IV Aeppli and Bott-Chern classes

- X cpt cplx mfd.

Aeppli cohomology:

$$H_A^{p,q} = \frac{\operatorname{Ker} \partial\bar\partial \cap \Omega^{p,q}}{\operatorname{Im} \partial \oplus \bar\partial}$$

Bott-Chern cohomology:

$$H_{BC}^{p,q} = \frac{\operatorname{Ker} d \cap \Omega^{p,q}}{\operatorname{Im} \partial\bar\partial}$$

Dolbeault cohomology:

$$H_{\bar\partial}^{p,q} = \frac{\operatorname{ker} \bar\partial \cap \Omega^{p,q}}{\operatorname{Im} \bar\partial}$$

Note: If X is Kähler (just need $\partial\bar\partial$-lemma), then

$$H_A^{p,q} \cong H_{BC}^{p,q} \cong H_{\bar\partial}^{p,q}.$$

true for Kähler mfd

$\partial\bar\partial$-lemma: X satisfies the $\partial\bar\partial$-lemma if there holds:

Suppose $\eta \in \Omega^{p,q}$ with $d\eta = 0$. TFAE:

1. $\eta = d\alpha$
2. $\eta = \partial\beta$
3. $\eta = \bar\partial\gamma$
4. $\eta = \partial\bar\partial\chi$.

Lem: If X satisfies $\partial\bar\partial$-lem, then:

$$\begin{aligned} H_{BC}^{p,q} &\to H_{\bar\partial}^{p,q} \\ [\eta]_{BC} &\mapsto [\eta]_{\bar\partial} \end{aligned}$$

isomorphism.

injective: exercise

surjective: Let $[\eta]_{\bar\partial} \in H^{p,q}_{\bar\partial}$. Solve $\partial\eta = \partial\bar\partial x$.

$[\eta]_{\bar\partial} = [\eta - \bar\partial x]_{\bar\partial}$

$[\eta - \bar\partial x]_{BC} \mapsto [\eta - \bar\partial x]_{\bar\partial}$ ✓

Lem: If X satisfies $\partial\bar\partial$-lem, then

$$\begin{aligned} H_{\bar\partial}^{p,q} &\to H_A^{p,q} \\ [\eta]_{\bar\partial} &\mapsto [\eta]_A \end{aligned} \quad \text{is an isomorphism.}$$

Pf: injective: if $\begin{array}{l}\bar\partial\eta_1 = 0 \\ \bar\partial\eta_2 = 0\end{array}$, $\eta_1 = \eta_2 + \partial\alpha_1 + \bar\partial\alpha_2$

then: $\bar\partial\partial\alpha_1 = 0 \Rightarrow \partial\alpha_1 = \partial\bar\partial x \Rightarrow [\eta_1]_{\bar\partial} = [\eta_2]_{\bar\partial}$.

surjective: exercise. ✓

Note: In general $H_A, H_{BC}, H_{\bar\partial}$ are all different, though there is a Poincaré duality

$$\begin{aligned} H_A^{p,q} \times H_{BC}^{n-p,n-q} &\to \mathbb{C} \\ [\alpha]_A \quad [\beta]_{BC} &\mapsto \int_X \alpha\wedge\beta \end{aligned}$$

$\Rightarrow H_A^{p,q} \cong (H_{BC}^{n-p,n-q})^*$.

Back to Strominger system :

$$d(|\Omega|_\omega \omega^2) \rightsquigarrow \begin{cases} \underline{b} \in H^{2,2}_{BC}(X) \\ \underline{b} = [|\Omega|_\omega \omega^2] \end{cases}$$

$$i\partial\bar\partial\omega = \alpha'(\mathrm{Tr}\, F\wedge F - \mathrm{Tr}\, R\wedge R)$$

$$\rightsquigarrow \begin{cases} \underline{a} \in H^{1,1}_A(X) \\ \underline{a} = [\omega - \alpha' R_2[h,\hat h] - \alpha' R_2[g,\hat g] - \alpha'\hat\beta] \end{cases}$$

(Compare with Kähler CY:
$d\omega = 0 \rightsquigarrow [\omega] \in H^{1,1}(X)$

Yau's Thm: $\exists!\ \omega_{cy} \in [\omega]$.)

On a non-Kähler CY,
Notion of Kähler class breaks into two:

(A) Can look for soln in given $\underline{a} \in H^{1,1}_A(X)$

(B) Can look for soln in given $\underline{b} \in H^{2,2}_{BC}(X)$

Both approaches have been pursued in the litterature.

How to define Aeppli class [a] ?

1. Choose reference metrics $(\hat g, \hat h)$ on $T^{1,0} \times E$.

Solve $E(\hat\gamma) \overset{(*)}{=} \mathrm{Tr}\,\hat F\wedge\hat F - \mathrm{Tr}\,\hat R\wedge\hat R$.

