Differential Geometry I: Worksheet 1

- **Problem:** (Stereographic Projection) Let $S^2 \subset \mathbb{R}^3$ be the 2-sphere.

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ denote the north and south poles. Define

$$U = S^2 \backslash \{N\}, \quad \tilde{U} = S^2 \backslash \{S\}$$

and consider the open cover

$$S^2 = U \cup \tilde{U}.$$

Define local coordinates by

$$\varphi : U \to \mathbb{R}^2, \quad \varphi(x, y, z) = (u, v),$$

where $(u, v, 0)$ is the unique point where the line through (x, y, z) and N intersects the $\{z = 0\}$ plane.

We can also define

$$\tilde{\varphi} : \tilde{U} \to \mathbb{R}^2, \quad \tilde{\varphi}(x, y, z) = (\tilde{u}, \tilde{v}),$$

where $(\tilde{u}, \tilde{v}, 0)$ is the point of intersection of the plane $\{z = 0\}$ and the line through (x, y, z) and S.

Give explicit formulas for φ and $\tilde{\varphi}$, and compute the coordinate change $\tilde{\varphi} \circ \varphi^{-1}$.

- **Problem:** Let

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \backslash \{0\}) / \sim$$

where $\sim$ is the equivalence relation

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$$

if

$$(x_0, \ldots, x_n) = (\lambda y_0, \ldots, \lambda y_n)$$

for some $\lambda \in \mathbb{R} \backslash \{0\}$. The equivalence class of a point $x = (x_0, \ldots, x_n)$ will be denoted

$$[x_0, \ldots, x_n],$$

with square brackets. We can cover $\mathbb{RP}^n$ by the open sets

$$U_i = \{[x_0, \ldots, x_n] : x_i \neq 0\}.$$

Define coordinates $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i[x_0, \ldots, x_n] = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right) = (w^1, \ldots, w^n).$$

We then have two different coordinates charts on $U_0 \cap U_1$: let us denote these by (U_0, w) and $(U_1, \tilde{w})$. Compute the coordinate change formula $\tilde{w}^i = f^i(w)$ on $U_0 \cap U_1$.

- **Problem:** Let $G(2,4)$ be the set of all 2-dimensional subspaces of $\mathbb{R}^4$. A point V in $G(2,4)$ can be represented by a matrix

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix},$$

where $v = (v_1, v_2, v_3, v_4)$ and $w = (w_1, w_2, w_3, w_4)$ span $V \subset \mathbb{R}^4$. If $V \in G(2,4)$ is represented by a 2×4 matrix A, we will write $V = [A]$.

Let $V \in G(2,4)$ and $\epsilon > 0$. A topology on $G(2,4)$ is given by the open neighborhoods

$$\mathcal{O}_{V,\epsilon} = \left\{ V' \in G(2,4) : |\Pi_V(v') - v'| < \epsilon |v'| \right\}$$

for all $v' \neq 0$ in V'. Here $\Pi_V : \mathbb{R}^n \to V$ is the orthogonal projection.

(a) Suppose a 2×4 matrix A can be brought to the form

$$\begin{bmatrix} 1 & 0 & x_1 & x_3 \\ 0 & 1 & x_2 & x_4 \end{bmatrix}$$

by row operations. Show that such a matrix is unique, i.e. the x_1, x_2, x_3, x_4 are uniquely determined.

(b) Show that if A can be brought to the form above by row operations, then any other matrix whose coeffcients are close to the coefficients of A can also be brought to that form.

(c) Show that any 2×4 matrix of rank 2 can be brought to one of the following matrices

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & 0 \\ 0 & * & * & 1 \end{bmatrix},$$

$$\begin{bmatrix} * & 1 & 0 & * \\ * & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} * & 1 & * & 0 \\ * & 0 & * & 1 \end{bmatrix}, \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix},$$

by row operations. Conclude that any point $V \in G(2,4)$ can be represented by $V = [A]$, where A has the form of one of the matrices above.

(d) Cover $G(2,4)$ by the sets

$$U_1 = \left\{ V \in G(2,4) : V = [\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}] \right\},$$

$$U_2 = \left\{ V \in G(2,4) : V = [\begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{bmatrix}] \right\},$$

and so on for $U_3, \ldots, U_6$. By the argument in (b), these sets are open. Define coordinates $\varphi_1 : U_1 \to \mathbb{R}^4$ by

$$\varphi_1\left(\begin{bmatrix} 1 & 0 & x_1 & x_3 \\ 0 & 1 & x_2 & x_4 \end{bmatrix}\right) = (x_1, x_2, x_3, x_4),$$

$$\varphi_2\left(\begin{bmatrix} 1 & y_1 & 0 & y_3 \\ 0 & y_2 & 1 & y_4 \end{bmatrix}\right) = (y_1, y_2, y_3, y_4),$$

and similarly for the other (U_i, φ_i). Compute the coordinate change $\varphi_1 \circ \varphi_2^{-1}$.

