Differential Geometry I: Worksheet 2

- **Problem:** Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $g \in G$, let $C_g : G \to G$, defined by
$$C_g(h) = ghg^{-1},$$
denote the conjugacy map.

  Recall that a Lie group representation on a vector space V is a smooth map $\rho : G \to GL(V)$ with the property that $\rho(e) = I_n$ and $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$.

  (a) Show that $\mathrm{Ad} : G \to GL(\mathfrak{g})$ defined by $\mathrm{Ad}_g(x) = (dC_g)_e(x)$ is a Lie group representation.

  (b) Show that in the case of $G = GL(n, \mathbb{R})$, the adjoint representation is given by matrix conjugation.

  (c) For $x \in \mathfrak{g}$, let $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ be defined by $\mathrm{ad}_x = d(\mathrm{Ad})_e(x)$. For $G = GL(n, \mathbb{R})$, show that
$$\mathrm{ad}_x(y) = [x, y] = xy - yx.$$

- **Problem:**

  (a) Let σ_i denote the Pauli matrices
$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$
  Consider the map $\mathrm{Ad} : SU(2) \to GL(i\mathfrak{su}(2))$
$$\mathrm{Ad}_g(x) = gxg^{-1}.$$
  Identify $i\mathfrak{su}(2)$ with $\mathbb{R}^3$ by identifying the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ with the standard basis e_1, e_2, e_3. Show that the map φ which maps $g \in SU(2)$ to the matrix of Ad_g in the basis $\sigma_1, \sigma_2, \sigma_3$ is a homomorphism of Lie groups
$$\varphi : SU(2) \to SO(3).$$
  As a first step, show that
$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \cdot \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} = \frac{1}{2}\mathrm{Tr}\, xy$$
  where $x = x^i \sigma_i$ and $y = y^i \sigma_i$.

  (b) Show that the path
$$\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

in $SU(2)$, corresponds under φ, to the path

$$\begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in $SO(3)$. We see that a loop in $SU(2)$ corresponds to two full rotations in $SO(3)$.

(c) Compute the matrices $E_\alpha \in M_{3\times 3}(\mathbb{R})$ such that

$$d\varphi_e(\sigma_\alpha/2i) = E_\alpha, \quad \alpha = 1, 2, 3.$$

Note: the entries of each E_α should only be either 0, 1, or -1.

(d) Denote an element $g \in SU(2)$ by

$$g = \begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix}, \quad u, v \in \mathbb{C}, \quad |u|^2 + |v|^2 = 1.$$

Compute the matrix $\varphi(g) \in SO(3)$.

(e) Show that $\varphi : SU(2) \to SO(3)$ is a double cover. To do this, you can first compute the kernel

$$\ker \varphi = \{g \in SU(2) : \varphi(g) = I_{3\times 3}\},$$

and then deduce that φ is $2:1$. Next, to show surjectivity of φ, you can use that any open set of the identity in a Lie group generates the whole connected component containing the identity.

- **Problem:** Let X be the vector field on

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

defined by

$$X = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

Sketch this vector field and compute the coordinate expression for X using the stereographic projection. Recall that these coordinates are setup such that

$$S^1 = U \cup \tilde{U},$$

with $U = S^1 \backslash \{N\}$, $\tilde{U} = S^1 \backslash \{S\}$, and

$$\varphi : U \to \mathbb{R}, \quad u = \varphi(x, y) = \frac{x}{1-y}$$
$$\tilde{\varphi} : \tilde{U} \to \mathbb{R}, \quad \tilde{u} = \tilde{\varphi}(x, y) = \frac{x}{1+y}.$$

The change of coordinates is $\tilde{u} = 1/u$. The problem is then to compute the local expressions

$$X|_U = X(u)\frac{\partial}{\partial u}, \quad X|_{\tilde{U}} = \tilde{X}(\tilde{u})\frac{\partial}{\partial \tilde{u}}.$$

Compute $X|_U$, and then use $\tilde{X} = \frac{\partial \tilde{u}}{\partial u} X$ to compute $X|_{\tilde{U}}$.

- **Problem 2:** Find a flow $\theta_t : T^2 \to T^2$ on T^2 corresponding to the vector field

$$X = a\frac{\partial}{\partial \theta^1} + b\frac{\partial}{\partial \theta^2}$$

where a, b are constants. Here $T^2 = S^1 \times S^1$ and we write

$$S^1 = \{e^{i\theta} : 0 \le \theta \le 2\pi\}.$$

- **Problem:** Consider the family of maps

$$\hat{\theta}_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} : \mathbb{R}^3 \to \mathbb{R}^3.$$

(a) Prove that $\hat{\theta}_t$ induces diffeomorphisms $\theta_t : S^2 \to S^2$. To show smoothness of the maps, you can be brief. Smoothness follows immediately from the definition expanded in the hemisphere coordinate charts such as

$$U = \{x > 0\} \cap S^2, \quad \varphi(x, y, z) = (y, z)$$

and is routine to check.

