Aeppli and Bott-Chern classes

- $X$ cpt cplx mfd.

Aeppli cohomology:

$H_A^{p,q} = \frac{\ker \partial \bar{\partial} \cap \Omega^{p,q}}{\text{Im } \partial \bar{\partial}}$

Bott-Chern cohomology:

$H_{BC}^{p,q} = \frac{\ker d \cap \Omega^{p,q}}{\text{Im } \partial \bar{\partial}}$

Dolbeault cohomology:

$H_{\bar{\partial}}^{p,q} = \frac{\ker \bar{\partial} \cap \Omega^{p,q}}{\text{Im } \bar{\partial}}$

Note: If $X$ is Kähler (just need $\partial \bar{\partial}$-lemma), then $H_A^{p,q} \cong H_{BC}^{p,q} \cong H_{\bar{\partial}}^{p,q}$.

$\partial \bar{\partial}$-lemma: $X$ satisfies the $\partial \bar{\partial}$-lemma if there holds:

Suppose $\eta \in \Omega^{p,q}$ with $d \eta = 0$. TFAE:

1. $\eta = \partial \alpha$
2. $\eta = \bar{\partial} \beta$
3. $\eta = \partial \bar{\partial} \gamma$
4. $\eta = \partial \bar{\partial} \chi$.

Lem: If $X$ satisfies $H_{BC}^{p,q} \rightarrow H_{\bar{\partial}}^{p,q}$ isomorphism. $\partial \bar{\partial}$-lem, then:

$[\eta]_{BC} \mapsto [\eta]_{\bar{\partial}}$
injective: exercise

surjective: Let \([\eta]_\overline{\alpha} \in H^{p,q}_\overline{\alpha}\). Solve 
\(\eta = \alpha \overline{\alpha} x\).

\([\eta]_\overline{\alpha} = [\eta - \overline{\alpha} x]_\overline{\alpha}\)

\([\eta - \overline{\alpha} x]_{BC} \mapsto [\eta - \overline{\alpha} x]_\overline{\alpha}\)

\(\overline{\text{Lem}}\): If \(X\) satisfies \(\overline{\alpha}\)-lem, then

\[H^{p,q}_\overline{\alpha} \rightarrow H^{p,q}_A\]

is an isomorphism.

\([\eta]_\overline{\alpha} \mapsto [\eta]_A\)

\(\overline{\text{Pf}}\): injective: if \(\overline{\alpha} \eta_1 = 0\), \(\eta_1 = \eta_2 + \alpha \alpha_1 + \overline{\alpha} \alpha_2\)

then: \(\overline{\alpha} \alpha_1 = 0 \Rightarrow \alpha_1 = \alpha \overline{\alpha} x \Rightarrow [\eta_1]_\overline{\alpha} = [\eta_2]_\overline{\alpha}\).

surjective: exercise.

\(\overline{\text{Note}}\): In general \(H_A, H_{BC}, H_\overline{\alpha}\) are all different, though there is a Poincaré duality:

\[H^{p,q}_A \times H^{n-p,n-q}_{BC} \rightarrow \mathbb{C}\]

\([\alpha]_A \quad [\beta]_{BC} \mapsto \int x^{\alpha} \wedge \beta\]

\(\Rightarrow H^{p,q}_A \cong (H^{n-p,n-q}_{BC})^*\).
Back to Strominger system:
\[ \Omega_\omega \approx \begin{cases} \frac{b \in H^{n,2}_{BC}(X)}{b = [\Omega_\omega \omega^2]} \\ i\Theta\omega = \alpha'(\text{Tr} F \wedge F - \text{Tr} \mathcal{R} \wedge \mathcal{R}) \end{cases} \]

where
\[ a \in H^{n,n}_A(X) \]
\[ a = [\omega - \alpha' R_2 [h, \hat{h}] - \alpha' R_2 [\hat{g}, \hat{g}] - \alpha' \hat{\beta}] \]

(compare with Kähler CY:
\[ d\omega = 0 \Rightarrow [\omega] \in H^{n,n}(X) \]

You's Thm: \( \exists! \omega_{CY} \in [\omega] \).

On a non-Kähler CY, notion of Kähler class breaks into two:

(A) Can look for soln in given \( a \in H^{n,n}_A(X) \)

(B) Can look for soln in given \( b \in H^{n,2}_{BC}(X) \)

Both approaches have been pursued in the literature.

How to define Aeppli class \([a]\) ?

