Notes on Spinors and Non-Kähler Threefolds

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These are expository notes on spin and complex geometry in dimension $n = 6$. We will discuss how natural equations on spinors lead to the notion of a non-Kähler Calabi-Yau threefold.

Contents

1 Clifford Algebra

In this section, we review the basics of Clifford algebras. Some references for this section are the lecture notes of O'Farrill [7] and Woit [12].

1.1 Spin group

Let $\mathbb{R}^n$ be Euclidean space with Euclidean inner product $\langle \cdot, \cdot \rangle$. Let $e_1, \dots, e_n$ be the standard orthonormal basis for $\mathbb{R}^n$. The Clifford algebra $\mathrm{Cliff}(n)$ is the $\mathbb{R}$-algebra generated by $e_1, \dots, e_n$ subject to the relation

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij} 1. \tag{1.1}$$

Here 1 is the unit in the algebra. In particular,

$$e_i^2 = -1, \quad e_i e_j = -e_j e_i, \quad i \neq j. \tag{1.2}$$

So for example, elements of $\mathrm{Cliff}(2)$ look like $\mathbb{R}$-linear combinations of

$$1,\, e_1,\, e_2,\, e_1 e_2. \tag{1.3}$$

In general, the $\mathbb{R}$-vector space underlying $\mathrm{Cliff}(n)$ has dimension 2^n.

The algebra $\mathrm{Cliff}(n)$ is independent of the choice of orthonormal basis $\{e_i\}$. In particular, $a^2 = -1$ for any $a = a^i e_i$ with $\|a\|^2 := \sum (a^i)^2 = 1$. This implies that for $k \in \mathbb{Z}_{\geq 1}$, the inverse of

$$u = a_1 \cdot a_2 \cdots a_{2k} \tag{1.4}$$

is

$$u^{-1} = a_{2k} \cdot a_{2k-1} \cdots a_1 \tag{1.5}$$

for $\|a_i\| = 1$. The spin group $\mathrm{Spin}(n)$ is defined as

$$\mathrm{Spin}(n) = \{a_1 \cdots a_{2k} \in \mathrm{Cliff}(n) : a_i \in \mathbb{R}^n,\ \|a_i\| = 1,\ k \in \mathbb{Z}_{\geq 1}\}, \tag{1.6}$$

where the group operation is Clifford multiplication. Here are some examples.

- $\mathrm{Spin}(1) \cong \{+1, -1\}$.
- $\mathrm{Spin}(2) \cong U(1)$. Let $a_1 = (\cos\theta, \sin\theta)$ and $a_2 = (\cos\phi, \sin\phi)$, so that

$$\begin{aligned} a_1 a_2 &= (\cos\theta\, e_1 + \sin\theta\, e_2) \cdot (\cos\phi\, e_1 + \sin\phi\, e_2) \\ &= -\cos\theta\cos\phi - \sin\theta\sin\phi + (\cos\theta\sin\phi - \sin\theta\cos\phi)e_1e_2 \\ &= \cos(\theta - \phi + \pi) + \sin(\theta - \phi + \pi)e_1e_2. \end{aligned} \tag{1.7}$$

Thus pairs a_1a_2 can be identified with elements of the form $\cos\psi + \sin\psi\, e_1e_2$. A product of these satisfies

$$\begin{aligned} & (\cos\psi_1 + \sin\psi_1\, e_1e_2) \cdot (\cos\psi_2 + \sin\psi_2\, e_1e_2) \\ = \ & \cos(\psi_1 + \psi_2) + \sin(\psi_1 + \psi_2)\, e_1e_2. \end{aligned} \tag{1.8}$$

Thus we can identify

$$(\cos\psi + \sin\psi\, e_1e_2) \mapsto e^{i\psi} \in U(1), \tag{1.9}$$

and this identification is compatible with multiplication.

- To do: $\mathrm{Spin}(3) \cong SU(2)$.

Theorem 1 *Let $u \in \mathrm{Spin}(n)$. The action of u on vectors $v \in \mathbb{R}^n$ given by*

$$v \mapsto uvu^{-1} \tag{1.10}$$

is well-defined and a rotation in $SO(n)$. This construction gives rise to a surjective homomorphism

$$\varphi : \mathrm{Spin}(n) \to SO(n), \tag{1.11}$$

which is two-to-one.

Proof: We only sketch the proof. Let a be a unit vector and $v \in \mathbb{R}^n$. The Clifford algebra relation implies

$$ava^{-1} = -vaa^{-1} - 2\langle a, v\rangle a^{-1} \tag{1.12}$$

and so since $a^{-1} = -a$, the conjugation action is

$$v \mapsto -(v - 2\langle a, v\rangle a). \tag{1.13}$$

The linear transformation $v \mapsto v - 2\langle a, v\rangle a$ is a reflection in the hyperplane with normal a, as it fixes all vectors orthogonal to a and sends $a \mapsto -a$.

Therefore the action of $u = a_1 \cdots a_{2k} \in \mathrm{Spin}(n)$ given by $v \mapsto uvu^{-1}$ is an even product of reflections, which is a matrix in $SO(n)$. To show that φ is surjective, one can use the Cartan-Dieudonné Theorem, which states that any orthogonal transformation is a product of reflections. We leave the computation of the kernel of φ, which is $\ker\varphi = \{\pm 1\}$. □

Note that both u and $-u$ give rise to the same rotation in $SO(n)$. It can be shown that for $n \geq 3$, $\mathrm{Spin}(n)$ is the universal cover of $SO(n)$.

The theorem attaches to each $u \in \mathrm{Spin}(n)$ a matrix $M_u \in SO(n)$ and gives the identity

$$ue_iu^{-1} = (M_u)^k{}_i e_k, \tag{1.14}$$

where $\{e_i\}$ is the standard basis in $\mathbb{R}^n$. Here we use the Einstein summation convention, where an index which is repeated implies a summation.

We now prove an identity for lifting paths on $SO(n)$ to paths on $\mathrm{Spin}(n)$. Since $\mathrm{Spin}(n)$ is a double-cover of $SO(n)$, there are two lifted paths, but if we find one lift $u(t) \in \mathrm{Spin}(n)$ of $A(t) \in SO(n)$, then the other lifted path is just $-u(t)$. Let $j < k$ and ε_{jk} be an $n \times n$ matrix with jk entry -1, kj entry 1, and all other entries 0. A rotation by angle θ in the jk plane is given by

$$\exp\theta\varepsilon_{jk}. \tag{1.15}$$

We will prove the following lifting identity:

Proposition 1

$$\varphi\left(\exp\frac{t}{2}e_je_k\right) = \exp t\varepsilon_{jk}. \tag{1.16}$$

Proof: To be concrete, we consider $j = 1$ and $k = 2$. We must compute the action

$$v \mapsto \exp^{\frac{\theta}{2}e_1e_2} v \exp^{-\frac{\theta}{2}e_1e_2}. \tag{1.17}$$

We start with

$$\exp\left(\frac{\theta}{2}e_1e_2\right) = 1 + \frac{\theta}{2}(e_1e_2) + \frac{1}{2}(\frac{\theta}{2}e_1e_2)^2 + \frac{1}{3!}(\frac{\theta}{2}e_1e_2)^3 + \dots. \tag{1.18}$$

Noting $(e_1e_2)^2 = -1$, we obtain

$$\exp\left(\frac{\theta}{2}e_1e_2\right) = \left(1 - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^2 + \dots\right) + \left(\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \dots\right)e_1e_2. \tag{1.19}$$

Therefore

$$\exp\left(\frac{\theta}{2}e_1e_2\right) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}e_1e_2. \tag{1.20}$$

As an aside, since

$$\cos\frac{\theta}{2} + \sin\frac{\theta}{2}e_1e_2 = (\cos\frac{\theta}{2}\,e_1 + \sin\frac{\theta}{2}\,e_2)\cdot(-e_1) \tag{1.21}$$

we see that

$$\exp\left(\frac{\theta}{2}e_1e_2\right) \in \mathrm{Spin}(n). \tag{1.22}$$

Let $v = v^1e_1 + v^2e_2$. Then

$$\begin{aligned} & \exp^{\frac{\theta}{2}e_1e_2}(v^1e_1 + v^2e_2)\exp^{-\frac{\theta}{2}e_1e_2} \\ = \;& (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}e_1e_2)(v^1e_1 + v^2e_2)(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}e_1e_2) \end{aligned} \tag{1.23}$$

This becomes

$$(\cos\frac{\theta}{2}v^1e_1 + \cos\frac{\theta}{2}v^2e_2 + \sin\frac{\theta}{2}v^1e_2 - \sin\frac{\theta}{2}v^2e_1)(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}e_1e_2) \tag{1.24}$$

which becomes

$$\begin{aligned} & \cos^2\frac{\theta}{2}v^1e_1 + \cos^2\frac{\theta}{2}v^2e_2 + \sin\frac{\theta}{2}\cos\frac{\theta}{2}v^1e_2 - \sin\frac{\theta}{2}\cos\frac{\theta}{2}v^2e_1 \\ & + \sin\frac{\theta}{2}\cos\frac{\theta}{2}v^1e_2 - \sin\frac{\theta}{2}\cos\frac{\theta}{2}v^2e_1 - \sin^2\frac{\theta}{2}v^1e_1 - \sin^2\frac{\theta}{2}v^2e_2. \end{aligned} \tag{1.25}$$

Using double-angle formulas, this becomes

$$(\cos\theta v^1 - \sin\theta v^2)e_1 + (\cos\theta v^2 + \sin\theta v^1)e_2. \tag{1.26}$$

Therefore, the path

$$u(t) = \exp((t/2)e_1e_2) \in \mathrm{Spin}(n) \tag{1.27}$$

acts by

$$u(t)(v^1e_1 + v^2e_2)u(t)^{-1} = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \tag{1.28}$$

We also have that

$$u(t)e_ku(t)^{-1} = e_k, \quad k \geq 3. \tag{1.29}$$

Thus

$$\varphi(u(t)) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & I \end{bmatrix} \tag{1.30}$$

and $u(t)$ corresponds under φ to the path

$$\exp(t\varepsilon_{12}) \in SO(n). \tag{1.31}$$

For example, we note

$$\exp\left(t\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \tag{1.32}$$

which shows that $\exp(\theta\varepsilon_{12})$ is indeed a rotation by θ in the plane spanned by e_1, e_2. $\square$

As a consequence, we can compute

$$\varphi_* : \mathfrak{spin}(n) \to \mathfrak{so}(n). \tag{1.33}$$

Differentiating the lifted path, we see that ε_{jk} should be identified with $\frac{1}{2}e_j{\cdot}e_k$.

$$\varphi_*\frac{1}{2}e_je_k = \varepsilon_{jk}. \tag{1.34}$$

1.2 Gamma matrices

To be concrete, in these notes we will view the Clifford algebra as an algebra of matrices. By this, we mean that we will use a homomorphism $\gamma : \mathrm{Cliff}(n) \to \mathrm{Mat}_{k\times k}(\mathbb{C})$, and we will denote $\gamma_i = \gamma(e_i)$ (the gamma matrices). We will build examples of gamma matrices by using the following building blocks:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{1.35}$$

and

$$\tau_1 = i\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \tau_2 = i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \tau_3 = i\sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{1.36}$$

We note the identities:

$$\begin{aligned} &\sigma_i^2 = I_{2\times 2}, \quad \tau_i^2 = -I_{2\times 2}, \\ &\sigma_i\sigma_j = -\sigma_j\sigma_i, \quad \tau_i\tau_j = -\tau_j\tau_i \ \text{ for } i \neq j. \\ &\sigma_1\sigma_2 = i\sigma_3, \quad \sigma_1\sigma_3 = -i\sigma_2, \quad \sigma_2\sigma_3 = i\sigma_1, \\ &\tau_1\tau_2 = -\tau_3, \quad \tau_1\tau_3 = \tau_2, \quad \tau_2\tau_3 = -\tau_1. \end{aligned} \tag{1.37}$$

Here are some examples of Clifford algebras represented as a matrix algebra.