(b) Compute the vector field X associated to the flow θ_t and find the points where X is zero. To describe X, use the description

$$TS^2 = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| = 1, \quad x \cdot v = 0\},$$

instead of local coordinate descriptions. In other words, write your answer as

$$X = \begin{bmatrix} V_1(x, y, z) \\ V_2(x, y, z) \\ V_3(x, y, z) \end{bmatrix},$$

and find the components V_i.

- **Problem:** Let X, Y, Z be the vector fields

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

defined on $\mathbb{R}^3$.

(a) Show that the map $(a, b, c) \mapsto aX + bY + cZ$ injects $\mathbb{R}^3$ onto its image, which is a subspace of the smooth vector fields on $\mathbb{R}^3$. Show that under this map, the bracket of vector fields corresponds to the cross product on $\mathbb{R}^3$.

(b) Compute the flow θ_t of the vector field Y.

- **Problem:** Let $E \subset S^1 \times \mathbb{R}^2$ be the set of points $(e^{i\theta}, v_1, v_2)$ such that

$$(\cos\theta)v_1 + (\sin\theta)v_2 = v_1$$
$$(\sin\theta)v_1 - (\cos\theta)v_2 = v_2.$$

This space forms the Möbius bundle $E \to S^1$ with projection $\pi(e^{i\theta}, v_1, v_2) = e^{i\theta}$. We will use the charts over $S^1 = U \cup \tilde{U}$ given by

$$U = \{e^{i\theta} : \theta \in (0, 2\pi)\}, \quad \tilde{U} = \{e^{i\tilde{\theta}} : \tilde{\theta} \in (-\pi, \pi)\}$$

with coordinates (U, θ) and $(\tilde{U}, \tilde{\theta})$.

(a) Show that when $0 < \theta < 2\pi$ and $(e^{i\theta}, v_1, v_2) \in E$, we can write

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = r \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$$

with

$$r = \frac{v_2}{\sin(\theta/2)}.$$

Similarly when $-\pi < \tilde{\theta} < \pi$ and $(e^{i\theta}, v_1, v_2) \in E$, we can write

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \rho \begin{bmatrix} \cos(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) \end{bmatrix}$$

with

$$\rho = \frac{v_1}{\cos(\tilde{\theta}/2)}.$$

(b) Give E the structure of a vector bundle over S^1 by defining a trivialization $\varphi_1 : \pi^{-1}(U) \to (0, 2\pi) \times \mathbb{R}$ by

$$\left(e^{i\theta}, r \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} \right) \mapsto (\theta, r),$$

and a trivialization $\varphi_2 : \pi^{-1}(\tilde{U}) \to (-\pi, \pi) \times \mathbb{R}$ by

$$\left(e^{i\tilde{\theta}}, \rho \begin{bmatrix} \cos(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) \end{bmatrix} \right) \mapsto (\tilde{\theta}, \rho).$$

Compute the transition function $\tau_{12} : U \cap \tilde{U} \to \mathbb{R}^*$.

(c) Consider the section $\sigma \in \Gamma(E)$ defined by

$$\sigma(e^{i\theta}) = (e^{i\theta}, \sin\theta, 1 - \cos\theta) \in E.$$

Write

$$\sigma|_U = (\theta, s(\theta)), \quad \sigma|_{\tilde{U}} = (\tilde{\theta}, \tilde{s}(\tilde{\theta})),$$

(recall the notation $\sigma|_U = \varphi_1 \circ \sigma$) and find the local smooth functions

$$s : (0, 2\pi) \to \mathbb{R}, \quad \tilde{s} : (-\pi, \pi) \to \mathbb{R}.$$

Verify the glueing relation $\tilde{s} = \tau_{12} s$.

- **Problem:** Recall that in a previous problem set, we covered $\mathbb{RP}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$ by the open sets
$$U_i = \{[x_0, \ldots, x_n] : x_i \neq 0\},$$
and equipped each set with local coordinates. Let
$$L = \{([x], w) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : w \in [x]\}.$$
Define the projection
$$\pi : L \to \mathbb{RP}^n$$
by $\pi([x], w) = [x]$. We can view L as a line bundle by trivializing L over U_i by $\Psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}$ with
$$\Psi_i([x], w) = ([x], w_i).$$
Here we write $w = (w_0, w_1, \ldots, w_n)$. Compute the transition function $\tau_{ij} : U_i \cap U_j \to \mathbb{R}^*$.

- **Problem:** Let $\pi : E \to M$ be a vector bundle and let $s : M \to E$ be a section. Show that
$$S = \{s(p) : p \in M\}$$
is a submanifold of E which is diffeomorphic to M.