I. Non-Kähler Complex Geometry

ex) Simple example of a non-Kähler metric:

G = complex Lie group, e.g. $G = GL(k,\mathbb{C})$
$G = SL(k,\mathbb{C})$

Choose hermitian inner prod on $T_e^{1,0}G$.
Choose $e_1,\dots,e_n$ global frame of left-invariant hol'c vector fields on G.

Let $\omega = i\sum_a e^a \wedge \overline{e^a}$, $\quad e^a(e_b) = \delta^a{}_b$ dual frame

hermitian (1,1)-form

$$\partial\omega = i\sum_a \underbrace{\partial e^a}_{} \wedge \overline{e^a}$$

$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$

$= -\frac{1}{2} c^a{}_{ij}\, e^i \wedge e^j$, $\quad [e_i, e_j] = \underbrace{c^a{}_{ij}}_{\text{structure constants}} e_a$

$\partial\omega \not\equiv 0$ unless $[\cdot,\cdot] \equiv 0$.

$\Rightarrow$ Left-invariant metric on cplx Lie group is non-Kähler.

General Setup + Notation

- X complex mfd, $\dim_{\mathbb{C}} X = n$.

$\rightsquigarrow$ hol'c coords: $\{z^\mu\}_{\mu=1}^n$.

- $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$

$\rightsquigarrow$ $V \in \Gamma(T^{1,0}X)$, $\quad V \overset{\text{loc}}{=} V^\mu \frac{\partial}{\partial z^\mu}$

$\rightsquigarrow V \in \Gamma(T_{\mathbb{C}}X)$

Roman
i,j,k $T_{\mathbb{C}}X$ indices

Greek
α,β,γ $T^{1,0}X$ indices

$$\sum_i a_i = \sum_\mu a_\mu + \sum_{\bar\mu} a_{\bar\mu}$$

$$V \overset{loc}{=} V^i \partial_i = V^\mu \frac{\partial}{\partial z^\mu} + V^{\bar\mu} \frac{\partial}{\partial z^{\bar\mu}}$$

(H) • Hermitian metric: $g \in \Gamma(T^{*(1,0)} \otimes T^{*(0,1)})$.

$$g \overset{loc}{=} g_{\mu\bar\nu}\, dz^\mu \otimes dz^{\bar\nu}, \text{ where } \begin{cases} [g_{\mu\bar\nu}] > 0 \text{ pos def,} \\ \overline{g_{\mu\bar\nu}} = g_{\nu\bar\mu}. \end{cases}$$

1) inner product on $T^{1,0}X$:

$$\langle V, W\rangle_g = g_{\mu\bar\nu} V^\mu \overline{W^\nu} \quad \forall\, V, W \in \Gamma(T^{1,0}X)$$

2) (H) induces:
metric tensor on $T_{\mathbb{C}}X$: $g \in \Gamma(T_{\mathbb{C}}^*X \otimes T_{\mathbb{C}}^*X)$.

For this, set $g_{\mu\nu} = 0$, $g_{\bar\mu\bar\nu} = 0$, $g_{\bar\mu\nu} = g_{\nu\bar\mu}$, and:

$$g(V,W) = g_{\underset{\text{runs } \alpha,\bar\alpha}{i}\,\underset{\text{runs } \beta,\bar\beta}{j}} V^i W^j \quad , \quad V, W \in \Gamma(T_{\mathbb{C}}X).$$

ex) $n=1$. Coord $z = x+iy$, hermitian metric $g_{z\bar z} > 0$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

$$g(\partial_x, \partial_x) = g(\partial_z + \partial_{\bar z}, \partial_z + \partial_{\bar z}) = 2 g_{z\bar z}.$$

• Hermitian form: $\omega \overset{loc}{=} i\, g_{\mu\bar\nu}\, dz^\mu \wedge dz^{\bar\nu} \in \Omega^{1,1}(X, \mathbb{R})$

Covariant Derivatives

$(X, g) \rightsquigarrow$ Natural ∇ on $T_{\mathbb{C}}X$?

Notation: $\nabla_i (V^\kappa \partial_\kappa) := (\nabla_i V^\kappa)\partial_\kappa$, $V \in \Gamma(T_{\mathbb{C}}X)$

$$\nabla_i V^k = \partial_i V^k + \Gamma_i{}^k{}_j V^j$$

1. Levi-Civita connection

$$\Gamma^{LC}{}_i{}^k{}_j = \frac{g^{k\ell}}{2}(-\partial_\ell g_{ij} + \partial_i g_{\ell j} + \partial_j g_{\ell i})$$

Exercise:

$$\Gamma^{LC}{}_{\bar\mu}{}^{\bar\alpha}{}_\beta = \frac{1}{2}(i\partial\omega)_{\bar\mu}{}^{\bar\alpha}{}_\beta .$$

$A_\mu{}^\alpha{}_{\bar\beta} = g^{\alpha\bar\sigma} A_{\mu\bar\sigma\bar\beta}$

$g^{\alpha\bar\sigma} g_{\bar\sigma\beta} = \delta^\alpha{}_\beta$

This means:
Let $V \in \Gamma(T^{1,0}X)$. Differentiate:

$$\nabla_{\bar\mu} V^{\bar\alpha} = \underset{=0}{\partial_{\bar\mu} V^{\bar\alpha}} + \Gamma_{\bar\mu}{}^{\bar\alpha}{}_\beta V^\beta + \underset{=0}{\Gamma_{\bar\mu}{}^{\bar\alpha}{}_{\bar\beta} V^{\bar\beta}} \quad \text{since } V \in T^{1,0}X$$

$$= \frac{1}{2}(i\partial\omega)_{\bar\mu}{}^{\bar\alpha}{}_\beta V^\beta .$$

$$\Rightarrow \nabla_{\bar\mu} V = \underset{\in T^{1,0}X}{\nabla_{\bar\mu} V^\alpha \frac{\partial}{\partial z^\alpha}} + \underset{\in T^{0,1}X}{\nabla_{\bar\mu} V^{\bar\alpha} \frac{\partial}{\partial z^{\bar\alpha}}} .$$

∇^{LC} does not preserve $T^{1,0}X$. Not good.