• $\gamma : \mathrm{Cliff}(1) \to \mathrm{Mat}_{1\times 1}(\mathbb{C})$. We can identify $\gamma_1 = i$ to obtain the complex numbers.

• $\gamma : \mathrm{Cliff}(2) \to \mathrm{Mat}_{2\times 2}(\mathbb{C})$. We can identify

$$\gamma_1 = \tau_1, \quad \gamma_2 = \tau_2. \tag{1.38}$$

In other words, the algebra contains

$$\gamma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \gamma_1\gamma_2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{1.39}$$

and their $\mathbb{R}$-linear combinations.

• $\gamma : \mathrm{Cliff}(3) \to \mathrm{Mat}_{4\times 4}(\mathbb{C})$. We can identify

$$\gamma_1 = \tau_1 \otimes I_{2\times 2}, \quad \gamma_2 = \tau_2 \otimes I_{2\times 2}, \quad \gamma_3 = -\tau_3 \otimes \sigma_1. \tag{1.40}$$

Here we use the notation

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \tag{1.41}$$

where $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ and $A \otimes B \in \mathrm{Mat}_{n^2\times n^2}(\mathbb{C})$. The product satisfies $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. The algebra contains

$$\gamma_1\gamma_2 = -\tau_3 \otimes I_{2\times 2}, \quad \gamma_1\gamma_3 = \tau_2 \otimes \sigma_1, \quad \gamma_2\gamma_3 = \tau_1 \otimes \sigma_1$$
$$\gamma_1\gamma_2\gamma_3 = -I_{2\times 2} \otimes \sigma_1, \quad 1 = I_{2\times 2} \otimes I_{2\times 2},$$

and their $\mathbb{R}$-linear combinations.

The focus of these notes is Calabi-Yau threefolds, and so we omit the calculation of Cliff(4) and Cliff(5), and move on to our main example: Cliff(6).

- $\gamma : \mathrm{Cliff}(6) \to \mathrm{Mat}_{8\times 8}(\mathbb{R})$. We can identify

$$\begin{aligned} \gamma_1 &= \tau_2 \otimes I_{2\times 2} \otimes \sigma_1 \\ \gamma_2 &= \tau_2 \otimes I_{2\times 2} \otimes \sigma_3 \\ \gamma_3 &= \sigma_1 \otimes \tau_2 \otimes I_{2\times 2} \\ \gamma_4 &= \sigma_3 \otimes \tau_2 \otimes I_{2\times 2} \\ \gamma_5 &= I_{2\times 2} \otimes \sigma_1 \otimes \tau_2 \\ \gamma_6 &= I_{2\times 2} \otimes \sigma_3 \otimes \tau_2. \end{aligned} \tag{1.42}$$

We note that all these matrices happen to be real, which did not occur in our previous examples. We also note that $\gamma_i^T = -\gamma_i$ is anti-symmetric.

Remark: This sort of explicit matrix representation of a Clifford algebra exists in arbitrary dimension: see the Weyl-Brauer matrices.

2 Spinors and Almost-Complex Structures

2.1 Spinor bundles

Let M be an oriented manifold of dimension n with metric g. Recall that orientability reduces the transition functions of the tangent bundle to matrices with positive determinant, and the existence of a metric reduces the transition functions of the tangent bundle to orthogonal matrices by a Gram-Schmidt process.

We can then choose a covering of M with transition functions Λ_{UV} for the tangent bundle TM such that $\Lambda_{UV} \in SO(n)$.

$$\Lambda_{UV} : U \cap V \to SO(n). \tag{2.1}$$

More concretely, let $\{e_i^U\}_{i=1}^n$ denote a local orthonormal oriented frame over U, and $\{e_i^V\}_{i=1}^n$ over V. On $U \cap V$, then

$$e_i^U(x) = e_k^V(x) {\Lambda_{VU}}^k{}_i(x). \tag{2.2}$$

In terms of components, an arbitrary tangent vector $X = X_U^i e_i^U = X_V^i e_i^V$ appears as a column vector X_U^i over U and as column vector X_V^i over V, and transforms as

$$X_U^i = {\Lambda_{UV}}^i{}_k X_V^k. \tag{2.3}$$

The Cech data $(U_\alpha \cap U_\beta, \Lambda_{U_\alpha U_\beta})$ satisfies the cocycle condition

$$\Lambda_{UU} = \text{id}, \quad \Lambda_{UV}\Lambda_{VW}\Lambda_{WU} = \text{id}. \tag{2.4}$$

A spin structure $\text{Spin}(n) \to M$ is a lift

$$\tilde{\Lambda}_{UV} : U \cap V \to \text{Spin}(n), \tag{2.5}$$

such that

$$\varphi(\tilde{\Lambda}_{UV}(x)) = \Lambda_{UV}(x), \quad x \in U \cap V \tag{2.6}$$

where φ is defined as in (1.11) and

$$\tilde{\Lambda}_{UV} = \text{id}, \quad \tilde{\Lambda}_{UV}\tilde{\Lambda}_{VW}\tilde{\Lambda}_{WU} = \text{id}. \tag{2.7}$$

Remark: Spin structures may not exist. Their obstruction is the vanishing of $w_2(M)$. To do: explain the $w_2(M) = 0$ condition, give example of spin structures on Riemann surfaces.

As discussed in the previous section, let us make a choice and represent the Clifford algebra by complex matrices. Let

$$\gamma : \mathrm{Cliff}(n) \to \mathrm{Mat}_{k\times k}(\mathbb{C}) \tag{2.8}$$

be an algebra homomorphism. We will denote

$$\rho : \mathrm{Spin}(n) \to \mathrm{Mat}_{k\times k}(\mathbb{C}) \tag{2.9}$$

to be the representation of the spin group which is the restriction of γ.

From a spin structure $\mathrm{Spin}(n) \to M$ and a Clifford algebra matrix representation $\gamma : \mathrm{Cliff}(n) \to \mathrm{Mat}_{k\times k}(\mathbb{C})$, we construct a rank k complex vector bundle $S \to M$ as follows: set the transition functions on the overlap $U \cap V$ to be $\rho(\tilde{\Lambda}_{UV})$. Sections $\psi \in \Gamma(S)$ of this bundle will be called spinors. Concretely, a spinor is defined by local functions

$$\psi_U : U \to \mathbb{C}^k \tag{2.10}$$

which transform on $U \cap V$ by

$$\psi_U = \rho(\tilde{\Lambda}_{UV})\psi_V. \tag{2.11}$$

We next discuss the gamma matrices in this geometric context. Let γ_a be the generators chosen in (2.8) to represent the Clifford algebra. The key relation (1.14) becomes then

$$\rho(\tilde{\Lambda})\gamma_a\rho(\tilde{\Lambda})^{-1} = \Lambda^b{}_a\gamma_b. \tag{2.12}$$

Let X be a vector field. The identity is then

$$\rho(\tilde{\Lambda})\gamma(X)\rho(\tilde{\Lambda})^{-1} = \gamma(\Lambda X). \tag{2.13}$$

Using X, we will now define an endomorphism of spinors denoted $\gamma(X)$. For this, we use a local orthonormal frame $E = \{e_a\}$ over U, write the vector field as $X = X^a_E e_a$ and a spinor locally as a column vector $\psi = \psi_E$ over U, and write

$$\gamma(X)\psi := X^a_E\gamma_a\psi_E. \tag{2.14}$$

We will sometimes also write $\gamma(X)$ locally as $\gamma(X_E) = X^a_E\gamma_a$. Here the γ_a are fixed chosen generators of the Clifford matrix algebra, such as the ones explicitly displayed in §1.2. We now prove that $\psi \mapsto \gamma(X)\psi$ sends sections of S to sections of S.

Proposition 2 *The above formula defines a section $\gamma \in \Gamma((TM)^* \otimes \text{End}\, S)$.*

Proof: Suppose we have two overlapping trivializations, one with frame $E = \{e_a\}$ and another with frame $F = \{f_a\}$, and let Λ be the transition function for TM on the overlap and $\tilde{\Lambda}$ the lift from the spin structure. A spinor ψ appears as ψ_E in the frame E and ψ_F in the frame F. A tangent vector appears as $X = X^a_E e_a$ in the frame E and $X = X^a_F f_a$ in the frame F.

To show that $\gamma(X)\psi$ produces another spinor, we need to show that it transforms correctly, namely

$$\gamma(X_E)\psi_E = \rho(\tilde{\Lambda})\gamma(X_F)\psi_F. \tag{2.15}$$

Our conventions as setup earlier give the transformation laws

$$\psi_E = \rho(\tilde{\Lambda})\psi_F, \quad X_E = \Lambda X_F. \tag{2.16}$$

Then

$$\gamma(X_E)\psi_E = \gamma(\Lambda X_F)\rho(\tilde{\Lambda})\psi_F = \rho(\tilde{\Lambda})\gamma(X_F)\psi_F, \tag{2.17}$$

by (2.13). This proves the formula (2.15). □

To summarize,

$$\psi \mapsto \gamma(X)\psi \tag{2.18}$$

is a legitimate operator on spinors. The strange thing to note here is that we always use the same constant matrices γ_a, even though the vector components X^a transform as (2.3) if we change frames. For any frame E then $\gamma(e_a) = \gamma_a$ are the same constant gamma matrices multiplying local spinor components ψ_E, even though the spinor components ψ_E transform in different frames.

2.2 Dimension $n = 6$

References for this section are Candelas-Horowitz-Strominger-Witten [3], Becker-Becker-Schwarz [1] and Lawson-Michelson [6]. Our application to differential geometry will take place in dimension $n = 6$.