A similar argument shows that all $\varphi_i \circ \varphi_j^{-1}$ are smooth functions, and hence $G(2,4)$ is given the structure of a smooth manifold. In fact, this argument can be generalized to the Grassmannian $G(k,n)$, which is the space of k-planes in $\mathbb{R}^n$.

- **Problem:** Let
$$S^1 = \{e^{i\theta} : \theta \in [0, 2\pi]\}.$$
Cover S^1 by the following two open sets:
$$U = \{e^{i\theta} : \theta \in (0, 2\pi)\}$$
$$\tilde{U} = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}.$$
Use the cover $S^1 = U \cup \tilde{U}$ to equip the tangent bundle TS^1 with coordinates. Compute the change of coordinates on TS^1, and conclude that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

- **Problem:** Equip S^2 with stereographic coordinates:
$$S^2 = U \cup \tilde{U}, \qquad U = S^2 \backslash \{N\}, \quad \tilde{U} = S^2 \backslash \{S\}$$
Let (u^1, u^2) denote stereographic coordinates on U and $(\tilde{u}^1, \tilde{u}^2)$ stereographic coordinates on $\tilde{U}$. In a previous homework, you computed the coordinate change
$$(\tilde{u}^1, \tilde{u}^2) = (f^1(u), f^2(u)).$$
Compute the change of coordinates on the tangent bundle TS^2. Recall that this is of the form:
$$(\tilde{u}^1, \tilde{u}^2, \tilde{q}^1, \tilde{q}^2) = \left(f^1(u), f^2(u), \frac{\partial \tilde{u}^1}{\partial u^i} q^i, \frac{\partial \tilde{u}^2}{\partial u^i} q^i\right).$$

- **Problem:** Show that the map $F : S^2 \to \mathbb{CP}^1$ given by
$$F(x,y,z) = \begin{cases} [x+iy, 1-z] & \text{if } z \neq 1, \\ [1,0] & \text{if } (x,y,z) = (0,0,1) \end{cases}$$
is a diffeomorphism.

- **Problem:** Consider the map $F : S^2 \to \mathbb{RP}^2$ given by
$$F(x_0, x_1, x_2) = [x_0, x_1, x_2].$$
Show that F is smooth. Is F surjective? Is F injective?

- **Problem:** Show that $F : S^1 \to \mathbb{RP}^1$ given by
$$F(e^{i\theta}) = [\cos(\theta/2), \sin(\theta/2)]$$
is a diffeomorphism.

- **Problem:** Let $F = (f, g) : \mathbb{R}^n \to \mathbb{R}^2$ be a smooth function. Let α be a regular value of f and let (α, β) be a regular value of F.

  (a) Let $Y = f^{-1}(\alpha)$ and $Z = F^{-1}(\alpha, \beta)$. Prove that $Z \subset Y$ is a submanifold of Y. Hint: to do this, let $p \in Z$, and you may choose coordinates near p such that

  $$f(x_1, \ldots, x_n) = x_1.$$

  Consider then $G : Y \to \mathbb{R}$ defined by $G = g|_Y$ and prove that p is a regular point of G.

  (b) Consider the function $h : \mathbb{R}^4 \to \mathbb{R}^2$ given by

  $$h(x, y, z, w) = (x^2 + y^2, z^2 + w^2).$$

  For any $\alpha \in (0, 1)$, show that $h^{-1}(\alpha, 1 - \alpha)$ is a 2-dimensional submanifold of S^3.

- **Problem:** Let $P : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree m. This means

  $$P(tx_1, \ldots, tx_k) = t^m P(x_1, \ldots, x_k).$$

  Assume m is an integer with $m \geq 2$.

  (a) Prove Euler's identity

  $$\sum_{i=1}^{k} x_i \frac{\partial P}{\partial x_i} = mP.$$

  Hint: consider $\frac{d}{dt}\big|_{t=1} P(tx) = \frac{d}{dt}\big|_{t=1} t^m P(x)$.

  (b) Let $a > 0$. Prove that

  $$X_a = \{x \in \mathbb{R}^k : P(x) = a\}$$

  is a $k - 1$ dimensional submanifold of $\mathbb{R}^k$.

  (c) Prove that X_a is diffeomorphic to X_1.

- **Problem:** Consider

  $$S = \{(e^{i\theta}, e^{2i\theta}) : \theta \in [0, 2\pi]\} \subset T^2$$

  where $T^2 = S^1 \times S^1$. Show that $S \subset T^2$ is a submanifold.