{P}^2)$ is well-defined on the overlap of two coordinate systems (for example $U_0 \cap U_1$). Next, write
$$\omega = i g_{j\bar{k}}\, dz^j \wedge d\bar{z}^k,$$
and compute the matrix $g_{j\bar{k}}$. Show that $g_{j\bar{k}}$ is positive definite: recall this means that
$$g_{j\bar{k}}\xi^j\overline{\xi^k} \geq 0$$
for all $\xi \in \mathbb{C}^n$ with equality if and only if $\xi = 0$.