TOPICS IN COMPLEX GEOMETRY: PROBLEM SET 3

Due Nov 21

1. Let $(E,H) \to (X,\omega)$ be a holomorphic vector bundle with metric over a compact Kähler manifold. Recall that the curvature $F \in \Lambda^{1,1}(\mathrm{End}\, E)$ is given by $F = \bar\partial(\partial H H^{-1})$.

- (a) Show that $i\mathrm{Tr}\, F \in \Lambda^{1,1}(X,\mathbb{R})$ is given by

$$i\mathrm{Tr}\, F = -i\partial\bar\partial \log\det H.$$

Note that $d\mathrm{Tr}\, F = 0$.

- (b) Let $\hat H$ be another metric with curvature $\hat F$. Show that

$$i\mathrm{Tr}\, F - i\mathrm{Tr}\, \hat F = i\partial\bar\partial \log f$$

where $f : X \to (0,\infty)$ is a well-defined function. Therefore

$$c_1(E) := \frac{1}{2\pi}[i\mathrm{Tr} F] \in H^2(X,\mathbb{R})$$

defines a de Rham cohomology class which is independent of the choice of metric H. (In fact, it defines a class in Bott-Chern cohomology, where the equivalence is up to the image of $i\partial\bar\partial$.)

- (c) Show that the pairing

$$c_1(E)\cdot[\omega]^{n-1} := \int_X c_1(E)\wedge\omega^{n-1} \in \mathbb{R}$$

produces a number which does not depend on the metric H or the choice of Kähler metric in the class $\omega \in [\omega] \in H^2(X,\mathbb{R}) \cap \Lambda^{1,1}$.

2. Let $D \subset X$ be a smooth analytic hypersurface in a compact complex manifold X. Let $\mathcal{O}(D) \to X$ be the associated line bundle and let $s \in H^0(X,\mathcal{O}(Y))$ be the defining global section vanishing along D. We will work through the proof of

$$c_1(\mathcal{O}(D))\cdot[\chi] := \int_X c_1(\mathcal{O}(D))\wedge\chi = \int_D \chi,$$

for any $\chi \in \Lambda^{n-1,n-1}(X)$ with $d\chi = 0$. This is sometimes written

$$c_1(\mathcal{O}(D)) = [D].$$

For readability, let's assume that the complex dimension is $n = 2$ so that X has two local coordinates (z^1, z^2). The proof in higher dimensions is identical.

We start by covering D with open sets $U_\alpha \subset X$ such that

$$D \cap U = \{(z_1,z_2) : |z_1| < 1, \quad z_2 = 0\}.$$

A tubular neighborhood $D \subset D_\varepsilon \subset X$ is given by

$$D_\varepsilon \cap U = \{(z_1, z_2) : |z_1| < 1, \quad |z_2| < \varepsilon\}.$$

Let h be an arbitrary metric on $\mathcal{O}(D)$ with curvature $iF_h \in \Lambda^{1,1}(X, \mathbb{R})$. Since F_h is a smooth differential form,

$$\frac{1}{2\pi} \int_X iF_h \wedge \chi = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{X \backslash D_\varepsilon} iF_h \wedge \chi.$$

- (a) Show that

$$iF_h = -i\partial\bar{\partial} \log |s|_h^2 \quad \text{on } X \backslash D_\varepsilon.$$

For this, recall that over the trivialization U, then by definition of the defining section we have $s_U(z) = z_2$, and by definition $|s|_h^2 = s_U \overline{s_U} h_U$ in this trivialization.

Computing $c_1(\mathcal{O}(D)) \cdot [\chi]$ then leads to the limit

$$c_1(\mathcal{O}(D)) \cdot [\chi] = \lim_{\varepsilon \to 0} \frac{-1}{2\pi} \int_{X \backslash D_\varepsilon} i\partial\bar{\partial} \log |s|_h^2 \wedge \chi,$$

and by Stokes's theorem

$$c_1(\mathcal{O}(D)) \cdot [\chi] = \lim_{\varepsilon \to 0} \frac{-i}{2\pi} \int_{\partial D_\varepsilon} \bar{\partial} \log |s|_h^2 \wedge \chi.$$

In a local chart,

$$\partial D_\varepsilon \cap U = \{(z_1, z_2) : |z_1| < 1, \quad |z_2| = \varepsilon\},$$

and so the integral we would like to compute splits into two pieces:

$$\lim_{\varepsilon \to 0} \frac{-i}{2\pi} \int_{\partial D_\varepsilon \cap U} \bar{\partial} \log h \wedge \chi + \lim_{\varepsilon \to 0} \frac{-i}{2\pi} \int_{\partial D_\varepsilon \cap U} \bar{\partial} \log |z_2|^2 \wedge \chi.$$

You can parametrize integrals over $\partial D_\varepsilon \cap U$ by setting $z_2 = \varepsilon e^{2\pi i t}$ for $0 \leq t \leq 1$.

- (b) Show that

$$\lim_{\varepsilon \to 0} \frac{-i}{2\pi} \int_{\partial D_\varepsilon \cap U} \bar{\partial} \log h \wedge \chi = 0.$$

In fact $\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon \cap U} \eta = 0$ for any smooth 3-form η over U.

- (c) The non-zero contribution is picked up from the singular integrand

$$\frac{-i}{2\pi} \int_{\partial D_\varepsilon \cap U} \bar{\partial} \log |z_2|^2 \wedge \chi.$$

Write

$$\chi = \chi_{j\bar{k}} dz^j \wedge d\bar{z}^k$$

and show that this integral is given by

$$\frac{-i}{2\pi} \int_{\partial D_\varepsilon \cap U} \chi_{1\bar{1}}(z_1, z_2, \bar{z}_1, \bar{z}_2) \frac{d\bar{z}_2}{\bar{z}_2} \wedge dz^1 \wedge d\bar{z}^1.$$

Set $z_2 = \varepsilon e^{2\pi i t}$ for $0 \leq t \leq 1$, send $\varepsilon \to 0$, and deduce

$$\lim_{\varepsilon \to 0} \frac{-i}{2\pi} \int_{\partial D_\varepsilon \cap U} \bar{\partial} \log |z_2|^2 \wedge \chi = \int_{\{|z_1|<1\}} \chi_{1\bar{1}}(z_1, 0, \bar{z}_1, 0) dz^1 \wedge d\bar{z}^1$$

which is

$$\int_{D \cap U} \chi$$

as required.

3. Let $L_1, L_2 \to (X, \omega)$ be a pair of holomorphic line bundle over a compact Kähler manifold, equipped with metrics h_1, h_2.

- (a) Equip $\tilde{L} = L_1 \otimes L_2$ with the metric $\tilde{h} = h_1 \otimes h_2$, and show that

$$F_{\tilde{h}} = F_{h_1} + F_{h_2},$$

which implies $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. Similarly, L^k has metric h^k and $F_{h^k} = kF_h$. Here the product in the notation L^k is the tensor product of line bundles.

- (b) Now let L be a positive line bundle so that there exists a metric h with $iF_h > 0$. Using the Kodaira vanishing theorem $H^q(X, L \otimes K_X) = 0$ for $q \geq 1$, deduce the following vanishing theorem: there exists $k_0 \gg 1$ such that for all $k \geq k_0$, then

$$H^q(X, L^k) = 0, \quad q \geq 1.$$

To do this, write $L^k = (L^k \otimes K_X^{-1}) \otimes K_X$.