2. Chern Connection

$\exists! \, \nabla$ on $T^{1,0}X$ s.t.

① $\nabla_{\bar\mu} V^\alpha = \partial_{\bar\mu} V^\alpha$ $\quad (\Gamma_{\bar\mu}{}^\alpha{}_\beta = 0)$

② $\nabla_\mu g_{\alpha\bar\beta} = 0.$ $\quad (\partial_\mu \langle V, W\rangle = \langle \nabla_\mu V, W\rangle + \langle V, \nabla_{\bar\mu} W\rangle)$

From $(T^{1,0}X, \nabla)$, can induce ∇ on $T_{\mathbb{C}}X$ by: $\nabla_i V^{\bar\alpha} := \overline{\nabla_{i} V^\alpha}$.

Proof of Chern connection: expand ②:

$$\partial_\mu g_{\alpha\bar\beta} - \Gamma_\mu{}^\lambda{}_\alpha g_{\lambda\bar\beta} = 0.$$

e.g. $\Gamma_{\bar\mu}{}^{\bar\alpha}{}_{\bar\beta} = \overline{\Gamma_\mu{}^\alpha{}_\beta}$

$$\Rightarrow \boxed{\Gamma_\mu{}^{\lambda}{}_{\alpha} = g^{\lambda\bar\sigma}\partial_\mu g_{\alpha\bar\sigma}}$$

Formula for Chern connection

$$\Gamma_\mu{}^{\bar\lambda}{}_{\alpha} = 0$$

$$\Gamma_\mu{}^{\lambda}{}_{\bar\alpha} = 0$$

$$\Gamma_\mu{}^{\bar\lambda}{}_{\bar\alpha} = 0$$

All others obtained by conjugation.

3. "H-connection" (Yano, Strominger, Bismut)

$\exists!\, H \in \Omega^3(X)$ with ansatz

$$\hat\nabla = \nabla^{LC} - \frac{1}{2} g^{-1} H$$

$$\hat\Gamma_i{}^{k}{}_{j} = \Gamma^{LC}{}_i{}^{k}{}_{j} - \frac{1}{2} H_i{}^{k}{}_{j}$$

which preserves $T^{1,0}X$.

Note: $\hat\nabla g = 0$ is automatic by the ansatz.

Need:

$$(*)\ \begin{cases} \hat\Gamma_\mu{}^{\bar\nu}{}_{\rho} = 0 \\ \hat\Gamma_{\bar\mu}{}^{\bar\nu}{}_{\rho} = 0 . \end{cases}$$

Exercise:

1) Compute

$$\Gamma^{LC}{}_\mu{}^{\bar\nu}{}_{\rho} = 0$$

$$\Gamma^{LC}{}_\mu{}^{\nu}{}_{\bar\rho} = -\frac{1}{2}(i\bar\partial\omega)_\mu{}^{\nu}{}_{\bar\rho}$$

take conj for $\Gamma_{\bar\mu}{}^{\bar\nu}{}_{\rho}$

2) Solve for H in $(*)$ $\Rightarrow$ $\boxed{H = i(\partial - \bar\partial)\omega}$

Remark: all 3 connections agree when ω is Kähler.

$$\nabla^{LC} = \nabla^{chern} = \hat{\nabla} \iff d\omega = 0$$

Curvature tensor: from a covariant derivative ∇ on $T_{\mathbb{C}}X$, we define:

$$R \in \Omega^2(\text{End}\, T_{\mathbb{C}}X)$$

$$\underbrace{R_{pq}}_{\text{2-form}}{}^{\underbrace{i}_{}}{}_{\underbrace{j}_{\text{End}\,T}} = \partial_p \Gamma_q{}^i{}_j + \Gamma_p{}^i{}_\ell \Gamma_q{}^\ell{}_j - (p \leftrightarrow q)$$

Exercise: Compute curvature of Chern connection.

$$R_{\mu\bar{\nu}}{}^\alpha{}_\beta = -\partial_{\bar{\nu}} \Gamma_\mu{}^\alpha{}_\beta$$

$$= -\partial_{\bar{\nu}}\left(g^{\alpha\bar{\sigma}} \partial_\mu g_{\beta\bar{\sigma}}\right)$$

$$R = \bar{\partial}(g^{-1}\partial g)$$

$$\in \Omega^{1,1}(\text{End}\, T^{1,0}X) \quad \text{or} \quad \Omega^{1,1}(\text{End}\, T_{\mathbb{C}}X)$$

Other exercises:

- Compute $R \in \Omega^2(\text{End}\, T_{\mathbb{C}}X)$ for:
  (a) Levi-Civita connection
  (b) H-connection

- G = complex Lie group as before with:
  $[e_i, e_j] = c^k{}_{ij}\, e_k$, $\quad \omega = i \sum e^k \wedge \overline{e^k}$

Compute R^{LC}, R^{ch}, R^H in terms of $c^k{}_{ij}$.

II. Non-Kähler Calabi-Yau Geometry

- Let X be compact complex manifold with $\dim_{\mathbb{C}} X = n$ admitting a holomorphic volume form:

$$\Omega \in \Lambda^{n,0}(X, \mathbb{C}), \quad d\Omega = 0,$$

$$\Omega \overset{loc}{=} f(z)\, dz^1 \wedge \cdots \wedge dz^n, \quad f \text{ local nowhere vanishing hol'c fun'c}$$

- Terminology: for a hermitian metric ω, we say:

(a) ω is Kähler if $d\omega = 0$

$d\omega^{n-1} = 0 \Leftrightarrow$ (b) ω is balanced if $d * \omega = 0$

(c) ω is conformally balanced if $d(|\Omega|_\omega \omega^{n-1}) = 0$

The norm of Ω: $|\Omega|^2_\omega \overset{loc}{=} \dfrac{f\bar{f}}{\det g_{\mu\bar{\nu}}}$.

cond found by Strominger in string theory

☆ **claim:** $d(|\Omega|_\omega \omega^{n-1}) = 0 \iff \hat{\nabla}\left(\dfrac{\Omega}{|\Omega|_\omega}\right) = 0.$

Proof of claim: later.

note: can go between $d\tilde{\omega}^{n-1} = 0$ / $d(|\Omega|_\omega \omega^{n-1}) = 0$ by $\tilde{\omega} = e^u \omega$.

note: Kähler CY geometry is conformally balanced.

e.g. $X = \{\sum_{i=0}^{4} z_i^5 = 0\} \subseteq \mathbb{P}^4$ is Kähler CY 3.