$\hat F = \bar\partial(\hat h^{-1}\partial\hat h)$
$\hat R = \bar\partial(\hat g^{-1}\partial\hat g)$

Here:
$$E = (\partial\bar\partial)(\partial\bar\partial)^\dagger + (\partial\bar\partial)^\dagger(\partial\bar\partial) + (\partial^\dagger\bar\partial)^\dagger\partial^\dagger\bar\partial + (\partial^\dagger\bar\partial)(\partial^\dagger\bar\partial)^\dagger + \bar\partial^\dagger\bar\partial + \partial^\dagger\partial.$$

$E =$ Kodaira-Spencer operator.
$=$ 4^{th} order elliptic operator

$\partial^{\dagger}$ L^2-adjoint wrt g.

Exercise: $\operatorname{Ker} E = \operatorname{Ker} d \cap \operatorname{Ker} (\partial\bar{\partial})^{\dagger}$

Since $c_2^{BC}(X) = c_2^{BC}(E)$,

$\operatorname{Tr} \hat{F} \wedge \hat{F} - \operatorname{Tr} \hat{R} \wedge \hat{R} \in \operatorname{Im} \partial\bar{\partial}$.

Fredholm alternative $\Rightarrow$ can solve (*)

2. If $E(\hat{\gamma}) = i\partial\bar{\partial}\eta$, then $d\hat{\gamma} = 0$.
(Exercise)

3. Define $\hat{\beta} = \frac{1}{i}(\partial\bar{\partial})^{\dagger}\hat{\gamma}$. $\quad i\partial\bar{\partial}\hat{\beta} = E(\hat{\gamma})$.

$$i\partial\bar{\partial}\hat{\beta} = \operatorname{Tr} \hat{F} \wedge \hat{F} - \operatorname{Tr} \hat{R} \wedge \hat{R}. \quad (a)$$

4. Solve
$$E(\gamma) = \operatorname{Tr} R \wedge R - \operatorname{Tr} \hat{R} \wedge \hat{R}$$

$R = \bar{\partial}(g^{-1}\partial g)$
$\hat{R} = \bar{\partial}(\hat{g}^{-1}\partial \hat{g})$

Define $R[g,\hat{g}] = \frac{1}{i}(\partial\bar{\partial})^{\dagger}\gamma$.

$$i\partial\bar{\partial} R[g,\hat{g}] = \operatorname{Tr} R \wedge R - \operatorname{Tr} \hat{R} \wedge \hat{R} \quad (b)$$

5. Define $R[h,\hat{h}]$ similarly.

$$i\partial\bar{\partial} R[h,\hat{h}] = \operatorname{Tr} F \wedge F - \operatorname{Tr} \hat{F} \wedge \hat{F}. \quad (c)$$

6. From (a, b, c)

$$i\partial\bar{\partial}\left(\omega - \alpha' R[h,\hat{h}] + \alpha' R[g,\hat{g}] - \alpha'\hat{\beta}\right) = 0.$$

7. Class independent of choice of ref $(\hat{g},\hat{h})$.
Take a pair $(\hat{g}_1,\hat{h}_1)$, $(\hat{g}_2,\hat{h}_2)$.

$$\begin{matrix} -R[h,\hat h_1] & +R[g,\hat g_1] & -\hat\beta_1 \\ +R[h,\hat h_2] & -R[g,\hat g_2] & +\hat\beta_2 \end{matrix} = (\partial\bar\partial)^\dagger(\underbrace{\cdots}_{:=\Psi})$$

Note

$$\begin{aligned}\partial\bar\partial\,(\partial\bar\partial)^\dagger\Psi = & -\mathrm{Tr}\,F\wedge F + \mathrm{Tr}\,\hat F_1\wedge\hat F_1 + \mathrm{Tr}\,R\wedge R - \mathrm{Tr}\,\hat R_1\wedge\hat R_1 \\ & -\mathrm{Tr}\,\hat F_1\wedge\hat F_1 + \mathrm{Tr}\,\hat R_1\wedge\hat R_1 + \mathrm{Tr}\,F\wedge F - \mathrm{Tr}\,\hat F_2\wedge\hat F_2 \\ & -\mathrm{Tr}\,R\wedge R + \mathrm{Tr}\,\hat R_2\wedge\hat R_2 + \mathrm{Tr}\,\hat F_2\wedge\hat F_2 - \mathrm{Tr}\,\hat R_2\wedge R_2 \\ = & \; 0\end{aligned}$$

$$\Rightarrow \langle \partial\bar\partial(\partial\bar\partial)^\dagger\Psi, \Psi\rangle = 0$$

$$\Rightarrow (\partial\bar\partial)^\dagger\Psi = 0$$

$\Rightarrow R[h,\hat h] - R[g,\hat g] + \hat\beta \in \Omega^{1,1}$ indep of reference $(\hat g, \hat h)$.