1. Choose reference metrics \((\hat{g}, \hat{h})\) on \( T^{n,0} \times E \).

Solve \( E(\hat{\gamma}) = \text{Tr} \hat{F} \wedge \hat{F} - \text{Tr} \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} \).

Here:
\[ E = (\partial \bar{\partial}) (\partial \bar{\partial})^\dagger + (\partial \bar{\partial})^\dagger (\partial \bar{\partial}) + (\partial^* \bar{\partial})^\dagger \partial^* \bar{\partial} + (\partial^* \bar{\partial}) (\partial^* \bar{\partial})^\dagger + \bar{\partial}^* \partial \bar{\partial}^* + \partial^* \partial + \partial^* \partial \]
\[ E = \text{Kodaira- Spencer operator} \quad \Theta^+ \text{ L}^2 \text{-adjoint} \text{ wrt } g. \]

**Exercise:** \( \text{Ker } E = \text{Ker } d \cap \text{Ker } (\Theta \bar{\Theta})^+ \)

Since \( c_{2, \text{bc}}(X) = c_{2, \text{bc}}(E) \),
\[ \text{Tr} \hat{F} \wedge \hat{F} - \text{Tr} \hat{R} \wedge \hat{R} \in \text{Im } \Theta \bar{\Theta}. \]

Fredholm alternative \( \Rightarrow \) can solve \((*)\)

2. If \( E(\hat{\theta}) = i \Theta \bar{\Theta} \eta \), then \( d \hat{\theta} = 0. \)  
(Exercise)

3. Define \( \hat{\beta} = \frac{1}{i} (\Theta \bar{\Theta})^+ \hat{\theta} \). \( i \Theta \bar{\Theta} \hat{\beta} = E(\hat{\theta}) \).
\[ i \Theta \bar{\Theta} \hat{\beta} = \text{Tr} \hat{F} \wedge \hat{F} - \text{Tr} \hat{R} \wedge \hat{R}. \quad (a) \]

4. Solve
\[ E(\hat{\gamma}) = \text{Tr} \hat{R} \wedge \hat{R} - \text{Tr} \hat{\bar{R}} \wedge \hat{\bar{R}} \]
Define \( \hat{R}[g, \hat{g}] = \frac{1}{i} (\Theta \bar{\Theta})^+ \hat{\gamma} \).
\[ i \Theta \bar{\Theta} \hat{R}[g, \hat{g}] = \text{Tr} \hat{R} \wedge \hat{R} - \text{Tr} \hat{\bar{R}} \wedge \hat{\bar{R}} \quad (b) \]

5. Define \( \hat{R}[h, \hat{h}] \) similarly.
\[ i \Theta \bar{\Theta} \hat{R}[h, \hat{h}] = \text{Tr} \hat{F} \wedge \hat{F} - \text{Tr} \hat{\bar{F}} \wedge \hat{\bar{F}}. \quad (c) \]

6. From \((a, b, c)\)
\[ i \Theta \bar{\Theta} \left( \omega - \alpha' \hat{R}[h, \hat{h}] + \alpha' \hat{R}[g, \hat{g}] - \alpha' \hat{\beta} \right) = 0. \]

7. Class independent of choice of ref \((\hat{g}, \hat{h})\).
Take a pair \((\hat{g}_1, \hat{h}_1), (\hat{g}_2, \hat{h}_2)\).
\[- R[h, \hat{h}_1] + R[g, \hat{g}_1] - \hat{\rho}_1 = (\exists \exists)^{\dagger}(\cdots) \equiv \psi \]

Note
\[
\exists \exists (\exists \exists)^{\dagger} = -\text{Tr} \hat{F}_1 \hat{F} + \text{Tr} \hat{F}_1 \hat{F} + \text{Tr} \hat{R}_1 \hat{R} - \text{Tr} \hat{R}_1 \hat{R}_1
\]
\[
- \text{Tr} \hat{F}_1 \hat{F}_1 + \text{Tr} \hat{R}_1 \hat{R}_1 \hat{R}_1 + \text{Tr} \hat{F} \hat{F}_1
\]
\[
- \text{Tr} \hat{F}_1 \hat{F}_1 + \text{Tr} \hat{R}_1 \hat{R}_1 \hat{R}_1 + \text{Tr} \hat{F}_1 \hat{F}_1
\]
\[
= 0
\]
\[
\Rightarrow (\exists \exists)^{\dagger} = 0
\]
\[
\Rightarrow R[h, \hat{h}_1] - R[g, \hat{g}] + \hat{\rho}_1 \in \Omega^n \text{ indep of reference } (\hat{g}, \hat{h}).