2.2.1 Setup

Let $\gamma_1, \ldots, \gamma_6$ be the 8×8 real matrices exhibited in (1.42) representing generators of Cliff(6). These satisfy

$$\{\gamma_a, \gamma_b\} = -2\delta_{ab} I, \quad \gamma_a^T = -\gamma_a. \tag{2.19}$$

It follows that $\gamma_a^T \gamma_a = I$. The representation of Spin(6) by matrices obtained by using these gamma matrices, denoted ρ, is contained in the orthogonal matrices.

$$\rho : \mathrm{Spin}(6) \to O(8) \tag{2.20}$$

This is because

$$\rho(a_1 \cdots a_{2k})\rho(a_1 \cdots a_{2k})^T = \gamma(a_1) \cdots \gamma(a_{2k})\gamma(a_{2k})^T \cdots \gamma(a_1)^T = I \tag{2.21}$$

for any $a_i \in \mathbb{R}^n$ with $\|a_i\| = 1$.

We will use the notation

$$\gamma_7 := i^3 \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6. \tag{2.22}$$

We can check from the definition that

$$\gamma_7^2 = I, \quad \gamma_7^\dagger = \gamma_7, \tag{2.23}$$

and

$$\gamma_7\gamma_a = -\gamma_a\gamma_7, \quad \gamma_7^T = -\gamma_7, \quad \gamma_7^\dagger\gamma_7 = I. \tag{2.24}$$

Let $V = \mathbb{C}^8$ be the complex vector space on which the γ_i matrices act by matrix multiplication. From (2.23), the eigenspaces of γ_7 have eigenvalues ± 1, and we may split

$$V = V^+ \oplus V^-. \tag{2.25}$$

This decomposition in orthogonal, since for $\psi_+ \in V^+$ and $\psi_- \in V^-$, then $\psi_+^\dagger\psi_- = \psi_+^\dagger\gamma_7^\dagger\gamma_7\psi_- = -\psi_+^\dagger\psi_-$. We note that

$$\gamma_a : V^+ \to V^- \tag{2.26}$$

is an isomorphism. To show γ_a is surjective, write $v \in V^-$ as $v = \gamma_a(-\gamma_a v)$ and note that $-\gamma_a v \in V^+$.

2.2.2 Positive chirality spinors

Going back to the geometric setup, let (M, g) be a Riemannian manifold of dimension $n = 6$ equipped with a spin structure, and use the representation $\rho : \mathrm{Spin}(6) \to O(8)$ from the previous section. Let $S \to M$ be the associated spinor bundle: it is a rank 8 complex vector bundle with transition functions

$\rho(\tilde{\Lambda}_{UV})$, where $\tilde{\Lambda}_{UV}$ is the lift of the $SO(n)$ transition functions Λ_{UV} on the tangent bundle. We note that given a spinor ψ, then the bilinear

$$\psi^\dagger \psi \tag{2.27}$$

gives a well-defined function on the manifold, since on an overlap $U \cap V$, then

$$\psi_U^\dagger \psi_U = \psi_V^\dagger \rho(\tilde{\Lambda}_{UV})^\dagger \rho(\tilde{\Lambda}_{UV}) \psi_V = \psi_V^\dagger \psi_V. \tag{2.28}$$

Here the dagger notation refers to the conjugate transpose.

Next, we check that if ψ is a spinor, then $\gamma_7 \psi$ is a well-defined spinor. For this, just like Proposition 2, we show $\gamma_7 \rho(\tilde{\Lambda}) = \rho(\tilde{\Lambda})\gamma_7$. This can be done by repeatedly applying $\rho(\tilde{\Lambda})\gamma_a \rho(\tilde{\Lambda})^{-1} = \Lambda^b{}_a \gamma_b$ and using that Λ is orthogonal and determinant 1. Thus

$$\gamma_7 \in \Gamma(\mathrm{End} S) \tag{2.29}$$

and the spinor bundle S breaks into two subbundles $S = S_+ \oplus S_-$. Spinors which are sections of S_+, i.e. satisfying $\gamma_7 \eta = \eta$, are said to have positive chirality.

Next, we'll need in the next section the notion of a pure spinor. For a spinor η, let

$$W_\eta = \{v \in (T_{\mathbf{C}} M) : \gamma(v)\eta = 0\}. \tag{2.30}$$

On a manifold of even dimension n, a spinor η is pure if at each point p then

$$\dim (W_\eta)|_p = n/2. \tag{2.31}$$

A special property of dimension $n = 6$ is that

Lemma 1 *Let (M, g) be a spin manifold of dimension $n = 6$. Let η_+ be a nowhere vanishing spinor of positive chirality. Then η_+ is a pure spinor.*

Proof: To do: include proof. See Lawson-Michelsohn [6]. □

Note that if $v, w \in W_\eta$, from

$$(\gamma(v)\gamma(w) + \gamma(w)\gamma(v))\eta = -2g(v, w)\eta \tag{2.32}$$

we conclude $g(v, w) = 0$ and $g(v, v) = 0$. Since v is a complexified vector, this does not imply that $v = 0$. (The subspace W_η is isotropic.) If we let $H(v, w) = g(v, \bar{w})$ be the induced Hermitian inner product, then

$$H(v, \bar{w}) = 0 \tag{2.33}$$

for all $v, w \in W_\eta$. This means that for a pure spinor η, we can break the complexified tangent bundle into two subbundles

$$T_{\mathbf{C}}M = W_\eta \oplus \overline{W_\eta}, \tag{2.34}$$

by taking the orthogonal complement of W_η using the inner product $H(\cdot, \cdot)$.

2.2.3 Almost-complex structure

First, some notation on orthonormal frames and coordinates: let $\{e_a\}$ be a local orthonormal frame of TM in (M, g) and let x^i be local coordinates. We can write one basis in terms of the other, and we will use the notation

$$\frac{\partial}{\partial x^i} = e^a{}_i e_a, \quad e_a = e^i{}_a \frac{\partial}{\partial x^i}. \tag{2.35}$$

This leads to the identities $e^b{}_i e^i{}_a = \delta_{ab}$, $e^j{}_a e^a{}_i = \delta_{ij}$, and

$$g_{ij} = \sum_a e^a{}_i e^a{}_j, \quad \delta_{ab} = e^i{}_a e^j{}_b g_{ij}. \tag{2.36}$$

We denote

$$\gamma_i = e^a{}_i \gamma_a. \tag{2.37}$$

This notation can be confusing. When denoted with indices i, j, k, the matrices γ_i are not constant, but the matrices γ_a with indices a, b, c are the explicit constant matrices exhibited earlier in (2.19). In other words, using (2.36), our conventions are such that

$$\{\gamma_i, \gamma_j\} = -2g_{ij}I, \quad \{\gamma_a, \gamma_b\} = -2\delta_{ab}I. \tag{2.38}$$

We use the notation

$$\gamma_{ij} = \frac{1}{2}(\gamma_i\gamma_j - \gamma_j\gamma_i). \tag{2.39}$$

We can also raise indices

$$\gamma^k{}_j = g^{ki}\gamma_{ij}. \tag{2.40}$$

Recall that an almost-complex structure is an endomorphism $J : TM \to TM$ which satisfies $J^2 = -I$. The main result of this section is the almost-complex structure constructed by Candelas-Horowitz-Strominger-Witten [3].

Proposition 3 *Let (M, g) be a spin manifold of dimension $n = 6$. Let η_+ be a nowhere vanishing spinor of positive chirality (such that $\gamma_7\eta_+ = \eta_+$) normalized by $\eta_+^\dagger\eta_+ = 1$. Then*

$$J^k{}_j = i\eta_+^\dagger\gamma^k{}_j\eta_+ \tag{2.41}$$

equips M with an almost-complex structure compatible with g, meaning that

$$g(JX, JY) = g(X, Y). \tag{2.42}$$

Proof: We first verify J is real using $\gamma_a^\dagger = -\gamma_a$, $(\gamma^k{}_j)^\dagger = -\gamma^k{}_j$, which gives

$$\overline{i\eta_+^\dagger\gamma^k{}_j\eta_+} = -i\eta_+^\dagger(\gamma^k{}_j)^\dagger\eta_+ = i\eta_+^\dagger\gamma^k{}_j\eta_+. \tag{2.43}$$

Next, we verify that J is a well-defined endomorphism. Indeed,

- $\gamma_i \in \Gamma(\mathrm{End}\, S \otimes \Lambda^1(M))$
- $\gamma^i{}_j \in \Gamma(\mathrm{End}\, S \otimes \mathrm{End}\, TM)$
- $\eta_+^\dagger\gamma^i{}_j\eta_+ \in \Gamma(\mathrm{End}\, TM)$.

This is because we noted in Proposition 2 that γ_i is an endomorphism of spinors with cotangent bundle index i, and (2.28) allows us to form bilinears with η_+.

Next, we define

$$T^{1,0}M = \{v \in T_{\mathbf{C}}M : \gamma(v)\eta_+ = 0\}. \tag{2.44}$$

We noted earlier that in this dimension, then η_+ is a pure spinor, and by (2.34) we can decompose

$$T_{\mathbf{C}}M = T^{1,0}M \oplus \overline{T^{1,0}M}. \tag{2.45}$$

We claim that for $v \in T^{1,0}M$, then $Jv = iv$ and $J\bar{v} = -i\bar{v}$. This implies the defining identity of an almost-complex structure

$$J^k{}_\ell J^\ell{}_p = -\delta^k{}_p, \tag{2.46}$$

which is a long computation to check directly. To see $Jv = iv$, we compute

$$J^k{}_j v^j = i\eta^\dagger\gamma^k{}_j v^j\eta = i\eta^\dagger(\gamma^k\gamma_j + \delta^k{}_j)v^j\eta, \tag{2.47}$$

using the identity

$$\gamma^k\gamma_j = \gamma^k{}_j - \delta^k{}_j. \tag{2.48}$$

Since $(\gamma_j v^j)\eta = 0$,
$$J^k{}_j v^j = i(\eta^\dagger \eta) v^k = i v^k, \tag{2.49}$$
as desired. In fact, $Jv = iv$ implies that $\gamma(v)\eta_+ = 0$, so that
$$T^{1,0}M = \{v \in T_{\mathbb{C}}M : Jv = iv\}. \tag{2.50}$$
Indeed, if $Jv = iv$ then from (2.47) we obtain $\eta_+^\dagger \gamma^k \gamma(v)\eta_+ = 0$, and contracting with v_k gives $\eta_+^\dagger \gamma(v)\gamma(v)\eta_+ = 0$ which implies $(\gamma(v)\eta_+)^\dagger \gamma(v)\eta_+ = 0$ since $\gamma_a^\dagger = -\gamma_a$.

