TOPICS IN COMPLEX GEOMETRY: PROBLEM SET 1

Due Oct 3

1. Consider $\mathbb{P}^2$ with coordinate charts $U_0 = \{Z_0 \neq 0\}$, $U_1 = \{Z_1 \neq 0\}$, $U_2 = \{Z_2 \neq 0\}$.

• On $U_0 \cap U_1$, let

$$(z^1, z^2) = \left(\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0} \right), \quad (\tilde{z}^1, \tilde{z}^2) = \left(\frac{Z_0}{Z_1}, \frac{Z_2}{Z_1} \right).$$

Compute the change of coordinates $z = f(\tilde{z})$.

• Recall the tangent bundle has the transformation law $\tilde{X}^p = \frac{\partial \tilde{z}^p}{\partial z^\ell} X^\ell$. In other words, the transition matrix t_{10} on $U_0 \cap U_1$ is $\frac{\partial \tilde{z}^p}{\partial z^\ell}$. Compute

$$\frac{\partial \tilde{z}}{\partial z} = \begin{bmatrix} \frac{\partial \tilde{z}^1}{\partial z^1} & \frac{\partial \tilde{z}^1}{\partial z^2} \\ \frac{\partial \tilde{z}^2}{\partial z^1} & \frac{\partial \tilde{z}^2}{\partial z^2} \end{bmatrix}$$

and compute $\det \frac{\partial \tilde{z}}{\partial z}$.

• Recall that $K_X = (\det T^{1,0} X)^*$. Compute the transition function t_{10} on $U_0 \cap U_1$ for $K_{\mathbb{P}^2}$.

• Recall that the line bundle $\mathcal{O}(k) \to \mathbb{P}^2$ is defined by the transition functions $t_{ij} = (Z_j/Z_i)^k$. Denote the transition functions of $\mathcal{O}(-3)$ by $\tilde{t}_{ij}$. Match this with your previous computation of t_{10} on $K_{\mathbb{P}^2}$ to show

$$\tilde{t}_{10} = -t_{10}.$$

We see that the transition functions for these two bundles are off by a sign.

• Nevertheless, the bundles are isomorphic

$$K_{\mathbb{P}^2} \cong \mathcal{O}(-3).$$

To show this, compute t_{21} on $U_1 \cap U_2$ and t_{20} on $U_0 \cap U_2$. For example, to compute t_{20} on $U_0 \cap U_2$ you can let

$$(z^1, z^2) = \left(\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0} \right), \quad (\tilde{z}^1, \tilde{z}^2) = \left(\frac{Z_0}{Z_2}, \frac{Z_1}{Z_2} \right).$$

and proceed as before. You should find

$$\tilde{t}_{10} = -t_{10}, \quad \tilde{t}_{20} = t_{20}, \quad \tilde{t}_{21} = -t_{21}.$$

To show these two bundles are isomorphic, construct an isomorphism $\{h_i : U_i \to \mathbb{C}^*\}$ such that

$$h_i = \tilde{t}_{ij} h_j t_{ij}^{-1}$$

This amounts to specifying h_0, h_1, h_2.

• In general, the formula is $K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$ (no need to hand-in this calculation for general n).

2. Consider $\mathbb{C}^2\backslash\{0\}$ with holomorphic coordinates z, w. Compute

$$i\partial\bar{\partial}\log|z|^2 = 0$$

and

$$\left(i\partial\bar{\partial}\log(|z|^2 + |w|^2)\right)^2 = 0.$$

Remark: in general on $\mathbb{C}^n\backslash\{0\}$, the identity is $(i\partial\bar{\partial}\log(\sum_i |z_i|^2))^n = 0$. These formulas become more interesting when the origin is included. In this case the equation should be interpreted as a current and a Dirac delta charge is placed at the origin.