I. Non-Kähler Complex Geometry

ex) Simple example of a non-Kähler metric:

G = complex Lie group, e.g. $G = GL(k,\mathbb{C})$
$G = SL(k,\mathbb{C})$

Choose hermitian inner prod on $T^{1,0}_e G$.
Choose $e_1,\dots,e_n$ global frame of left-invariant hol'c vector fields on G.

Let $\omega = i \sum_a e^a \wedge \overline{e^a}$, $\quad e^a(e_b) = \delta^a{}_b$ dual frame

hermitian (1,1)-form

$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$

$$\partial\omega = i \sum_a \partial e^a \wedge \overline{e^a}$$

$$\partial e^a = -\frac{1}{2} c^a{}_{ij}\, e^i \wedge e^j, \quad [e_i, e_j] = \underbrace{c^a{}_{ij}}_{\text{structure constants}} e_a$$

$\partial\omega \not\equiv 0$ unless $[\cdot,\cdot] \equiv 0$.

$\Rightarrow$ Left-invariant metric on cplx Lie group is non-Kähler.

General Setup + Notation

- X complex mfd, $\dim_{\mathbb{C}} X = n$.

$\rightsquigarrow$ hol'c coords: $\{z^\mu\}_{\mu=1}^n$.

- $T_{\mathbb{C}} X = T^{1,0}X \oplus T^{0,1}X$

$\rightsquigarrow V \in \Gamma(T^{1,0}X)$, $V \overset{\text{loc}}{=} V^\mu \dfrac{\partial}{\partial z^\mu}$

$\rightsquigarrow V \in \Gamma(T_{\mathbb{C}}X)$

$V \overset{loc}{=} V^i \partial_i$
$= V^\mu \frac{\partial}{\partial z^\mu} + V^{\bar\mu} \frac{\partial}{\partial z^{\bar\mu}}$

Roman
i, j, k $T_{\mathbb{C}}X$ indices
Greek
α, β, γ $T^{1,0}X$ indices
$\sum_i a_i = \sum_\mu a_\mu + \sum_{\bar\mu} a_{\bar\mu}$

(H) • Hermitian metric: $g \in \Gamma(T^{*(1,0)} \otimes T^{*(0,1)})$.

$g \overset{loc}{=} g_{\mu\bar\nu}\, dz^\mu \otimes dz^{\bar\nu}$, where $\begin{cases} [g_{\mu\bar\nu}] > 0 \text{ pos def,} \\ \overline{g_{\mu\bar\nu}} = g_{\nu\bar\mu}. \end{cases}$

1) inner product on $T^{1,0}X$:

$$\langle V, W\rangle_g = g_{\mu\bar\nu} V^\mu \overline{W^\nu} \quad \forall\, V, W \in \Gamma(T^{1,0}X)$$

2) (H) induces:
metric tensor on $T_{\mathbb{C}}X$: $g \in \Gamma(T^*_{\mathbb{C}}X \otimes T^*_{\mathbb{C}}X)$.

For this, set $g_{\mu\nu} = 0$, $g_{\bar\mu\bar\nu} = 0$, $g_{\bar\mu\nu} = g_{\nu\bar\mu}$, and:

$$g(V,W) = g_{ij}\, V^i W^j, \quad V, W \in \Gamma(T_{\mathbb{C}}X).$$

runs $\alpha, \bar\alpha$ runs $\beta, \bar\beta$

ex) $n=1$. Coord $z = x + iy$, hermitian metric $g_{z\bar z} > 0$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

$$g(\partial_x, \partial_x) = g(\partial_z + \partial_{\bar z}, \partial_z + \partial_{\bar z}) = 2 g_{z\bar z}.$$

• Hermitian form: $\omega \overset{loc}{=} i\, g_{\mu\bar\nu}\, dz^\mu \wedge dz^{\bar\nu} \in \Omega^{1,1}(X, \mathbb{R})$

Covariant Derivatives

$(X, g) \rightsquigarrow$ Natural ∇ on $T_{\mathbb{C}}X$?

Notation: $\nabla_i (V^k \partial_k) := (\nabla_i V^k)\partial_k$, $V \in \Gamma(T_{\mathbb{C}}X)$

$$\nabla_i V^k = \partial_i V^k + \Gamma_i{}^k{}_j V^j$$

1. Levi-Civita connection

$$\Gamma^{LC}{}_i{}^k{}_j = \frac{g^{k\ell}}{2}(-\partial_\ell g_{ij} + \partial_i g_{\ell j} + \partial_j g_{\ell i})$$

Exercise:

$$\Gamma^{LC}{}_{\bar\mu}{}^{\bar\alpha}{}_\beta = \frac{1}{2}(i\partial\omega)_{\bar\mu}{}^{\bar\alpha}{}_\beta .$$

$A_\mu{}^\alpha{}_{\bar\beta} = g^{\alpha\bar\sigma} A_{\mu\bar\sigma\bar\beta}$

$g^{\alpha\bar\sigma} g_{\bar\sigma\beta} = \delta^\alpha{}_\beta$

This means:

Let $V \in \Gamma(T^{1,0}X)$. Differentiate:

$$\nabla_{\bar\mu} V^{\bar\alpha} = \underbrace{\partial_{\bar\mu} V^{\bar\alpha}}_{=0} + \Gamma_{\bar\mu}{}^{\bar\alpha}{}_\beta V^\beta + \underbrace{\Gamma_{\bar\mu}{}^{\bar\alpha}{}_{\bar\beta} V^{\bar\beta}}_{=0} \quad \text{since } V \in T^{1,0}X$$

$$= \frac{1}{2}(i\partial\omega)_{\bar\mu}{}^{\bar\alpha}{}_\beta V^\beta.$$

$$\Rightarrow \nabla_{\bar\mu} V = \underbrace{\nabla_{\bar\mu} V^\alpha \frac{\partial}{\partial z^\alpha}}_{\in T^{1,0}X} + \underbrace{\nabla_{\bar\mu} V^{\bar\alpha} \frac{\partial}{\partial z^{\bar\alpha}}}_{\in T^{0,1}X} .$$

∇^{LC} does not preserve $T^{1,0}X$. Not good.