Finally, we need to verify
$$g_{mn} = J^k{}_m J^\ell{}_n g_{k\ell}. \tag{2.51}$$
First, we note
$$J_{mn} = -J_{nm}, \quad J_{mn} = g_{np} J^p{}_m, \tag{2.52}$$
since
$$J_{mn} = \eta_+^\dagger \gamma_{mn} \eta_+ \tag{2.53}$$
and γ_{mn} is skew-symmetric by definition. Therefore
$$J^k{}_m J^\ell{}_n g_{k\ell} = J^k{}_m J_{kn} = -J_{nk} J^k{}_m = -g_{np} J^p{}_k J^k{}_m = +g_{nm}. \tag{2.54}$$
Here we used $J^2 = -I$. Therefore g satisfies
$$g(X,Y) = g(JX, JY) \tag{2.55}$$
as claimed. □

We can also define
$$\omega(X,Y) = g(JX, Y) \tag{2.56}$$
which is a skew-symmetric 2-tensor, i.e. $\omega \in \Omega^2(X)$. In terms of the spinor η_+, this is $\omega_{jk} = i\eta_+^\dagger \gamma_{jk} \eta_+$.

By the identities for γ_7 (see (2.23), and the equation below), many other tensors formed by bilinears vanish.

- $\eta_+^\dagger \gamma_i \eta_+ = 0$
- $\eta_+^T \gamma_i \eta_+ = 0$
- $\eta_+^T \gamma_{ij} \eta_+ = 0$
- $\eta_+^\dagger \gamma_{ijk} \eta_+ = 0$.

The 3-form $\eta_+^T \gamma_{ijk} \eta_+$ is non-zero, and later in these notes we will discuss its properties. To show e.g. $\eta_+^\dagger \gamma_i \eta_+ = 0$, we note
$$\eta_+^\dagger \gamma_i \eta_+ = (\gamma_7 \eta_+)^\dagger \gamma_7 \gamma_i \eta_+ = -\eta_+^\dagger \gamma_i \eta_+, \tag{2.57}$$
since $\gamma_7 \gamma_i = -\gamma_i \gamma_7$.

2.3 Nijenhuis tensor

In the previous section, we showed that a nowhere vanishing positive chirality spinor η_+ on (M, g) with $\dim M = 6$ produces an almost-complex structure J compatible with the metric g. Thus spinors have brought us into the field of almost-complex geometry. In this section, we review some basics of the general theory of almost-complex geometry.

Let M be a manifold of dimension $2n$. An almost complex structure $J : TM \to TM$ satisfies $J^2 = -I$. In components, it acts on tangent vectors $V = V^i \partial_i$ by

$$(JV)^i = J^i{}_p V^p. \tag{2.58}$$

Since $J^2 = -I$, we can split

$$T_{\mathbb{C}} M = T^{1,0} M \oplus T^{0,1} M \tag{2.59}$$

where $T^{1,0}M$ is the $+i$ eigenspace of J and $T^{0,1}M$ is the $-i$ eigenspace of J. Explicitly,

$$T^{1,0}_p M = \text{span}\,\{X - iJX : X \in T_p M\}. \tag{2.60}$$

Here TM is the real tangent bundle, and $T_{\mathbb{C}} M = TM \otimes \mathbb{C}$ is the complexified tangent bundle, where we allow linear combinations of vectors in TM with complex coefficients. We see that an almost-complex structure produces a complex vector bundle $T^{1,0}M \to M$ of rank n.

From the decomposition (2.59), we can decompose differential forms into (p, q) type.

$$\Lambda^k (T_{\mathbb{C}})^* = \sum_{p+q=k} \Lambda^p (T^{1,0} M)^* \otimes \Lambda^q (T^{0,1} M)^* := \sum_{p+q=k} \Omega^{p,q}(M). \tag{2.61}$$

In terms of local frames, if $\{e_i\}$ is a local frame of $T^{1,0}M$ then

$$\{e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n\} \tag{2.62}$$

locally generates $T_{\mathbb{C}} M$ and

$$e^{i_1} \wedge \cdots \wedge e^{i_p} \wedge \overline{e^{j_1}} \wedge \cdots \wedge \overline{e^{j_q}} \tag{2.63}$$

locally generates $\Omega^{p,q}(M)$. Here $\{e^i\}$ is the dual frame to $\{e_i\}$. For example, a 3-form η can be written as

$$\eta = \eta^{3,0} + \eta^{2,1} + \eta^{1,2} + \eta^{0,3}, \tag{2.64}$$

where

$$\begin{aligned}\eta^{3,0}(e_i, e_j, e_k) &= \eta(e_i, e_j, e_k), \\ \eta^{3,0}(\overline{e_i}, e_j, e_k) &= \eta^{3,0}(\overline{e_i}, \overline{e_j}, e_k) = \eta^{3,0}(\overline{e_i}, \overline{e_j}, \overline{e_k}) = 0,\end{aligned} \tag{2.65}$$

and similarly for the other components, e.g. $\eta^{2,1}(e_i, e_j, \bar{e}_k) = \eta(e_i, e_j, \bar{e}_k)$.

Let g be a metric on M which is compatible with J, which means that $g(JV, JW) = g(V, W)$. We will denote

$$g_{ij} = g(e_i, e_j), \quad g_{\bar{i}j} = g(\overline{e}_i, e_j), \quad g_{\bar{i}\bar{j}} = g(\overline{e}_i, \overline{e}_j). \tag{2.66}$$

Compatibility with J implies that

$$g_{ij} = 0, \quad g_{\bar{i}\bar{j}} = 0, \tag{2.67}$$

and hence only the metric components $g_{\bar{k}j} = g_{j\bar{k}}$ are non-zero. Since g is real and symmetric, then

$$\overline{g_{\bar{k}j}} = g_{\bar{j}k}. \tag{2.68}$$

The Nijenhuis tensor $N : TM \times TM \to TM$ is defined by

$$N(X,Y) = \frac{1}{4}([JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]). \tag{2.69}$$

The interpretation of N is that it measures the failure of $T^{1,0}M$ being closed under taking the Lie bracket: let $U, V \in T^{1,0}M$, so that

$$\begin{aligned}N(U,V) &= \frac{1}{4}(i^2[U,V] - iJ[U,V] - iJ[U,V] - [U,V]) \\ &= -\frac{1}{2}([U,V] + iJ[U,V]).\end{aligned} \tag{2.70}$$

For any vector X, we can write

$$X = \frac{1}{2}(X - iJX) + \frac{1}{2}(X + iJX), \tag{2.71}$$

and this gives the decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, i.e.

$$X^{1,0} = \frac{1}{2}(X - iJX), \quad X^{0,1} = \frac{1}{2}(X + iJX). \tag{2.72}$$

With this interpretation, then

$$N(U,V) = -[U,V]^{0,1}. \tag{2.73}$$

Similarly,

$$N(\overline{U}, V) = 0. \tag{2.74}$$

It follows that $N = 0$ if and only if $[U, V] \in T^{1,0}M$ for all $U, V \in T^{1,0}M$. In a frame $\{e_i\}$ generating $T^{1,0}M$, then this discussion implies that N is determined by the components $N_{ij}^{\bar{k}}$.

Next, let's write

$$N = \frac{1}{2} N^p{}_{mn} \, dx^m \wedge dx^n \otimes \partial_p \tag{2.75}$$

and look at the components of the Nijenhuis tensor. We have

$$4N(\partial_m, \partial_n) = [J^i{}_m \partial_i, J^k{}_n \partial_k] - J[J^i{}_m \partial_i, \partial_n] - J[\partial_m, J^i{}_n \partial_i], \tag{2.76}$$

which becomes

$$4N(\partial_m, \partial_n) = J^i{}_m \partial_i J^p{}_n \partial_p - J^k{}_n \partial_k J^i{}_m \partial_i + J(\partial_n J^p{}_m \partial_p) - J(\partial_m J^i{}_n \partial_i). \tag{2.77}$$

which simplifies to

$$4N(\partial_m, \partial_n) = (J^q{}_m \partial_q J^p{}_n - J^q{}_n \partial_q J^p{}_m + J^p{}_q \partial_n J^q{}_m - J^p{}_q \partial_m J^q{}_n) \partial_p. \tag{2.78}$$

Then

$$N^p{}_{mn} = \frac{1}{4} (J^q{}_m \partial_q J^p{}_n + J^p{}_q \partial_n J^q{}_m - J^q{}_n \partial_q J^p{}_m - J^p{}_q \partial_m J^q{}_n). \tag{2.79}$$

Writing

$$\nabla_q J^p{}_n = \partial_q J^p{}_n + \Gamma^p_{qr} J^r{}_n - \Gamma^r_{qn} J^p{}_r \tag{2.80}$$

for the Levi-Civita connection, a calculation gives the expression

$$N^p{}_{mn} = \frac{1}{4} (J^q{}_m \nabla_q J^p{}_n + J^p{}_q \nabla_n J^q{}_m - J^q{}_n \nabla_q J^p{}_m - J^p{}_q \nabla_m J^q{}_n) \tag{2.81}$$

since $\Gamma^k_{ij} = \Gamma^k_{ji}$.

The Newlander-Nirenberg theorem relates almost-complex structures with $N = 0$ to holomorphic coordinate charts. A proof can be found in e.g. [5].

Theorem 2 *Let J be an almost-complex structure on a real manifold M. If $N_J = 0$, then M admits a complex structure: there are holomorphic coordinates making M a complex manifold.*

As a consequence of (2.81), we obtain:

Corollary 1 *Let J be an almost-complex structure on a manifold (M, g). Let ∇ denote the Levi-Civita connection. If $\nabla J = 0$, then M admits holomorphic coordinates making M a complex manifold.*

We will denote holomorphic coordinates by $\{z^i\}$. The corresponding real coordinates $\{(x^i, y^i)\}$ will be denoted

$$z^k = x^k + iy^k, \quad \bar{z}^k = x^k - iy^k, \tag{2.82}$$

and we denote

$$\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right), \quad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right). \tag{2.83}$$

Note that the dual vector field to $dz^k = dx^k + idy^k$ is $\frac{\partial}{\partial z^k}$. From a complex manifold, we obtain an almost-complex structure J by setting

$$J\frac{\partial}{\partial z^k} = i\frac{\partial}{\partial z^k}, \quad J\frac{\partial}{\partial \bar{z}^k} = -i\frac{\partial}{\partial \bar{z}^k}. \tag{2.84}$$

In other words,

$$T_p^{1,0}M = \mathrm{Span}\left\{\frac{\partial}{\partial z^1}\bigg|_p, \ldots, \frac{\partial}{\partial z^n}\bigg|_p\right\} \tag{2.85}$$

and this is well-defined since change of coordinates are holomorphic. J is a real endomorphism of the real tangent bundle, since in terms of real coordinates $\{(x^i, y^i)\}$ it is

$$J\frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}, \quad J\frac{\partial}{\partial y^k} = -\frac{\partial}{\partial x^k}. \tag{2.86}$$

In $\{(x^i, y^i)\}$ coordinates, the endomorphism J appears as the constant matrix

$$[J^i{}_j] = \begin{bmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{bmatrix}. \tag{2.87}$$

From (2.79), we see that $N_J = 0$.