Yau's thm $\Rightarrow \exists\, \omega_{cy}$ Kähler Ricci-flat.

(Exercise: (Kähler 1933) Show if ω Kähler, then

$$Ric_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \log |\Omega|^2_\omega$$

where $Ric_{p\bar{j}} = -R_{pm}{}^{m}{}_{\bar{j}}$.)

$\therefore$ ω_{cy} Kähler Ricci-flat solves $\Delta \log |\Omega|^2_\omega = 0$.

$\Rightarrow$ $|\Omega|_{\omega_{cy}} \equiv$ const and $d(|\Omega|_{\omega_{cy}} \, \omega_{cy}^{\,n-1}) = 0$.

(max princ exercise)

So for Kähler CY, $|\Omega|_\omega$ is constant.
For non-Kähler metrics, $|\Omega|_\omega$ may fluctuate.

note: non-Kähler metrics with $\hat\nabla\left(\frac{\Omega}{|\Omega|_\omega}\right) = 0$ are not Ricci-flat.

Correct eqn from string theory is:

(*) $$R_{mn} + 2\nabla_m \nabla_n \Phi - \frac{1}{4} H_{mpq} H_n{}^{pq} = \frac{1}{2} (dH)^p{}_{pmn}$$

on (M,g) Riem mfd where: $H \in \Omega^3(M)$
Φ scalar function

In complex geometry, take:

$H = i(\partial - \bar\partial)\omega$ 3-form field
$\Phi = -\frac{1}{2}\log|\Omega|_\omega$ dilaton function

Kähler CY soln to (*): set $H \equiv 0$, $\Phi \equiv$ const
Non-Kähler CY soln to (*): need $\hat\nabla\left(\frac{\Omega}{|\Omega|_\omega}\right) = 0$.

note: If $\hat\nabla\left(\frac{\Omega}{|\Omega|_\omega}\right) = 0$,

then: $\hat R_{ij}{}^{\mu}{}_{\mu} = 0$ ($\hat\nabla$ connection is "Ricci-flat")

Proof of claim ☆

(1) Recall $\hat\nabla$ connection:

$$\hat\nabla_\mu V^\alpha = \partial_\mu V^\alpha + (g^{\alpha\bar\sigma}\partial_\mu g_{\beta\bar\sigma}) V^\beta - H_\mu{}^\alpha{}_\lambda V^\lambda$$

$$\hat\nabla_{\bar\mu} V^\alpha = \partial_{\bar\mu} V^\alpha - H_{\bar\mu}{}^\alpha{}_\beta V^\beta$$

$\nabla = d + A$ on $E^* \rightsquigarrow -A$

$\nabla = d + A$ on $\det E \rightsquigarrow \mathrm{Tr} A$

(2) Action on $(n,0)$-forms: $\Psi \in \Lambda^{n,0}(X)$ (negative trace connection)

$$\hat{\nabla}_\mu \Psi = \partial_\mu \Psi - (g^{\alpha\bar{\sigma}} \partial_\mu g_{\alpha\bar{\sigma}}) \Psi + H_\mu{}^\alpha{}_\alpha \Psi$$

$$\hat{\nabla}_{\bar{\mu}} \Psi = \partial_{\bar{\mu}} \Psi + H_{\bar{\mu}}{}^\alpha{}_\alpha \Psi$$

(3) Derivative of determinant:

$$\partial_\mu \log |\Omega|^2_\omega = \partial_\mu \log \frac{|f|^2}{\det g}$$

$$= \frac{\partial_\mu f}{f} - (g^{\alpha\bar{\sigma}} \partial_\mu g_{\alpha\bar{\sigma}})$$

(4) $\hat{\nabla}_\mu \Omega = (\partial_\mu \log |\Omega|^2_\omega + H_\mu{}^\alpha{}_\alpha)\, \Omega$ from (2)+(3)

$$\hat{\nabla}_{\bar{\mu}} \Omega = (H_{\bar{\mu}}{}^\alpha{}_\alpha)\, \Omega$$

$$\hat{\nabla}_\mu \left(\frac{\Omega}{|\Omega|_\omega} \right) = \frac{1}{|\Omega|_\omega} \left(\hat{\nabla}_\mu \Omega - \partial_\mu \log |\Omega|_\omega \Omega \right)$$

$$= \frac{1}{|\Omega|_\omega} \left(\partial_\mu \log |\Omega|_\omega + H_\mu{}^\alpha{}_\alpha \right) \Omega$$

$$\hat{\nabla}_{\bar{\mu}} \left(\frac{\Omega}{|\Omega|_\omega} \right) = \frac{1}{|\Omega|_\omega} \left(-\partial_{\bar{\mu}} \log |\Omega|_\omega + H_{\bar{\mu}}{}^\alpha{}_\alpha \right) \Omega$$

(5) $d(|\Omega|_\omega \omega^{n-1}) = 0$

$\Leftrightarrow$ Exercise $\begin{cases} H_\mu{}^\alpha{}_\alpha = -\partial_\mu \log |\Omega|_\omega \\ H_{\bar{\mu}}{}^\alpha{}_\alpha = \partial_{\bar{\mu}} \log |\Omega|_\omega \end{cases}$ where as before $H = i(\partial - \bar{\partial})\omega$.

(4) + (5) prove the claim.

ex) $X = SL(2,\mathbb{C})/\Lambda$ compact quotient

Take left-invariant basis $\{e_a\}$ of $sl(2,\mathbb{C})$.

e.g.

$e_1 = X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad [H,X] = 2X \quad$ $[e_i, e_j] = c^k{}_{ij} e_k$

$e_2 = Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad [H,Y] = -2Y \quad$ $c^1{}_{31} = 2$

$e_3 = H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad [X,Y] = H \quad$ $c^2{}_{32} = -2$, $c^3{}_{12} = 1$

$\omega = \sum i e^k \wedge \overline{e^k}$ left-invariant metric

$\Omega = e^1 \wedge e^2 \wedge e^3$ hol'c with $|\Omega|_\omega = 1$.

