Differential Geometry I: Worksheet 3

- **Problem:**

  (a) Let $f : (a, b) \to \mathbb{R}$ and consider the submanifold
  $$S = \{(x, f(x)) : x \in (a, b)\} \subset \mathbb{R}^2.$$
  In other words, S is the graph of $y = f(x)$. Equip S with coordinates $\varphi : S \to U$, $U = (a, b)$ given by
  $$\varphi(x, f(x)) = x.$$
  Denote by g the Euclidean metric on $\mathbb{R}^2$ restricted to S. Find the expression for g in the coordinate chart (U, x). To be clear, this means $g = (\varphi^{-1})^* g_{\text{Euc}}$. Derive the formula for the arclength by simplifying the general formula
  $$\int_U \sqrt{\det g_{ij}}\, dx^1 \cdots dx^n$$
  in this context.

  (b) Next, let $f : U \to \mathbb{R}$ be a function of two variables $f(x, y)$, where $U \subset \mathbb{R}^2$ is an open set. Consider the submanifold
  $$S = \{(x, y, f(x, y)) : (x, y) \in U\} \subset \mathbb{R}^3.$$
  Equip S with coordinates $\varphi : S \to U$. Compute the metric g in these coordinates, where g is the pullback of the Euclidean metric. Derive the formula for the area of a surface by simplifying the general formula
  $$\int_U \sqrt{\det g_{ij}}\, dx^1 \cdots dx^n$$
  in this context.

  (c) Suppose the level set of the function $F : \mathbb{R}^3 \to \mathbb{R}$ defines a submanifold:
  $$S = \{F(x, y, z) = 0\} \subseteq \mathbb{R}^3.$$
  Let $(a, b, c) \in S$ and suppose $\partial_z F(a, b, c) \neq 0$. By the implicit function theorem, there exists a neighborhood $U \subset \mathbb{R}^2$ of (a, b), $V \subset \mathbb{R}^3$ of (a, b, c) and a function $f : U \to \mathbb{R}$ such that
  $$S \cap V = \{(x, y, f(x, y)) : (x, y) \in U\}.$$
  Use the results from part (b) to derive the formula for area of $S \cap V$:
  $$\iint_U \frac{|\nabla F|}{|\partial_z F|}\, dxdy.$$
  Verify this formula by computing the area of the hemisphere by taking $F = x^2 + y^2 + z^2 - 1$ and $U = \{x^2 + y^2 < 1\}$.

- **Problem:** Let $0 < b < a$. The torus of revolution T^2 is defined as a parametrized surface element by $f : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$, where

$$f(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u).$$

(a) Compute the metric g_{ij} on T^2 in (u, v)-coordinates, where the metric is induced by the Euclidean metric on $\mathbb{R}^3$. Give your answer as a 2×2 matrix function of (u, v).

(b) Compute the length of the curves $c_1 : [0, 2\pi] \to T^2$ given by

$$c_1(t) = f(\pi, t),$$

and $c_2 : [0, 2\pi] \to T^2$ given by $c_2(t) = f(t, \pi)$.

(c) Compute the area of the torus.

- **Problem:** Consider the chart $U = S^2 \cap \{z > 0\}$ on $S^2 = \{x^2 + y^2 + z^2 = 1\}$ with coordinates

$$\varphi(x, y, \sqrt{1 - x^2 - y^2}) = (x, y).$$

Let $\iota : S^2 \to \mathbb{R}^3$ be inclusion.

(a) Compute the local coordinate expression for ι^*dx, ι^*dy and ι^*dz over (U, φ).

(b) Compute the local coordinate expression for $\iota^*(xdx + ydy + zdz)$ over (U, φ) by using part (a). Is there another way to anticipate this result?

- **Problem:**

(a) Let $\alpha = xdx + ydy$ and $\beta = ydx + xdy$ be 1-forms on $\mathbb{R}^2 \backslash \{(x, y) : x = \pm y\}$. Find the frame of vector fields $\{X, Y\}$ such that $\{\alpha, \beta\}$ is the dual frame of $\{X, Y\}$.

(b) Let $h(x, y) = x^2y$ and $g(x, y) = \sin(xy)$ be two functions in $\mathbb{R}^2$. Compute $dh \wedge dg$, which is a 2-form in $\mathbb{R}^2$.

- **Problem:** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that

$$\frac{\partial f}{\partial z}(p) \neq 0$$

at all points $p \in S$, where $S = f^{-1}(c)$.

(a) State a theorem from which you can deduce that S is a manifold.

(b) Let $\omega \in \Omega^2(\mathbb{R}^3)$ be $\omega = dx \wedge dy$. Let $\iota : S \to \mathbb{R}^3$. Show that $\iota^*\omega \in \Omega^2(S)$ is a nowhere vanishing top form.

- **Problem:** Let $S^2 = \{x^2 + y^2 + z^2 = 1\}$ and $\iota : S^2 \to \mathbb{R}^3$ be inclusion. Consider

$$\tilde{\omega} = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(\mathbb{R}^3)$$
$$\tilde{\eta} = -ydx + xdy \in \Omega^1(\mathbb{R}^3).$$

Define $\omega = \iota^*\tilde{\omega} \in \Omega^2(S^2)$ and $\eta = \iota^*\tilde{\eta} \in \Omega^1(S^2)$.

(a) Find $\lambda(x, y, z) \in C^\infty(S^2)$ such that $d\eta = \lambda\,\omega$.

(b) Find $\phi(x,y,z) \in C^\infty(S^2)$ such that $X \lrcorner \omega \wedge Y \lrcorner \omega = \phi\, \omega$, where

$$\tilde{X} = -zx\frac{\partial}{\partial x} - zy\frac{\partial}{\partial y} + (x^2+y^2)\frac{\partial}{\partial z}, \quad \tilde{Y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

are vector fields on $\mathbb{R}^3$ which when restricted to the sphere define vector fields X and Y on S^2.

- **Problem:** Show that the manifold TM is always orientable, even if M is not.
- **Problem:** Let $T^2 = S^1 \times S^1 \subseteq \mathbb{R}^4$ be given by

  $$T^2 = \{(w,x,y,z) \in \mathbb{R}^4 : w^2 + x^2 = y^2 + z^2 = 1\}.$$

  Let

  $$\omega = xz\, dw \wedge dy, \quad \omega \in \Omega^2(\mathbb{R}^4).$$

  Compute the integral

  $$\int_{T^2} \omega.$$

  You should declare an orientation on T^2 before evaluating this integral.
- **Problem:** Let $\tilde{\omega} \in \Omega^2(\mathbb{R}^3)$ be defined by

  $$\tilde{\omega} = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

  Let $\iota : S^2 \to \mathbb{R}^3$ and $\omega = \iota^*\tilde{\omega} \in \Omega^2(S^2)$. Give S^2 the Riemannian metric $g = \iota^* g_{\text{Euc}}$.

  (a) Show that $\omega = d\text{vol}_g$.

  (b) Compute the integral

  $$\int_{S^2} \omega,$$

  where S^2 is given the outward-pointing orientation via $S^2 = \partial B^3$.

  (c) Show that there does not exist $\alpha \in \Omega^1(S^2)$ such that $\omega = d\alpha$.