3 Connections on Bundles

3.1 Notation

Let (M, g) be an oriented Riemannian manifold. Let $E \to M$ be a complex vector bundle of rank k trivialized by coordinate charts $\{U_\alpha\}$ and with transition matrices $\{(U_\alpha \cap U_\beta, t_{U_\alpha U_\beta})\}$.

$$t_{UV} : U \cap V \to \mathrm{Mat}_{k\times k}(\mathbb{C}). \tag{3.1}$$

Recall that this means that sections $s \in \Gamma(E)$ appear as a collection of local functions $\{U_\alpha, s_{U_\alpha}\}$ with the $s_U : U \to \mathbb{C}^k$ satisfying

$$s_U = t_{UV} s_V \tag{3.2}$$

on overlaps $U \cap V$. From the point of view of trivializing local frames $\{e^U_\alpha\}_{\alpha=1}^k$, this means a section can be written as $s = (s_U)^\alpha (e^U)_\alpha$ over U and $(e^U)_\alpha = (e^V)_\beta {t_{UV}}^\beta{}_\alpha$.

A connection ∇ on E is given by a collection of local matrix-valued 1-forms $\{U_\alpha, A_{U_\alpha}\}$, so that $A_U = (A_U)_i dx^i$ with $(A_U)_i : U \to \mathrm{Mat}_{k\times k}(\mathbb{C})$, satisfying

$$(A_U)_i = t_{UV}(A_V)_i t_{UV}^{-1} - \partial_i t_{UV} t_{UV}^{-1}, \tag{3.3}$$

on overlaps $U \cap V$. This definition is so that $\nabla = d + A$ defines a derivative of sections of E: if we define the derivative on local components by

$$\nabla_i s_U = \partial_i s_U + (A_U)_i s_U, \tag{3.4}$$

then

$$\nabla_i s_U = t_{UV} \nabla_i s_V. \tag{3.5}$$

This transformation law implies that $\nabla_X s \in \Gamma(E)$ for any vector field X. Here we let $\nabla_X s = X^i \nabla_i s_U$ where $X = X^i \partial_i$ over U. If α, β indices track the column vector index, then we write the derivative on local components (3.4) as

$$\nabla_i s^\alpha = \partial_i s^\alpha + {A_i}^\alpha{}_\beta s^\beta. \tag{3.6}$$

We can also understand connections from the point of view of local frames rather than local components. Using local frames, we write $s = s^\alpha e_\alpha$ and the connection acts by

$$\nabla_{\partial_i} s = \partial_i s^\alpha e_\alpha + s^\alpha \nabla_{\partial_i} e_\alpha = (\partial_i s^\alpha + {A_i}^\alpha{}_\beta s^\beta) e_\alpha, \tag{3.7}$$

where $\nabla_{\partial_i} e_\alpha = A_i{}^\beta{}_\alpha e_\beta$.

Here is how this notation appears in the case of the tangent bundle. Let ∇ be a metric compatible covariant derivative such as the Levi-Civita connection. In a coordinate chart $\{x^i\}$ a vector field $V = V^i \frac{\partial}{\partial x^i}$ has covariant derivative

$$\nabla_{\partial_k} V = (\nabla_k V^i)\,\partial_i, \quad \nabla_k V^i = \partial_k V^i + \Gamma^i_{kp} V^p, \tag{3.8}$$

and the connection coefficients are denoted by Γ^k_{ij} rather than $A_i{}^k{}_j$. We will also use trivializations of the tangent bundle which do not come from local coordinates, but instead from a local orthonormal frame $\{e_a\}$. Let $e_a = e^i{}_a \partial_i$ be an orthonormal frame of TM and write vector fields as $V = V^a e_a$.

$$\nabla_{\partial_i} V = (\partial_i V^b + \omega_i{}^b{}_a V^a) e_b \tag{3.9}$$

where $\nabla_{\partial_i} e_a = \omega_i{}^b{}_a e_b$ denote the connection coefficients in this frame. Let ω_i be the matrix with entries $[\omega_i]^b{}_a = \omega_i{}^b{}_a$. Since ∇ is metric compatible with g, it follows that $\omega_i{}^b{}_a = -\omega_i{}^a{}_b$. Indeed, metric compatibility means

$$0 = g(\nabla_{\partial_i} e_a, e_b) + g(e_a, \nabla_{\partial_i} e_b), \quad \omega_i{}^b{}_a = g(\nabla_{\partial_i} e_a, e_b). \tag{3.10}$$

This is one difference from working with an orthonormal frame $\{e_a\}$ and $\omega_i{}^b{}_a$ rather than a coordinate frame $\{\partial_i\}$ and Γ^k_{ij}. Thus we can write ω_i in terms of a basis of skew-symmetric matrices. For example, in dimension $n = 3$ then

$$\omega_i = \omega_i{}^2{}_1 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \omega_i{}^3{}_1 \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \omega_i{}^3{}_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{3.11}$$

and in general we have

$$\omega_i = \sum_{a<b} \omega_i{}^{ba} \varepsilon_{ab}. \tag{3.12}$$

Here ε_{ij} with $i < j$ is the $n \times n$ matrix with a -1 at the (ij) entry and 1 at the (ji) entry, and $\varepsilon_{ii} = 0$. The coefficients $\omega_i{}^{ba}$ in this matrix sum are just $\omega_i{}^{ba} = \omega_i{}^b{}_a$. If we let $\varepsilon_{ba} = -\varepsilon_{ab}$ for $a < b$, then we can write this using the full sum over all a, b:

$$\omega_i = \frac{1}{2} \sum_{a,b} \omega_i{}^{ba} \varepsilon_{ab}. \tag{3.13}$$

In parallel with the notation in (3.4), in an orthonormal frame the Levi-Civita connection can be written as acting on components of vector fields by

$$\nabla_i V = \partial_i V + \frac{1}{2} \omega_i{}^{ba} \varepsilon_{ab} V. \tag{3.14}$$

We now motivate the definition of the spin connection. Let (M, g) be a manifold of dimension n with spin structure $\mathrm{Spin}(n) \to M$ and let $\gamma : \mathrm{Cliff}(n) \to \mathrm{Mat}_{k\times k}(\mathbb{C})$ be a matrix representation. As discussed earlier, this produces a spinor bundle $S \to M$. Since the double cover $\varphi : \mathrm{Spin}(n) \to SO(n)$ identifies ε_{ab} with $\frac{1}{2}\gamma_a\gamma_b$ via φ_* (1.34), we guess that the connection induced by the Levi-Civita connection on sections ψ of the spin bundle $S \to M$ acts on components by

$$\nabla_i\psi = \partial_i\psi + \frac{1}{4}\omega_i{}^{ab}\gamma_b\gamma_a\psi. \tag{3.15}$$

Recall γ_a denotes the fixed γ-matrices $\gamma_1, \ldots, \gamma_n$ corresponding to the standard basis of $\mathbb{R}^n$ in the representation $\gamma : \mathrm{Cliff}(n) \to \mathrm{Mat}_{k\times k}(\mathbb{C})$. We will verify that this formula gives a well-defined connection on spinors in the following section.

But before that, we introduce more notation. It will sometimes be useful to write the spin connection using curved, varying gamma matrices. Recall our conventions for coefficients relating the orthonormal frame $\{e_a\}$ and the coordinate frame $\{\partial_i\}$: $e_a = e^i{}_a\partial_i$ and $\partial_i = e^a{}_ie_a$. We write $\gamma_i = e^a{}_i\gamma_a$ for the curved gamma matrices which vary from point to point. We also write $\omega_i{}^{jk} = e^j{}_ae^k{}_b\omega_i{}^{ab}$. Then the spin connection in coordinates $\{x^i\}$ is

$$\nabla_i\psi = \partial_i\psi + \frac{1}{4}\omega_i{}^{jk}\gamma_k\gamma_j\psi. \tag{3.16}$$

Verifying this is a straight-forward calcultion using $e^b{}_ie^i{}_a = \delta^b{}_a$.

3.2 Spin connection

Let's verify that the spin connection

$$\nabla_i\psi = \partial_i\psi + \frac{1}{4}\omega_i{}^{ab}\gamma_b\gamma_a\psi. \tag{3.17}$$

is a legitimate connection. Here the indices a, b represent an orthonormal frame. Let

$$A(X) = \frac{1}{4}\omega^{ba}(X)\gamma_a\gamma_b. \tag{3.18}$$

Let $(U, \{e_a^U\}_{a=1}^n)$, $(V, \{e_a^V\}_{a=1}^n)$ be an overlap of trivializations by orthonormal frames of TM. We need to show

$$A_U = \rho A_V\rho^{-1} - d\rho\,\rho^{-1}. \tag{3.19}$$

Here $\rho : U \cap V \to GL(k, \mathbb{C})$ is given by $\rho(x) = \gamma(\tilde{\Lambda}(x))$, where

$$\Lambda : U \cap V \to SO(n), \quad e_a^U = e_b^V \Lambda^b{}_a \tag{3.20}$$

is the transition function for TM, and the spin structure gives us lifts $\tilde{\Lambda}(x) \in \mathrm{Spin}(n)$. If ω_U is the local connection form of the Levi-Civita connection over U, then we have

$$\omega_U = \Lambda \omega_V \Lambda^T - d\Lambda \Lambda^T. \tag{3.21}$$

Therefore

$$A_U = \frac{1}{4}(\Lambda \omega_V \Lambda^T - (d\Lambda)\Lambda^T)^{ba} \gamma_a \gamma_b. \tag{3.22}$$

The defining identity for the spin group lift of Λ is (1.14), which we rewrite here

$$\rho \gamma_a \rho^{-1} = \Lambda^b{}_a \gamma_b. \tag{3.23}$$

We start by proving

$$\frac{1}{4}(\Lambda \omega_V \Lambda^T)^{ba} \gamma_a \gamma_b = \rho A_V \rho^{-1}. \tag{3.24}$$

Converting matrix multiplication to index notation, we have

$$\frac{1}{4}(\Lambda \omega_V \Lambda^T)^{ba} \gamma_a \gamma_b = \frac{1}{4} {\omega_V}^{cd} \, \Lambda^a{}_d \gamma_a \, \Lambda^b{}_c \gamma_b. \tag{3.25}$$