Last lecture: $i \bar{\partial} \omega = \frac{1}{2} c^a{}_{bd}\, e^a \wedge \overline{e^d} \wedge \overline{e^b}$.

Exercise: $d\omega^2 = 0 \iff \sum_p c^p{}_{ip} = 0 \;\; \forall i.$

✓ true for $SL(2,\mathbb{C})$

ex) Iwasawa mfd: $(a,b,c) \in \mathbb{Z}[i] \circlearrowright \mathbb{C}^3$ by

$(x,y,z) \mapsto (x+a,\, y+c,\, z+\bar{a}y+b)$

$X = \mathbb{C}^3/\sim$

$\pi : X \xrightarrow{T^2} T^4 = \mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$

$\pi(x,y,z) = (x,y)$

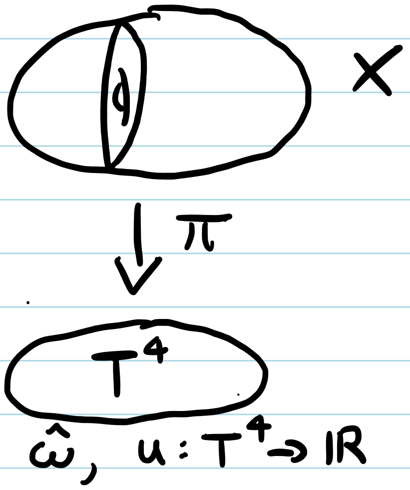
$X = T^2$ fibration over CY2

$\Omega = dz \wedge dx \wedge dy$ hol'c volume form

$$\omega = e^{u}\hat{\omega} + i\,\theta\wedge\bar{\theta}$$

$$\hat{\omega} = i\,dx\wedge d\bar{x} + i\,dy\wedge d\bar{y}$$
$$\theta = dz - \bar{x}\,dy$$

$$u : T^4 \to \mathbb{R}$$



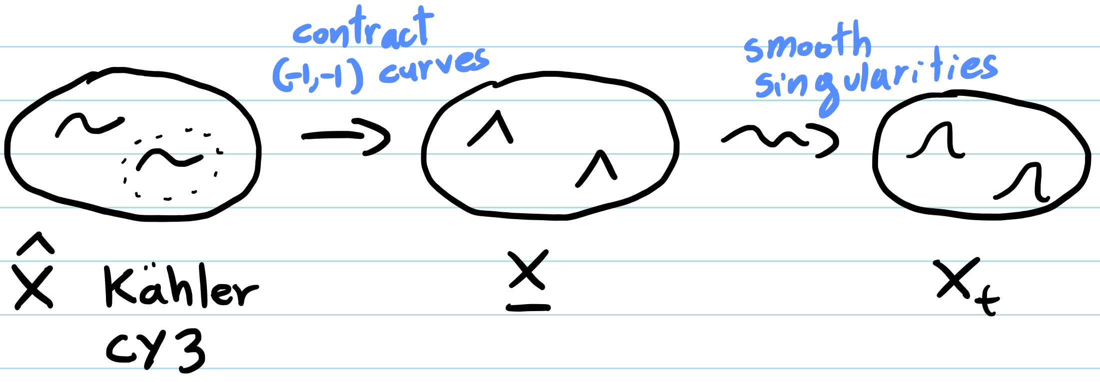
Exercise:

- $|\Omega|_{\omega} = e^{-u}$
- $d(|\Omega|_{\omega}\,\omega^2) = 0$
- X does not admit any Kähler metric:

hint: show $i\partial\bar{\partial}\omega_0 = \frac{\hat{\omega}^2}{2}$ where $\omega_0 = \hat{\omega} + i\,\theta\wedge\bar{\theta}$,

and consider: $\int_X i\partial\bar{\partial}\omega_0 \wedge \omega_{Kah}$.

ex) Fu-Li-Yau: $\hat{X} \to \underline{X} \rightsquigarrow X_t$ conifold transition.



X_t may or may not support a Kähler metric, but:

$$\exists\,(X_t, \omega_t, \Omega_t) \text{ with: } d\Omega_t = 0$$
$$d(|\Omega_t|_{\omega_t}\,\omega_t^2) = 0.$$

III. Strominger System

X cpt cplx mfd $\dim_{\mathbb{C}} X = 3$

Ω hol'c vol form.

ω hermitian metric solving

$$\begin{cases} d(|\Omega|_\omega \, \omega^2) = 0 \\ i\partial\bar\partial\omega = \alpha'(\mathrm{Tr}\, F\wedge F - \mathrm{Tr}\, R\wedge R) \end{cases}$$

where: $R = \bar\partial(g^{-1}\partial g)$ chern curvature

$F = \bar\partial(h^{-1}\partial h)$ curv of HYM metric h on hol'c bundle $E \to (X, \omega)$

$\alpha' > 0$

↑ can be arbitrary by rescaling $\omega \mapsto \lambda\omega$

Note: $ch_2(E) = ch_2(X)$. Role of bundle E is to cancel $ch_2(X)$ so can solve eqn when $ch_2(X) \neq 0$.

