Differential Topology: Homework 3

Due Date: October 25

- **Problem 1:**

  (a) Let $T^3 = S^1 \times S^1 \times S^1$ and equip T^3 with oriented coordinates. Compute the Poincaré dual of $S \subset T^3$, where

  $$S = \{(e^{i\theta}, e^{2i\theta}, e^{-i\theta}) : 0 \leq \theta \leq 2\pi\} \subseteq T^3.$$

  To compute $\dim H^i(T^3)$, you may use the Künneth formula

  $$H^k(M \times N) = \bigoplus_{p+q=k} H^p(M) \otimes H^q(N),$$

  which we have not yet proved in-class.

  (b) Compute the Poincaré dual of $\mathbb{CP}^1 \subset \mathbb{CP}^3$ embedded as

  $$S = \{[Z_0, Z_1, 0, 0] \in \mathbb{CP}^3\}.$$

  Recall that the cohomology of $\mathbb{CP}^n$ is

  $$H^k(\mathbb{CP}^n; \mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

  and you can write your answer in terms of the Fubini-Study form $\omega_{FS} \in \Omega^{1,1}(\mathbb{CP}^n)$ which satisfies $\int_{\mathbb{CP}^n} \omega_{FS}^n = 1$.

- **Problem 2:** Show that the direct sum of two copies of the Möbius bundle M is isomorphic to a trivial bundle. To do this, you may use the cover $S^1 = U_1 \cup U_2$ with

  $$U_1 = \{e^{i\theta} : \theta \in (0, 2\pi)\}, \quad U_2 = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}$$

  with change of coordinates

  $$\tilde{\theta} = \begin{cases} \theta, & \text{if } 0 < \theta < \pi \\ \theta - 2\pi, & \text{if } \pi < \theta < 2\pi. \end{cases}$$

  The bundle $M \to S^1$ is defined by the data $(U_1 \cap U_2, g_{12})$ with

  $$g_{12} = \begin{cases} 1, & \text{if } 0 < \theta < \pi \\ -1, & \text{if } \pi < \theta < 2\pi. \end{cases}$$

To show $M \oplus M \to S^1$ is the trivial bundle, find

$$\lambda_\alpha : U_\alpha \to GL(2, \mathbb{R})$$

such that on $U_1 \cap U_2$ there holds

$$I_{2\times 2} = \lambda_1 g_{12} \lambda_2^{-1}.$$

Hint: the λ_α will be of the form

$$\begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix},$$

where $\psi(\theta)$ on U_1 and $\psi(\tilde{\theta})$ on U_2.

- **Problem 3:** Let $E \to M$ be a vector bundle of rank k. A metric on E is a section $h \in \Gamma(E^* \otimes E^*)$ that restricts to each fiber as a symmetric positive-definite quadratic form. Concretely, let $M = \cup_\alpha U_\alpha$ be a trivialization of the bundle E with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ satisfying

$$g_{\alpha\beta} = g_{\alpha\mu} g_{\mu\beta}.$$

A metric is given by a collection $\{h_\alpha\}$

$$h_\alpha : U_\alpha \to GL(k, \mathbb{R})$$

such that the matrices h_α are symmetric and positive definite and on overlaps the relation

$$h_\alpha = (g_{\alpha\beta}^{-1})^T h_\beta g_{\alpha\beta}^{-1},$$

holds.

(a) Show that if $s, u \in \Gamma(E)$ (so that e.g. $u_\alpha = g_{\alpha\beta} u_\beta$), then

$$\langle u, s \rangle_h = u_\alpha^T h_\alpha s_\alpha$$

is independent of the choice of trivialization U_α.

(b) Show that any bundle admits a metric. To do this, let ρ_α be a partition of unity subordinate to $\{U_\alpha\}$. Define

$$h_\alpha = \sum_\mu \rho_\mu \, g_{\mu\alpha}^T g_{\mu\alpha}.$$

and show that this defines a metric.

(c) Show that if $E \to M$ is equipped with a metric h, then E is isomorphic to a bundle with transition functions in $O(k)$. Recall that different transition functions $\check{g}_{\alpha\beta}$ define an isomorphic bundle if there exists local matrices

$$\lambda_\alpha : U_\alpha \to GL(k, \mathbb{R})$$

such that on overlaps

$$\check{g}_{\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}.$$

To find the suitable λ_α, equip E with a metric and decompose the local positive-definite matrices as $h_\alpha = \lambda_\alpha^T \lambda_\alpha$. We see that a metric reduces the structure group of E from $GL(k)$ to $O(k)$.

(d) Let $L \to M$ be an orientable line bundle (rank 1). Prove that L is isomorphic to the trivial bundle.