From (3.23), we obtain

$$\frac{1}{4}(\Lambda \omega_V \Lambda^T)^{ba} \gamma_a \gamma_b = \frac{1}{4} {\omega_V}^{ba} (\rho \gamma_a \rho^{-1}) \, (\rho \gamma_b \rho^{-1}) \tag{3.26}$$

and hence (3.24) follows. Next, we need to show

$$\frac{1}{4}(d\Lambda \Lambda^T)^{ba} \gamma_a \gamma_b = d\rho(\Lambda) \rho(\Lambda)^{-1}. \tag{3.27}$$

Let $\Lambda \in SO(n)$, and recall $T_\Lambda SO(n) = \{\Lambda X : X \in \mathrm{Lie}(SO(n))\}$. Any path $\Lambda(t) \in SO(n)$ with $\Lambda(0) = \Lambda$ has tangent vector of the form $\dot{\Lambda}(0) = \Lambda X^{ab} \varepsilon_{ab}$ where ε_{ab} is the basis of skew-symmetric matrices given in (3.12). So we must show

$$\frac{1}{4}\left(\left[\frac{d}{dt}\bigg|_{t=0} \Lambda \right] \Lambda(0)^T \right)^{ba} \gamma_a \gamma_b = \left[\frac{d}{dt}\bigg|_{t=0} \rho(\Lambda(t)) \right] \rho(\Lambda(0))^{-1} \tag{3.28}$$

for a path with $\dot{\Lambda}(0) = \Lambda\varepsilon_{cd}$. A possible such path is

$$\Lambda(t) = \Lambda e^{t\varepsilon_{cd}} \in SO(n), \tag{3.29}$$

and by Proposition 1 and $c < d$

$$\rho(\Lambda(t)) = \rho e^{(t/2)\gamma_c\gamma_d}. \tag{3.30}$$

Write $\delta = \frac{d}{dt}|_{t=0}$, so that

$$\delta\Lambda = \Lambda\varepsilon_{cd}, \quad \delta\rho = \frac{1}{2}\rho\gamma_c\gamma_d. \tag{3.31}$$

We compute

$$\frac{1}{4}(\delta\Lambda\Lambda^T)^{ba}\gamma_a\gamma_b = \frac{1}{4}\Lambda^b{}_\ell(\varepsilon_{cd})^\ell{}_q\Lambda^a{}_q\gamma_a\gamma_b \tag{3.32}$$

which becomes by (3.23)

$$\frac{1}{4}(\delta\Lambda\Lambda^T)^{ba}\gamma_a\gamma_b = \frac{1}{4}\rho(\varepsilon_{cd})^\ell{}_q\gamma_q\gamma_\ell\rho^{-1} = \frac{1}{4}\rho(-\gamma_d\gamma_c + \gamma_c\gamma_d)\rho^{-1}. \tag{3.33}$$

Therefore

$$\frac{1}{4}(\delta\Lambda\Lambda^T)^{ba}\gamma_a\gamma_b = \left[\frac{1}{2}\rho\gamma_c\gamma_d\right]\rho^{-1} = \delta\rho\rho^{-1} \tag{3.34}$$

by (3.31). This concludes the derivation of (3.28).

3.3 Derivative of gamma matrices

Denote the spin connection by $\nabla + A$. Recall that $\gamma \in \Gamma(T^*M \otimes \mathrm{End}(S))$. For a vector field X, then $\gamma(X) \in \Gamma(\mathrm{End}(S))$. The covariant derivative is defined on $\Gamma(\mathrm{End}(S))$ such that it satisfies the product rule

$$\nabla_i(\gamma(X)\eta) = \nabla_i(\gamma(X))\eta + \gamma(X)\nabla_i\eta. \tag{3.35}$$

The right definition for this is

$$\nabla_i\gamma(X) = \partial_i\gamma(X) + A_i\gamma(X) - \gamma(X)A_i. \tag{3.36}$$

In components, the induced connection on a endomorphism-valued 1-form is

$$\nabla_i\gamma_a = \partial_i\gamma_a - \omega_i{}^b{}_a\gamma_b + A_i\gamma_a - \gamma_a A_i. \tag{3.37}$$

Recall that γ_a are the constant components of $\gamma = \gamma_a\, e^a$ in an orthonormal frame $\{e_a\}$. Therefore the first ∂_i term in the above formula vanishes. In fact, this geometry is setup such that

$$\nabla_i \gamma_b = 0. \tag{3.38}$$

Another way to write this identity (without indices) is

$$\nabla(\gamma(X)\eta) = \gamma(\nabla X)\eta + \gamma(X)\nabla\eta. \tag{3.39}$$

To check $\nabla_i \gamma_a = 0$, we expand

$$A\gamma_c - \gamma_c A = \frac{1}{4}(\omega^{ba}\gamma_a\gamma_b\gamma_c - \omega^{ba}\gamma_c\gamma_a\gamma_b) \tag{3.40}$$

We have

$$\gamma_a\gamma_b\gamma_c = \gamma_a(-\gamma_c\gamma_b - 2\delta_{cb}) = \gamma_c\gamma_a\gamma_b + 2\delta_{ac}\gamma_b - 2\delta_{cb}\gamma_a, \tag{3.41}$$

hence

$$\omega^{ba}\gamma_a\gamma_b\gamma_c = \omega^{ba}\gamma_c\gamma_a\gamma_b + 4\omega^{ac}\gamma_a. \tag{3.42}$$

Therefore, since $\omega^{ac} = \omega^a{}_c$ in an orthonormal frame,

$$A\gamma_c - \gamma_c A = \omega^a{}_c\gamma_a \tag{3.43}$$

Since γ_b is a constant matrix, then $\partial_i \gamma_b = 0$ and putting everything together shows

$$\nabla_i \gamma_b = 0. \tag{3.44}$$

3.4 Holonomy

For further details relating to this section, see [9].

Let $E \to M$ be a vector bundle with connection ∇. Let $\gamma : [0,1] \to M$ be a curve with $\dot{\gamma} \neq 0$. The pullback bundle $\gamma^* E$ is a vector bundle over $[0,1]$. It can be described as follows. Let E be trivialized by a cover $M = \bigcup U_\mu$ with transition functions $t_{\mu\nu}$. Then $\gamma^* E$ is trivialized by $\bigcup \gamma^{-1}(U_\mu)$ with transition functions $t_{\mu\nu} \circ \gamma$. If $\nabla = d + A$, then the pullback connection is

$$\gamma^* \nabla = d + \gamma^* A. \tag{3.45}$$

Let s be a local section of E. Then $s \circ \gamma$ is a local section of $\gamma^* E$. We say s is parallel along γ if

$$(\gamma^* \nabla)(s \circ \gamma) = 0. \tag{3.46}$$

In a local trivialization, then

$$s \circ \gamma = \begin{bmatrix} s^1(t) \\ \vdots \\ s^k(t) \end{bmatrix} \tag{3.47}$$

and this condition reads

$$\dot{s}^\alpha + \dot{\gamma}^i A_i{}^\alpha{}_\beta s^\beta = 0. \tag{3.48}$$

This is an ODE for $s(t)$, which admits a unique solution given initial conditions. If the path $\gamma(t)$ crosses two trivializations of E, we can stop the ODE on the overlap and restart the corresponding ODE in the next trivialization.

Let e be a point in the fiber over $\gamma(0)$. We define the parallel transport map P_γ by

$$P_\gamma(e) = s(1), \tag{3.49}$$

where $s(t)$ is the unique section of $\gamma^* E$ such that $(\gamma^* \nabla) s(t) = 0$ and $s(0) = e$.

Remark: parallel transport allows us to add/subtract sections at different basepoints and make sense of expressions such as $s(q) - s(p)$ evaluated at p. To do this, connect p and q by a geodesic γ. Choose a frame $\{e_a(p)\}$ for the fiber $E|_p$ and parallel transport this frame along γ. We can write s in this basis so that $s(q) = s(q)^a e_a(q)$ and $s(p) = s(p)^a e_a(p)$. To parallel transport $s(q)$ along γ means to write $s(t) = s(q)^a e_a(t)$, so transporting $s(q)^a e_a(t)$ to p produces $s(q)^a e_a(p)$ and we can interpret

$$s(q) - s(p) := (s^a(q) - s^a(p)) e_a(p). \tag{3.50}$$

Next, we define holonomy. Let $p \in M$. We define

$$\mathrm{Hol}_p(\nabla) = \{P_\gamma \text{ with } \gamma : [0,1] \to M \text{ such that } \gamma(0) = \gamma(1) = p\}. \tag{3.51}$$

The claim is this is a group with the operation being composition $P_\gamma P_\eta = P_\gamma \circ P_\eta$. Furthermore, P_γ is an endomorphism of E_x, so after choosing a trivialization we can view

$$\mathrm{Hol}_p(\nabla) \subseteq GL(r, \mathbb{R}). \tag{3.52}$$

Here r is the rank of E. A different choice of trivialization will produce a different subgroup of $GL(r, \mathbb{R})$, but these two groups agree up to conjugation by the change of basis formula. Also, we will omit the basepoint p in the notation from now on, since it can be verified that different choices of basepoints only change the group by conjugation of a matrix in $GL(r, \mathbb{R})$.

To summarize our conventions, if we find a point p and a choice of trivialization of E around p such that

$$\mathrm{Hol}_p(\nabla) = G \subseteq GL(r, \mathbb{R}), \tag{3.53}$$

we will say that $\mathrm{Hol}(\nabla) = G$, where G is an explicit group of matrices.

3.4.1 Orthogonal group

Let (M, g) be Riemannian manifold of dimension n. If ∇ is a connection on TM satisfying $\nabla g = 0$, then

$$\mathrm{Hol}(\nabla) \subseteq O(n). \tag{3.54}$$

Indeed, let $V \in T_pM$ such that $g(V, V) = 1$ and let γ be a loop based at p. We parallel transport V to obtain a section of $\gamma^* TM$ denoted $V(t)$. We also obtain a section $g \circ \gamma$ of $\gamma^*(T^*M \otimes T^*M)$ denoted $g_{ij}(t)$. Since $(\gamma^*\nabla)V(t) = 0$ and $\nabla g = 0$ imply

$$\frac{d}{dt}(g_{ij}(t)V^i(t)V^j(t)) = 0, \tag{3.55}$$

we conclude

$$g(P_\gamma V, P_\gamma V) = 1. \tag{3.56}$$

If we choose coordinates at p such that $g_{ij}(p) = \delta_{ij}$, then $P_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation.