If $ch_2(X) = 0$, can take E = trivial and try to solve

$$i\partial\bar\partial\omega = -\alpha' \,\mathrm{Tr}\, R\wedge R.$$

Exercise: when $\alpha' = 0$:

Fino-Grantcharov

$$\begin{cases} d(|\Omega|_\omega \, \omega^{n-1}) = 0 \\ i\partial\bar\partial\omega = 0 \end{cases} \Rightarrow \omega \text{ Kähler Ricci-flat}$$

Outline:

1) Compute (general non-Kähler identity)

$$(i\partial\bar\partial\omega)_\mu{}^{\mu\rho}{}_\rho = -R^{ch}{}_\mu{}^{\mu\rho}{}_\rho - \hat{R}_\mu{}^{\mu\rho}{}_\rho + |i\partial\omega|^2$$

2) Show $\hat{R}_{\alpha\bar\beta}{}^\rho{}_\rho = 0$ for conf bal metrics

3) $g^{\mu\bar\gamma}\partial_\mu\partial_{\bar\gamma} \log|\Omega|^2_\omega = |i\partial\omega|^2$ trace $i\partial\bar\partial\omega = 0$

4) Apply maximum principle

$d\omega = 0$ and $|\Omega|_\omega \equiv \text{const}.$

Interpretation as hol'c str

Bismut, Gualtieri, de la Ossa - Svanes, Mc Orist - P - Svanes

$$Q = T^{*(1,0)} \oplus \text{End}\,E \oplus T^{(1,0)}$$

$$\bar{D} : \Omega^{0,k}(Q) \to \Omega^{0,k+1}(Q)$$

$$\bar{D} = \begin{pmatrix} \bar\partial & \alpha' F^* & \mathcal{H} + \alpha'(R\cdot\nabla) \\ & \bar\partial & F \\ & & \bar\partial \end{pmatrix} \begin{pmatrix} T^{*(1,0)} \\ \text{End}\,E \\ T^{(1,0)} \end{pmatrix}$$

$$\bar{D}^2 \begin{pmatrix} z \\ a \\ V \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(-i\partial\bar\partial\omega + \frac{\alpha'}{2}(\text{Tr}\,F\wedge F - \text{Tr}\,R\wedge R)\right)_{\rho\bar\sigma\mu\bar\nu} dz^\rho \otimes dz^{\bar\sigma\bar\nu} \wedge V^\mu \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{D}^2 = 0 \iff i\partial\bar\partial\omega = \frac{\alpha'}{2}(\text{Tr}\,F\wedge F - \text{Tr}\,R\wedge R).$$

Note $(Q, \bar{D})$ is not a hol'c bundle since:

$$\bar{D}(fs) \neq \bar\partial f s + f \bar{D} s$$

$$\bar{D}(fs) = \bar\partial f s + f \bar{D} s + O(\alpha')$$

due to $\alpha'(R\cdot\nabla)$ term

Definitions: let $\begin{pmatrix} z \\ a \\ v \end{pmatrix} \in \Omega^{0,k}\begin{pmatrix} T^{*(1,0)} \\ \text{End}\,E \\ T^{(1,0)} \end{pmatrix}$ and define:

$$\mathcal{F}(V) = F_{\mu\bar\nu}\, d\bar z^{\bar\nu} \wedge V^\mu \quad \in \Omega^{0,k+1}(\text{End}\,E)$$

$$F^*(a) = \text{Tr}\, F_{\mu\bar\nu}\, dz^\mu \otimes dz^{\bar\nu} \wedge a \quad \in \Omega^{0,k+1}(T^{*(1,0)})$$

$$\mathcal{H}(V) = H_{\rho\bar{\nu}\mu}\, dz^{\rho} \otimes d\bar{z}^{\nu} \wedge V^{\mu} \quad \in \Omega^{0,k+1}(T^{*(1,0)})$$

$$(R\cdot\nabla)V = -\frac{1}{k!} R_{\rho\bar{\mu}}{}^{\sigma}{}_{\lambda}\, \hat{\nabla}_{\sigma} V^{\lambda}{}_{\bar{\alpha}_1\cdots\bar{\alpha}_k}\, dz^{\rho} \otimes d\bar{z}^{\bar{\mu}\bar{\alpha}_1\cdots\bar{\alpha}_k}$$

$\hat{\nabla}$ acts on bundle indices, not form-indices $\bar{\alpha}_i$

$\in \Omega^{0,k+1}(T^{*(1,0)})$

Instead of full calculation $\bar{D}^2 = 0$, we work out some simplified setups:

(A) $Q = \mathrm{End}\,E \oplus T^{1,0}$

$$\bar{D} = \begin{pmatrix} \bar{\partial} & F \\ 0 & \bar{\partial} \end{pmatrix}$$

Atiyah:

$H^{0,1}_{\bar{D}}(Q)$ = infinitesimal def of pair (X, E)

$\bar{D}^2 = 0$ follows from Bianchi: $\bar{\partial}F = 0$, $F \in \Omega^{1,1}(\mathrm{End}\,E)$.

$$\bar{D}^2 \begin{pmatrix} a \\ v \end{pmatrix} = \begin{pmatrix} \bar{\partial}(Fv) + F\bar{\partial}v \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a \\ v \end{pmatrix} \in \begin{pmatrix} \mathrm{End}\,E \\ T^{1,0} \end{pmatrix}$$

$$\bar{\partial}(FV) = \bar{\partial}(V^{\mu} F_{\mu\bar{\nu}}\, dz^{\bar{\nu}})$$

$$= \partial_{\bar{\alpha}} V^{\mu} F_{\mu\bar{\nu}}\, dz^{\bar{\alpha}} \wedge dz^{\bar{\nu}} + \underbrace{V^{\mu} \partial_{\bar{\alpha}} F_{\mu\bar{\nu}}\, dz^{\bar{\alpha}\bar{\nu}}}_{=0}$$

$$F(\bar{\partial}V) = F_{\mu\bar{\nu}}\, dz^{\bar{\nu}} \wedge (\partial_{\bar{\alpha}} V^{\mu}\, dz^{\bar{\alpha}})$$

$$\Rightarrow \bar{D}^2 \begin{pmatrix} a \\ v \end{pmatrix} = 0. \quad ✓$$

(B) $Q = T^{*(1,0)} \oplus \mathrm{End}\,E \oplus T^{(1,0)}$

$$\bar{D} = \begin{pmatrix} \bar{\partial} & F^{*} & \mathcal{H} \\ 0 & \bar{\partial} & F \\ 0 & 0 & \bar{\partial} \end{pmatrix} \begin{pmatrix} T^{*(1,0)} \\ \mathrm{End}\,E \\ T^{(1,0)} \end{pmatrix}$$

de la Ossa – Svanes

$$\bar{D} : \Omega^{0,k}(Q) \to \Omega^{0,k+1}(Q)$$

$$\bar{D}^2 \begin{pmatrix} z \\ a \\ V \end{pmatrix} = \begin{pmatrix} \bar{\partial}(\mathcal{H}V) + \mathcal{H}\bar{\partial}V + F^*FV \\ 0 \\ 0 \end{pmatrix}$$ follows from:

$$\begin{cases} \bar{\partial}(FV) + F\bar{\partial}V = 0 & \text{(checked before)} \\ \bar{\partial}(F^*a) + F^*\bar{\partial}a = 0. & \text{(similar)} \end{cases}$$

