Differential Topology: Homework 2

Due Date: October 11

- **Problem 1:** Let S^n be the n-sphere. Use the Mayer-Vietoris sequence and induction to compute $H^i(S^n)$ for all $0 \leq i \leq n$.

- **Problem 2:** Let $\mathbb{CP}^n = \mathbb{C}^{n+1}/\sim$ be complex projective space. We will denote points by $[Z_0, Z_1, \ldots, Z_n]$.

  (a) Compute $H^i(\mathbb{CP}^1)$ for $0 \leq i \leq 2$. You can use the Mayer-Vietoris sequence with respect to the open cover
  $$\mathbb{CP}^1 = U \cup V$$
  where $U = \{Z_0 \neq 0\}$ and $V = \mathbb{CP}^1 \backslash \{p\}$ with $p = [1, 0]$.

  (b) Compute $H^i(\mathbb{CP}^2)$ for $0 \leq i \leq 4$. You can use the Mayer-Vietoris sequence with respect to the open cover
  $$\mathbb{CP}^2 = U \cup V$$
  where $U = \{Z_0 \neq 0\}$ and $V = \mathbb{CP}^2 \backslash \{p\}$ with $p = [1, 0, 0]$.

  You may use that $F : V \times [0, 1] \to V$ given by
  $$F([Z_0, Z_1, Z_2], t) = [tZ_0, Z_1, Z_2]$$
  gives a retraction from V to $\mathbb{CP}^1 \subseteq \mathbb{CP}^2$ where
  $$\mathbb{CP}^1 = \{[0, Z_1, Z_2] \in \mathbb{CP}^2\}.$$
  Continuing on with this method, you can compute the cohomology $H^*(\mathbb{CP}^n)$, but there is no need to hand-in the write-up for general $\mathbb{CP}^n$.

- **Problem 3:** Compute the de Rham cohomology of $\Sigma_g \backslash \{p_1, p_2, \ldots, p_k\}$, where Σ_g is a genus g compact orientable surface, and $p_1, p_2, \ldots p_k \in \Sigma_g$ are k distinct points. You may use directly that $\chi(\Sigma_g) = 2 - 2g$.

- **Problem 4:** Consider a short exact sequence
  $$0 \to A \to B \to C \to 0$$
  of differential complexes where the arrows are chain maps. Let $H^i(A)$, $H^i(B)$, $H^i(C)$ be the respective cohomology groups, and suppose these H^i are all finite dimensional and zero for all $i \geq N$ for some integer N. Let $\chi(A)$, $\chi(B)$, $\chi(C)$ be the respective Euler characteristics, e.g.
  $$\chi(A) = \sum_{i=0}^{N} (-1)^i \dim H^i(A).$$
  Show that
  $$\chi(B) = \chi(A) + \chi(C).$$