3.4.2 Unitary group

Let (M, g) be equipped with an almost-complex structure J compatible with g and a connection ∇ such that

$$\nabla g = 0, \quad \nabla J = 0. \tag{3.57}$$

Let γ be a loop in M with $\gamma(0) = p$. Choose coordinates at p so that we have

$$P_\gamma : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad g_{ij}(p) = \delta_{ij}, \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{3.58}$$

where I_n is the $n \times n$ identity matrix. We want to show that P_γ is J-invariant, which means $P_\gamma J = JP_\gamma$. This is true since if $V(t)$ satisfies $(\gamma^*\nabla)V = 0$, so does

$$(\gamma^*\nabla)(JV) = 0, \tag{3.59}$$

so the transport of $JV(0)$ is given by $JV(t)$, as claimed. This implies that P_γ can be identified with a complex matrix in $GL(n, \mathbb{C})$ via

$$\iota : GL(n, \mathbb{C}) \to GL(2n, \mathbb{R}), \quad \iota(A + iB) = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}. \tag{3.60}$$

(Check ι is a well-defined isomorphism.) By the previous section

$$\mathrm{Hol}_p(\nabla) \subseteq O(2n). \tag{3.61}$$

A direct check shows that if $P \in O(2n)$ and $PJ = JP$ then $\iota^{-1}(P) \in U(n)$, where $U(n)$ is the set of $n \times n$ unitary matrices. Indeed, expanding

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}^T \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \tag{3.62}$$

yields

$$(A + iB)^\dagger (A + iB) = I_n. \tag{3.63}$$

Thus

$$\mathrm{Hol}(\nabla) \subseteq U(n). \tag{3.64}$$

Here is another way to view this. If $V \in T^{1,0}_p M$, then $P_\gamma J = JP_\gamma$ implies that $P_\gamma V \in T^{1,0}_p M$. This means parallel transport descends to

$$P_\gamma : T^{1,0}_p M \to T^{1,0}_p M. \tag{3.65}$$

This is represented by a matrix $[P_\gamma] \in GL(n, \mathbb{C})$, and since we showed earlier that

$$H(P_\gamma V, P_\gamma V) = H(V, V), \quad H(v, w) = g(v, \bar{w}), \tag{3.66}$$

then $[P_\gamma] \in U(n)$.

3.4.3 Special unitary group

Let (M, g) be equipped with a complex structure J and a nowhere vanishing smooth $(n, 0)$ form Ψ. Let ∇ be a connection on TM such that

$$\nabla g = 0, \quad \nabla J = 0, \quad \nabla \Psi = 0. \tag{3.67}$$

Let $p \in M$. Choose complex coordinates at p, and possibly rescale Ψ, such that

$$g|_p = \sum dz^k \otimes d\bar{z}^k, \quad \Psi|_p = dz^1 \wedge \cdots \wedge dz^n. \tag{3.68}$$

Let γ be a loop based at p. Let $U_1, \ldots, U_n \in T_p^{1,0}M$ be the tangent vectors which appear as the standard basis in $\mathbb{C}^n$ in the complex coordinates above. Let $U_1(t), \ldots, U_n(t)$ denote their parallel transport along $\gamma(t)$. We showed previously that

$$U_1(t), \ldots, U_n(t) \in T_p^{1,0}M \tag{3.69}$$

at all times t. We also have

$$\frac{d}{dt}\Psi(U_1(t), \ldots, U_n(t)) = 0. \tag{3.70}$$

It follows that

$$1 = (dz^1 \wedge \cdots \wedge dz^n)(P_\gamma U_1, \ldots, P_\gamma U_n) = \det [P_\gamma]. \tag{3.71}$$

Here $P_\gamma : T_p^{1,0}M \to T_p^{1,0}M$ is parallel transport on $T^{1,0}M$. In the previous section, we showed that $[P_\gamma] \in U(n)$, and so

$$\text{Hol}(\nabla) \subseteq SU(n). \tag{3.72}$$

4 Calabi-Yau Geometry

4.1 From spinors to complex manifolds

4.1.1 Candelas-Horowitz-Strominger-Witten

We return to the setup of §2.2. Let (M, g) be a spin manifold of dimension $n = 6$ with spinor bundle $S \to M$. Let $\eta \in \Gamma(S)$ be a nowhere vanishing spinor of positive chirality satisfying

$$\nabla \eta = 0, \quad \eta^\dagger \eta = 1. \tag{4.1}$$

Let $J^p{}_q = i\eta^\dagger \gamma^p{}_q \eta$ be the almost complex structure defined in §2.2. A calculation (given below) gives

$$\nabla J = 2i\eta^\dagger \gamma^p{}_q \nabla \eta. \tag{4.2}$$

From $\nabla \eta = 0$, it follows that

$$\nabla J = 0. \tag{4.3}$$

By (2.81) and the Newlander-Nirenberg theorem, it follows that M can be given the structure of a complex manifold.

In string theory, equation (4.1) arises from supersymmetry conditions on $\mathbb{R}^{3,1} \times M^6$. It was Candelas-Horowitz-Strominger-Witten [3] who observed that M^6 can be then be given the structure of a complex manifold, and this brought string theory into the world of complex geometry. In fact, [3] further showed that $\Omega_{ijk} = \bar{\eta}^T \gamma_{ijk} \bar{\eta}$ defines a nowhere vanishing holomorphic section of the canonical bundle (we will discuss this in §4.4 below), making M^6 a Calabi-Yau threefold.

To do: add discussion on a converse. If M is complex manifold with trivial canonical bundle, does M admit a spin structure? How to recover the spinor η from the holomorphic volume form Ω? Related reference: [Atiyah: Riemann surfaces and spin structures]

We now discuss the derivation of (4.2), which follows from the fact that ∇ obeys the Leibniz rule under Clifford multiplication by γ matrices. We showed earlier that $\nabla \gamma^p{}_q = 0$. Then

$$\nabla J = i\nabla \eta^\dagger \gamma^p{}_q \eta + i\eta^\dagger \gamma^p{}_q \nabla \eta = 0. \tag{4.4}$$

Next, we note $\eta^\dagger \in \Gamma(S^*)$ and $\nabla \eta^\dagger$ is the induced connection on the dual bundle S^*. If η is locally the column vector η^a, then $\eta^\dagger$ is locally the row vector $\eta_a = \overline{\eta^a}$, and

$$\nabla \eta_a = d\eta_a - \eta_b A^b{}_a. \tag{4.5}$$

The spin connection is $A = \frac{1}{4}\omega^{ba}\gamma_a\gamma_b$. It satisfies

$$A^\dagger = \frac{1}{4}\omega^{ba}\gamma_b^\dagger\gamma_a^\dagger = \frac{1}{4}\omega^{ba}\gamma_b\gamma_a = -A. \tag{4.6}$$

Here we used $\gamma_a^\dagger = -\gamma_a$ and $\omega^{ab} = -\omega^{ba}$. Therefore

$$\nabla \eta_a = d\eta_a + \bar{A}^a{}_b\eta_b = \overline{d\eta^a + A^a{}_b\eta^b}. \tag{4.7}$$

It follows that $\nabla\eta^\dagger = (\nabla\eta)^\dagger$.

4.1.2 Strominger

We will study a generalization of the above result, which is due to Strominger [11]. Let (M, g) be a spin manifold of dimension $n = 6$. Let H be a 3-form. We will encode this differential form into our geometry by introducing the connection

$$\hat{\nabla} = \nabla + \frac{1}{2}g^{-1}H. \tag{4.8}$$

Here ∇ denotes the Levi-Civita connection. If we write $\hat{\nabla} = d + A$, the connection forms are

$$A^k_{ij} = \Gamma^k_{ij} + \frac{1}{2}g^{kp}H_{pij}. \tag{4.9}$$

We also write $H^k{}_{ij} = g^{kp}H_{pij}$. The torsion of this connection is

$$A^k{}_{ij} - A^k{}_{jk} = \frac{1}{2}H^k{}_{ij} - \frac{1}{2}H^k{}_{ji} = H^k{}_{ij}. \tag{4.10}$$

We can induce $\hat{\nabla}$ on all associated bundles as usual. We note that $\hat{\nabla}g = 0$ for any 3-form H. Indeed,

$$\hat{\nabla}_i g_{kj} = \nabla_i g_{kj} - \frac{1}{2}H^m{}_{ik}g_{mj} - \frac{1}{2}H^m{}_{ij}g_{km} = 0, \tag{4.11}$$

since H_{ijk} is skew-symmetric. Let η be a nowhere vanishing spinor of positive chirality satisfying

$$\hat{\nabla}\eta = 0. \tag{4.12}$$

Written out in more detail, the induced spin connection is

$$\hat{\nabla}_i \eta = (\nabla_i + \frac{1}{8} H^a{}_i{}^b \gamma_b \gamma_a)\eta = 0. \tag{4.13}$$

using the constant frame gamma matrices, and

$$\hat{\nabla}_i \eta = (\nabla_i + \frac{1}{8} H_{jik} \gamma^k \gamma^j)\eta = 0. \tag{4.14}$$

using curved coordinate gamma matrices. If we let $J^p{}_q = i\eta^\dagger \gamma^p{}_q \eta$ as before, then

$$\hat{\nabla} J = 0, \quad g(JX, JY) = g(X, Y). \tag{4.15}$$

Unlike for the Levi-Civita connection, $\hat{\nabla} J = 0$ does not imply that J is integrable for a non-zero 3-form field H. Strominger [11] proved that J is in fact integrable for non-zero H provided the dilatino equation

$$(H + 2d\phi) \cdot \eta = 0, \tag{4.16}$$

holds for some function ϕ (this function is called the dilaton function).

In Candelas-Horowitz-Strominger-Witten's model, the 3-form field strength H was set to zero and the scalar field ϕ set to a constant. Strominger proved that we can arrive at complex geometry from the supersymmetric equations $\hat{\nabla}\eta = 0$, $(H + 2\phi) \cdot \eta = 0$ even with non-zero 3-form field H and non-constant scalar field ϕ. We will present the derivation of this result in the following sections. For another exposition of Strominger's integrability theorem aimed at mathematicians, see [8].

4.2 Connections in almost-complex geometry

The study of connections of the form $\hat{\nabla} = \nabla + H$ satisfying $\hat{\nabla} J = 0$ on almost-complex manifolds can be found in Chapter VI.8 of Yano's book [13]. We note that the condition $\hat{\nabla} J = 0$ implies that $\hat{\nabla}$ defines a connection on the complex vector bundle $T^{1,0}M$. Before returning to spinors, we will need the following lemma in the general setting of almost-complex geometry.

Lemma 2 *Let J be an almost-complex structure on a Riemannian manifold (M, g). Suppose J is compatible with g, meaning $g(JX, JY) = g(X, Y)$. Let H be a real 3-form and $\hat{\nabla} = \nabla + \frac{1}{2} g^{-1} H$. Suppose*

$$\hat{\nabla} J = 0. \tag{4.17}$$

If we let $\omega(X,Y) = g(JX,Y)$, then we must have

$$H = i(\partial - \bar{\partial})\omega + H^{3,0} + H^{0,3}, \tag{4.18}$$

where $H^{3,0} = \overline{H^{0,3}}$ is given by the Nijenhuis tensor: if $\{e_i\}$ is a frame for $T^{1,0}M$, then

$$H_{ijk} = N_{ijk} \tag{4.19}$$

in components of this frame.