1. $\bar{\partial}(\mathcal{H}V) = \bar{\partial}(H_{\rho\bar{\nu}\mu} V^\mu dz^\rho \otimes dz^{\bar{\nu}})$

$$\mathcal{H}\bar{\partial}V = H_{\rho\bar{\nu}\mu}\, \partial_{\bar{\beta}} V^\mu dz^\rho \otimes dz^{\bar{\nu}} \wedge dz^{\bar{\beta}}$$

$$\Rightarrow \bar{\partial}(\mathcal{H}V) + \mathcal{H}\bar{\partial}V = \partial_{\bar{\beta}} H_{\rho\bar{\nu}\mu} V^\mu dz^\rho \otimes dz^{\bar{\beta}} \wedge dz^{\bar{\nu}}$$

$$H = i(\partial - \bar{\partial})\omega$$

$$\bar{\partial}(\mathcal{H}V) + \mathcal{H}\bar{\partial}V = \frac{1}{2}(-i\partial\bar{\partial}\omega)_{\bar{\beta}\rho\bar{\nu}\mu} V^\mu dz^\rho \otimes dz^{\bar{\beta}\bar{\nu}}$$

(Conventions: $\bar{\partial}H \overset{2,1}{=} \frac{1}{2!2!} (\bar{\partial}H)_{\alpha\beta\bar{\mu}\bar{\nu}} dz^{\alpha\beta} \wedge dz^{\bar{\mu}\bar{\nu}}$

$$H^{2,1} = \frac{1}{2!} H_{\alpha\beta\bar{\nu}} dz^{\alpha\beta} \wedge dz^{\bar{\nu}}$$

$$\bar{\partial}H^{2,1} = \frac{1}{2!} \partial_{\bar{\mu}} H_{\alpha\beta\bar{\nu}} dz^{\bar{\mu}} \wedge dz^{\alpha\beta\bar{\nu}}$$)

2. $F^*FV = \mathrm{Tr}\, F_{\rho\bar{\beta}} F_{\mu\bar{\nu}} V^\mu dz^\rho \otimes dz^{\bar{\beta}} \wedge dz^{\bar{\nu}}$

$$(\mathrm{Tr}\, F \wedge F)_{\rho\bar{\beta}\mu\bar{\nu}} = 2(\mathrm{Tr}\, F_{\rho\bar{\beta}} F_{\mu\bar{\nu}} - \mathrm{Tr}\, F_{\rho\bar{\nu}} F_{\mu\bar{\beta}})$$

$$F^*FV = \frac{1}{4}(\mathrm{Tr}\, F \wedge F)_{\rho\bar{\beta}\mu\bar{\nu}} V^\mu dz^\rho \otimes dz^{\bar{\beta}} \wedge dz^{\bar{\nu}}$$

3. $$\bar{D}^2 \begin{pmatrix} z \\ a \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(-i\partial\bar{\partial}\omega + \frac{1}{2}\mathrm{Tr} F\wedge F\right)_{\rho\bar{\beta}\mu\bar{\gamma}} v^{\mu}\, dz^{\rho} \otimes dz^{\bar{\beta}} \wedge dz^{\bar{\nu}} \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{D}^2 = 0 \iff i\partial\bar{\partial}\omega = \frac{1}{2}\mathrm{Tr}\, F\wedge F.$$

Thus: soln to $\rightsquigarrow$ $(Q, \bar{D}) \to X$

$i\partial\bar{\partial}\omega = \alpha'(\mathrm{Tr} F\wedge F - \mathrm{Tr} R\wedge R)$ $\qquad \bar{D}^2 = 0$

$$\Big\Downarrow$$

$$0 \to \Gamma(Q) \xrightarrow{\bar{D}} \Omega^{0,1}(Q) \xrightarrow{\bar{D}} \Omega^{0,2}(Q) \xrightarrow{\bar{D}} \Omega^{0,3}(Q) \to 0$$

differential complex. Exercise: elliptic complex ✓

$$\Big\Downarrow$$

significance of cohomology $\qquad H^{0,k}_{\bar{D}}(Q)$?

Argument from string theory $\qquad$ McOrist - Svanes

$H^{0,1}_{\bar{D}}(Q)$ = infinitesimal deformations of Strominger system

IV Aeppli and Bott-Chern classes

- X cpt cplx mfd.

Aeppli cohomology:

$$H_A^{p,q} = \frac{\ker \partial\bar\partial \cap \Omega^{p,q}}{\operatorname{Im} \partial \oplus \bar\partial}$$

Bott-Chern cohomology:

$$H_{BC}^{p,q} = \frac{\ker d \cap \Omega^{p,q}}{\operatorname{Im} \partial\bar\partial}$$

Dolbeault cohomology:

$$H_{\bar\partial}^{p,q} = \frac{\ker \bar\partial \cap \Omega^{p,q}}{\operatorname{Im} \bar\partial}$$

Note: If X is Kähler (just need $\partial\bar\partial$-lemma), then

$$H_A^{p,q} \cong H_{BC}^{p,q} \cong H_{\bar\partial}^{p,q}.$$

true for Kähler mfd

$\partial\bar\partial$-lemma: X satisfies the $\partial\bar\partial$-lemma if there holds:

Suppose $\eta \in \Omega^{p,q}$ with $d\eta = 0$. TFAE:

1. $\eta = d\alpha$
2. $\eta = \partial\beta$
3. $\eta = \bar\partial\gamma$
4. $\eta = \partial\bar\partial\chi$.