2. Chern Connection

$\exists!\ \nabla$ on $T^{1,0}X$ s.t.

① $\nabla_{\bar\mu} V^\alpha = \partial_{\bar\mu} V^\alpha$ $\quad (\Gamma_{\bar\mu}{}^\alpha{}_\beta = 0)$

② $\nabla_\mu g_{\alpha\bar\beta} = 0.$ $\quad (\partial_\mu \langle V, W\rangle = \langle \nabla_\mu V, W\rangle + \langle V, \nabla_{\bar\mu} W\rangle)$

From $(T^{1,0}X, \nabla)$, can induce ∇ on $T_{\mathbb{C}}X$ by: $\nabla_i V^{\bar\alpha} := \overline{\nabla_{\bar i} V^\alpha}$.

Proof of Chern connection: expand ②:

$$\partial_\mu g_{\alpha\bar\beta} - \Gamma_\mu{}^\lambda{}_\alpha g_{\lambda\bar\beta} = 0.$$

e.g. $\Gamma_{\bar\mu}{}^{\bar\alpha}{}_{\bar\beta} = \overline{\Gamma_\mu{}^\alpha{}_\beta}$

$$\Rightarrow \boxed{\Gamma_\mu{}^\lambda{}_\alpha = g^{\lambda\bar\sigma}\partial_\mu g_{\alpha\bar\sigma}}$$

Formula for Chern connection

$$\Gamma_\mu{}^{\bar\lambda}{}_\alpha = 0$$

$$\Gamma_\mu{}^\lambda{}_{\bar\alpha} = 0$$

$$\Gamma_\mu{}^{\bar\lambda}{}_{\bar\alpha} = 0$$

All others obtained by conjugation.

3. "H-connection" (Yano, Strominger, Bismut)

$\exists!\ H \in \Omega^3(X)$ with ansatz

$$\hat\nabla = \nabla^{LC} - \tfrac{1}{2} g^{-1} H$$

$$\hat\Gamma_i{}^k{}_j = \Gamma^{LC}{}_i{}^k{}_j - \tfrac{1}{2} H_i{}^k{}_j$$

which preserves $T^{1,0}X$.

Note: $\hat\nabla g = 0$ is automatic by the ansatz.

Need:

$$(*)\ \begin{cases} \hat\Gamma_\mu{}^{\bar\nu}{}_\rho = 0 \\ \hat\Gamma_{\bar\mu}{}^{\bar\nu}{}_\rho = 0. \end{cases}$$

Exercise:

1) Compute
$$\Gamma^{LC}{}_\mu{}^{\bar\nu}{}_\rho = 0$$
$$\Gamma^{LC}{}_\mu{}^\nu{}_{\bar\rho} = -\tfrac{1}{2}(i\bar\partial\omega)_\mu{}^\nu{}_{\bar\rho}$$

take conj for $\Gamma_{\bar\mu}{}^{\bar\nu}{}_\rho$

2) Solve for H in $(*)$ $\Rightarrow$ $\boxed{H = i(\partial - \bar\partial)\omega}$

Remark: all 3 connections agree when ω is Kähler.

$$\nabla^{LC} = \nabla^{chern} = \hat{\nabla} \iff d\omega = 0$$

Curvature tensor: from a covariant derivative ∇ on $T_{\mathbb{C}}X$, we define:

$R \in \Omega^2(\mathrm{End}\, T_{\mathbb{C}} X)$

$$R_{\underbrace{pq}_{\text{2-form}}}{}^{\underbrace{i}_{}}{}_{\underbrace{j}_{\mathrm{End}\,T}} = \partial_p \Gamma_q{}^i{}_j + \Gamma_p{}^i{}_\ell \Gamma_q{}^\ell{}_j - (p \leftrightarrow q)$$

Exercise: Compute curvature of Chern connection.

$$\begin{aligned} R_{\mu\bar\nu}{}^\alpha{}_\beta &= -\partial_{\bar\nu} \Gamma_\mu{}^\alpha{}_\beta \\ &= -\partial_{\bar\nu}(g^{\alpha\bar\sigma} \partial_\mu g_{\beta\bar\sigma}) \end{aligned}$$

$$R = \bar\partial(g^{-1}\partial g)$$

$$\in \Omega^{1,1}(\mathrm{End}\, T^{1,0}X) \quad \text{or} \quad \Omega^{1,1}(\mathrm{End}\, T_{\mathbb{C}}X)$$

Other exercises:

- Compute $R \in \Omega^2(\mathrm{End}\, T_{\mathbb{C}} X)$ for:
  (a) Levi-Civita connection
  (b) H-connection

- G = complex Lie group as before with:
  $[e_i, e_j] = c^k{}_{ij}\, e_k, \quad \omega = i \sum e^k \wedge \overline{e^k}$

Compute R^{LC}, R^{ch}, R^H in terms of $c^k{}_{ij}$.