Lem: If X satisfies $\partial\bar\partial$-lem, then:

$$H_{BC}^{p,q} \to H_{\bar\partial}^{p,q} \qquad \text{isomorphism.}$$
$$[\eta]_{BC} \mapsto [\eta]_{\bar\partial}$$

injective: exercise

surjective: Let $[\eta]_{\bar\partial} \in H_{\bar\partial}^{p,q}$. Solve $\partial\eta = \partial\bar\partial x$.

$$[\eta]_{\bar\partial} = [\eta - \bar\partial x]_{\bar\partial}$$

$$[\eta - \bar\partial x]_{BC} \mapsto [\eta - \bar\partial x]_{\bar\partial}$$ ✓

Lem: If X satisfies $\partial\bar\partial$-lem, then

$$H_{\bar\partial}^{p,q} \to H_A^{p,q}$$
$$[\eta]_{\bar\partial} \mapsto [\eta]_A$$
is an isomorphism.

Pf: injective: if $\bar\partial\eta_1 = 0$, $\bar\partial\eta_2 = 0$, $\eta_1 = \eta_2 + \partial\alpha_1 + \bar\partial\alpha_2$

then: $\bar\partial\partial\alpha_1 = 0 \Rightarrow \partial\alpha_1 = \partial\bar\partial x \Rightarrow [\eta_1]_{\bar\partial} = [\eta_2]_{\bar\partial}$.

surjective: exercise. ✓

Note: In general H_A, H_{BC}, $H_{\bar\partial}$ are all different, though there is a Poincaré duality

$$H_A^{p,q} \times H_{BC}^{n-p,n-q} \to \mathbb{C}$$
$$[\alpha]_A \quad [\beta]_{BC} \mapsto \int_X \alpha \wedge \beta$$

$$\Rightarrow H_A^{p,q} \cong (H_{BC}^{n-p,n-q})^*.$$

Back to Strominger system :

$$d(|\Omega|_\omega \omega^2) \rightsquigarrow \begin{cases} \underline{b} \in H^{2,2}_{BC}(X) \\ \underline{b} = [|\Omega|_\omega \omega^2] \end{cases}$$

$$i\partial\bar\partial\omega = \alpha'(\mathrm{Tr}\, F\wedge F - \mathrm{Tr}\, R\wedge R)$$

$$\rightsquigarrow \begin{cases} \underline{a} \in H^{1,1}_A(X) \\ \underline{a} = [\omega - \alpha' R_2[h,\hat h] - \alpha' R_2[g,\hat g] - \alpha'\hat\beta] \end{cases}$$

(Compare with Kähler CY:

$d\omega = 0 \rightsquigarrow [\omega] \in H^{1,1}(X)$

Yau's Thm: $\exists !\ \omega_{CY} \in [\omega]$.)

On a non-Kähler CY,
Notion of Kähler class breaks into two:

(A) Can look for soln in given $\underline{a} \in H^{1,1}_A(X)$

(B) Can look for soln in given $\underline{b} \in H^{2,2}_{BC}(X)$

Both approaches have been pursued in the litterature.

How to define Aeppli class [a] ?

1. Choose reference metrics $(\hat g, \hat h)$ on $T^{1,0} \times E$.

Solve $E(\hat\gamma) \overset{(*)}{=} \mathrm{Tr}\, \hat F\wedge\hat F - \mathrm{Tr}\, \hat R\wedge\hat R$.

$\hat F = \bar\partial(\hat h^{-1}\partial\hat h)$
$\hat R = \bar\partial(\hat g^{-1}\partial\hat g)$

Here:

$$E = (\partial\bar\partial)(\partial\bar\partial)^\dagger + (\partial\bar\partial)^\dagger(\partial\bar\partial) + (\partial^\dagger\bar\partial)^\dagger\partial^\dagger\bar\partial + (\partial^\dagger\bar\partial)(\partial^\dagger\bar\partial)^\dagger + \bar\partial^\dagger\bar\partial + \partial^\dagger\partial.$$

E = Kodaira-Spencer operator.
= 4th order elliptic operator

$\partial^\dagger$ L^2-adjoint wrt g.

Exercise: $\operatorname{Ker} E = \operatorname{Ker} d \cap \operatorname{Ker} (\partial\bar\partial)^\dagger$

Since $c_2^{BC}(X) = c_2^{BC}(E)$,

$\operatorname{Tr} \hat F \wedge \hat F - \operatorname{Tr} \hat R \wedge \hat R \in \operatorname{Im} \partial\bar\partial$.

Fredholm alternative => can solve $(*)$

2. If $E(\hat\gamma) = i\partial\bar\partial\eta$, then $d\hat\gamma = 0$.
(Exercise)

3. Define $\hat\beta = \frac{1}{i}(\partial\bar\partial)^\dagger \hat\gamma$. $\quad i\partial\bar\partial\hat\beta = E(\hat\gamma)$.

$$i\partial\bar\partial\hat\beta = \operatorname{Tr} \hat F \wedge \hat F - \operatorname{Tr} \hat R \wedge \hat R. \quad (a)$$

4. Solve
$$E(\gamma) = \operatorname{Tr} R \wedge R - \operatorname{Tr} \hat R \wedge \hat R$$

$R = \bar\partial(g^{-1}\partial g)$
$\hat R = \bar\partial(\hat g^{-1}\partial \hat g)$

Define $R[g,\hat g] = \frac{1}{i}(\partial\bar\partial)^\dagger \gamma$.

$$i\partial\bar\partial R[g,\hat g] = \operatorname{Tr} R \wedge R - \operatorname{Tr} \hat R \wedge \hat R \quad (b)$$

5. Define $R[h,\hat h]$ similarly.

$$i\partial\bar\partial R[h,\hat h] = \operatorname{Tr} F \wedge F - \operatorname{Tr} \hat F \wedge \hat F. \quad (c)$$

6. From (a, b, c)

$$i\partial\bar\partial\left(\omega - \alpha' R[h,\hat h] + \alpha' R[g,\hat g] - \alpha'\hat\beta\right) = 0.$$

7. Class independent of choice of ref $(\hat g, \hat h)$.
Take a pair $(\hat g_1, \hat h_1)$, $(\hat g_2, \hat h_2)$.

$$\begin{array}{llll} - R[h, \hat{h}_1] & + R[g, \hat{g}_1] & - \hat{\beta}_1 & \\ + R[h, \hat{h}_2] & - R[g, \hat{g}_2] & + \hat{\beta}_2 & \end{array} = (\partial\bar{\partial})^{\dagger} \underbrace{(\cdots)}_{:= \Psi}$$

Note

$$\begin{aligned} \partial\bar{\partial}\, (\partial\bar{\partial})^{\dagger} \Psi = & - \mathrm{Tr}\, F \wedge F + \mathrm{Tr}\, \hat{F}_1 \wedge \hat{F}_1 + \mathrm{Tr}\, R \wedge R - \mathrm{Tr}\, \hat{R}_1 \wedge \hat{R}_1 \\ & - \mathrm{Tr}\, \hat{F}_1 \wedge \hat{F}_1 + \mathrm{Tr}\, \hat{R}_1 \wedge \hat{R}_1 + \mathrm{Tr}\, F \wedge F - \mathrm{Tr}\, \hat{F}_2 \wedge \hat{F}_2 \\ & - \mathrm{Tr}\, R \wedge R + \mathrm{Tr}\, \hat{R}_2 \wedge \hat{R}_2 + \mathrm{Tr}\, \hat{F}_2 \wedge \hat{F}_2 - \mathrm{Tr}\, \hat{R}_2 \wedge R_2 \end{aligned}$$

$$= 0$$

$$\Rightarrow \langle \partial\bar{\partial} (\partial\bar{\partial})^{\dagger} \Psi , \Psi \rangle = 0$$

$$\Rightarrow (\partial\bar{\partial})^{\dagger} \Psi = 0$$

$\Rightarrow R[h, \hat{h}] - R[g, \hat{g}] + \hat{\beta} \in \Omega^{1,1}$ indep of reference $(\hat{g}, \hat{h})$.

V Bott-Chern balanced perspective

Yau conj:
- X cplx cpt mfd $\dim_{\mathbb{C}} X = 3$
- X admits hol'c volume form Ω
- X admits conformally balanced ω_0
- $E \to (X, |\Omega|_{\omega_0} \omega_0^2)$ stable hol'c bundle with $c_1(E) = 0$
- $c_2^{BC}(X) = c_2^{BC}(E)$

Then $\forall \alpha' > 0$ small, $\exists$ soln (ω, h) to:

$$\begin{cases} d(|\Omega|_\omega \omega^2) = 0 \\ F_h \wedge \omega^2 = 0 \\ i\partial\bar\partial\omega = \alpha'(\mathrm{Tr}\, F_h \wedge F_h - \mathrm{Tr}\, R \wedge R) \\ [|\Omega|_\omega \omega^2]_{BC} = [|\Omega|_{\omega_0} \omega_0^2]_{BC} \end{cases}$$

Let's give an outline of proof in the special case

$[|\Omega|_{\omega_0} \omega_0^2] = [\omega_{cy}^2]$. (square of a Kähler class)

(General conj still open)

Setup in Kähler case: $\hat\omega$ Kähler-Ricci flat.

$E \to (X, \hat\omega)$ stable bundle, $c_1(E) = 0$.

Donaldson-Uhlenbeck-Yau: $\exists \hat h$ s.t. $F_{\hat h} \wedge \hat\omega^2 = 0$.

Deformation of $(\hat{h}, \omega_{cy})$: consider $\left(e^{u}\hat{h},\ \omega_{\Theta}\right)$, where:

a) $u \in H_0(E) = \left\{ u \in \Gamma(\mathrm{End}\,E) : u^{\dagger} = u,\ \mathrm{tr}\, u = 0 \right\}$

b) $|\Omega|_{\omega_\Theta}\ \omega_\Theta^2 = |\Omega|_{\hat{\omega}}\, \hat{\omega}^2 + \Theta, \qquad \Theta \in \mathcal{U}$

$$\mathcal{U} = \left\{ \Theta \in \Omega^{2,2} \ \text{ s.t. } \ \begin{matrix} \Theta = i\partial\bar{\partial}\beta \\ |\Omega|_{\hat{\omega}}\, \hat{\omega}^2 + \Theta > 0 \end{matrix} \right\}.$$

Define:

$$F : \mathbb{R} \times H_0(E) \times \mathcal{U} \to W$$

$$F(\alpha', u, \Theta) = \begin{pmatrix} e^{-u/2}\, \omega_\Theta^2 \wedge i F_u\, e^{u/2} \\ \\ i\partial\bar{\partial}\omega_\Theta - \alpha'\left(\mathrm{Tr}\, F_h \wedge F_h - \mathrm{Tr}\, R_\Theta \wedge R_\Theta\right) \end{pmatrix}$$

note: Kähler CY solves $F(0,0,0) = 0$.

want: $F(\alpha', u_{\alpha'}, \Theta_{\alpha'}) = 0 \quad \forall\, \alpha' \in (-\varepsilon, \varepsilon)$.

Compute linearization and use IFT:

$F(\overbrace{\alpha'}^{1}, \overbrace{u, \Theta}^{2})$. Derivative turns out to be:

$$D_2 F\big|_0 (\dot{u}, \dot{\Theta}) = \begin{pmatrix} L_1 & A \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{\Theta} \end{pmatrix}$$

$$L_1(\dot{u}) = -\hat{g}^{\mu\bar{\nu}} \nabla_{\bar{\nu}} \nabla_\mu \dot{u} \otimes \frac{\hat{\omega}^3}{3!}$$ linearization of HYM

$$L_2(\dot{\Theta}) = -\frac{1}{2|\Omega|_{\hat{\omega}}} \Delta_{\hat{\omega}} \dot{\Theta}$$ Laplacian on Kähler mfd

Can show L_1, L_2 are invertible in suitable spaces.

$\Rightarrow D_2 F|_0$ is invertible.

Implicit function thm $\Rightarrow \exists$ path of soln

$$F(\alpha', u_{\alpha'}, \theta_{\alpha'}) = 0$$

near $(0,0,0)$.

Full details: see Collins-Picard-Yau

(also earlier related work by: Li-Yau, Andreas-Garcia-Fernandez)