We note that $H^{0,3} = 0$ implies that M is a complex manifold, and can thus be interpreted as an integrability condition similar to $F^{0,2} = 0$ on holomorphic vector bundles. We also note the notation $\partial\omega = (d\omega)^{2,1}$ and $\bar{\partial}\omega = (d\omega)^{1,2}$.

Proof: Let x^μ be local coordinates and write $\hat{\nabla} = d + A$. Then if we write

$$\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{4.20}$$

we have

$$d\omega = \frac{1}{2}\partial_\alpha \omega_{\mu\nu}\, dx^\alpha \wedge dx^\mu \wedge dx^\nu, \tag{4.21}$$

which is

$$d\omega = \frac{1}{2}(\hat{\nabla}_\alpha \omega_{\mu\nu} + A^\beta{}_{\alpha\mu}\omega_{\beta\nu} + A^\beta{}_{\alpha\nu}\omega_{\mu\beta})\, dx^\alpha \wedge dx^\mu \wedge dx^\nu. \tag{4.22}$$

We note that $\hat{\nabla}\omega = 0$ since $\hat{\nabla}g = 0$ and $\hat{\nabla}J = 0$. Hence skew-symmetrizing $A^\beta{}_{\alpha\mu}\omega_{\beta\nu}$ in α, μ, ν and then again for the second term, we obtain

$$\begin{aligned} d\omega &= \frac{1}{3!}(A^\beta{}_{\alpha\mu}\omega_{\beta\nu} + A^\beta{}_{\nu\alpha}\omega_{\beta\mu} + A^\beta{}_{\mu\nu}\omega_{\beta\alpha}) dx^\alpha \wedge dx^\mu \wedge dx^\nu \\ &\quad + \frac{1}{3!}(A^\beta{}_{\alpha\nu}\omega_{\mu\beta} + A^\beta{}_{\mu\alpha}\omega_{\nu\beta} + A^\beta{}_{\nu\mu}\omega_{\alpha\beta}) dx^\alpha \wedge dx^\mu \wedge dx^\nu, \end{aligned} \tag{4.23}$$

and since $H^\beta{}_{\alpha\mu} = A^\beta{}_{\alpha\mu} - A^\beta{}_{\mu\alpha}$ (4.10), then

$$d\omega = \frac{1}{3!}(H^\beta{}_{\alpha\mu}\omega_{\beta\nu} + H^\beta{}_{\nu\alpha}\omega_{\beta\mu} + H^\beta{}_{\mu\nu}\omega_{\beta\alpha}) dx^\alpha \wedge dx^\mu \wedge dx^\nu. \tag{4.24}$$

Since g is metric compatible, we have $\omega(X,Y) = -g(X,JY)$. We write this as

$$\omega_{\alpha\beta} = -g_{\alpha,J\beta}. \tag{4.25}$$

Then

$$d\omega = \frac{1}{3!}(-H^{\beta}{}_{\alpha\mu}g_{\beta,J\nu} - H^{\beta}{}_{\nu\alpha}g_{\beta,J\mu} - H^{\beta}{}_{\mu\nu}g_{\beta,J\alpha})dx^{\alpha}\wedge dx^{\mu}\wedge dx^{\nu}, \quad (4.26)$$

and in components

$$(d\omega)(\partial_\alpha,\partial_\mu,\partial_\nu) = -(H(J\partial_\alpha,\partial_\mu,\partial_\nu) + H(J\partial_\nu,\partial_\alpha,\partial_\mu) + H(J\partial_\mu,\partial_\nu,\partial_\alpha)). \quad (4.27)$$

Let $\partial\omega = (d\omega)^{2,1}$ and $\bar\partial\omega = (d\omega)^{1,2}$, so that

$$d\omega = \partial\omega + \bar\partial\omega + (d\omega)^{(3,0)+(0,3)}. \quad (4.28)$$

Let $\{e_i\}$ be a local frame spanning $T^{1,0}M$. We then have

$$(\partial\omega)(e_i,e_j,e_{\bar k}) = -H(Je_{\bar k},e_i,e_j) - H(Je_j,e_{\bar k},e_i) - H(Je_i,e_j,e_{\bar k}), \quad (4.29)$$

which becomes

$$(\partial\omega)(e_i,e_j,e_{\bar k}) = iH(e_{\bar k},e_i,e_j) - iH(e_j,e_{\bar k},e_i) - iH(e_i,e_j,e_{\bar k}), \quad (4.30)$$

and we write this as which gives

$$H_{\bar k ij} = i(\partial\omega)_{\bar k ij}. \quad (4.31)$$

Since $\bar H = H$ and $\bar\omega = \omega$, taking conjugates gives

$$H_{\bar k\bar\ell j} = -i(\bar\partial\omega)_{\bar k\bar\ell j}. \quad (4.32)$$

This proves

$$H = i(\partial - \bar\partial)\omega + H^{3,0} + H^{0,3}. \quad (4.33)$$

Next, we use $\hat\nabla J = 0$ to relate H to N. The definition of $\hat\nabla J$ in $\{x^\mu\}$ coordinates is

$$\hat\nabla_\mu J^\alpha{}_\beta = \nabla_\mu J^\alpha{}_\beta + \frac{1}{2}H^\alpha{}_{\mu\nu}J^\nu{}_\beta - \frac{1}{2}J^\alpha{}_\nu H^\nu{}_{\mu\beta}, \quad (4.34)$$

which implies

$$\nabla_\mu J^\alpha{}_\beta = -\frac{1}{2}H^\alpha{}_{\mu\nu}J^\nu{}_\beta + \frac{1}{2}J^\alpha{}_\nu H^\nu{}_{\mu\beta}. \quad (4.35)$$

By (2.81), we obtain

$$\begin{aligned} 4N^\alpha{}_{\mu\nu} &= J^\gamma{}_\mu(-\frac{1}{2}H^\alpha{}_{\gamma\sigma}J^\sigma{}_\nu + \frac{1}{2}J^\alpha{}_\sigma H^\sigma{}_{\gamma\nu}) + J^\alpha{}_\gamma(-\frac{1}{2}H^\gamma{}_{\nu\sigma}J^\sigma{}_\mu + \frac{1}{2}J^\gamma{}_\sigma H^\sigma{}_{\nu\mu}) \\ &\quad -J^\gamma{}_\nu(-\frac{1}{2}H^\alpha{}_{\gamma\sigma}J^\sigma{}_\mu + \frac{1}{2}J^\alpha{}_\sigma H^\sigma{}_{\gamma\mu}) - J^\alpha{}_\gamma(-\frac{1}{2}H^\gamma{}_{\mu\sigma}J^\sigma{}_\nu + \frac{1}{2}J^\gamma{}_\sigma H^\sigma{}_{\mu\nu}) \end{aligned}$$

This is

$$4N^{\alpha}{}_{\mu\nu} = -H^{\alpha}{}_{J\mu,J\nu} + H^{J\alpha}{}_{J\mu,\nu} - H^{\alpha}{}_{\nu\mu} - H^{J\alpha}{}_{J\nu,\mu}. \tag{4.36}$$

In a frame for $T^{1,0}M$ denoted $\{e_i\}$ with indices i, j, k, we have

$$4N^{\bar{k}}_{ij} = -(i)^2 H^{\bar{k}}{}_{ij} + (-i)(i)H^{\bar{k}}{}_{ij} - H^{\bar{k}}{}_{ji} - (-i)(i)H^{\bar{k}}{}_{ji}. \tag{4.37}$$

Lowering the index, we obtain

$$N_{kij} = H_{kij} \tag{4.38}$$

as required. □

If we assume that the 3-form field strength satisfies $H^{0,3} = 0$, then by (2.74) and $N^{\bar{k}}_{ij} = H^{\bar{k}}{}_{ij} = 0$, we conclude that $N = 0$ and the Newlander-Nirenberg theorem gives the existence of holomorphic coordinates. In holomorphic coordinates $\{z^i\}$, we write $\omega = i g_{j\bar{k}} dz^j \wedge d\bar{z}^k$, and the theorem gives the expression

$$H = i^2 \partial_i g_{j\bar{k}} dz^i \wedge dz^j \wedge d\bar{z}^k - i^2 \partial_{\bar{i}} g_{j\bar{k}} d\bar{z}^i \wedge dz^j \wedge d\bar{z}^k \tag{4.39}$$

which is

$$H = \frac{1}{2}(-\partial_i g_{j\bar{k}} + \partial_j g_{i\bar{k}}) dz^i \wedge dz^j \wedge d\bar{z}^k + \frac{1}{2}(-\partial_{\bar{i}} g_{j\bar{k}} + \partial_{\bar{k}} g_{j\bar{i}}) dz^j \wedge d\bar{z}^i \wedge d\bar{z}^k \tag{4.40}$$

Our conventions are

$$H = \frac{1}{2} H_{ij\bar{k}} dz^i \wedge dz^j \wedge d\bar{z}^k + \frac{1}{2} H_{j\bar{i}\bar{k}} dz^j \wedge d\bar{z}^i \wedge d\bar{z}^k, \tag{4.41}$$

so that

$$H_{ij\bar{k}} = -\partial_i g_{j\bar{k}} + \partial_j g_{i\bar{k}}, \tag{4.42}$$

and the other components of H are determined by taking conjugates or skew-symmetry.

To do: derive the expression for $\hat{\nabla}$ in complex coordinates in terms of the metric $g_{\bar{k}j}$ and the Chern connection.

4.3 Complex geometry with H-flux

The references for this section are [1, 11].

Let us recall the setup. Let (M, g) be a manifold of dimension $n = 6$ with positive chirality spinor η with $\eta^\dagger \eta = 1$, let H be a 3-form and ϕ a scalar function. Suppose this geometry satisfies the supersymmetric equations

$$\hat{\nabla}\eta = 0, \quad (H + 2d\phi) \cdot \eta = 0. \tag{4.43}$$

Here we use the notation

$$(H + 2d\phi) \cdot \eta = \frac{1}{3!} H_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\eta + 2\partial_\mu \phi \gamma^\mu \eta, \tag{4.44}$$

where

$$\gamma_{ijk} = \frac{1}{3!}(\gamma_i\gamma_j\gamma_k + \gamma_k\gamma_i\gamma_j + \gamma_j\gamma_k\gamma_i - \dots). \tag{4.45}$$

We showed earlier that this structure equips M with an almost-complex structure $J^k{}_j = i\eta^\dagger \gamma^k{}_j \eta$. In this section, we will follow Strominger's [11] calculation that $N = 0$, which implies that M admits the structure of a complex manifold.

First, we noted in (2.44) that if the index i denote a frame $\{e_i\}$ of $T^{1,0}M$, then

$$\gamma_i \eta = \gamma^{\bar{i}} \eta = 0. \tag{4.46}$$

The gamma matrix identity $\{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu} I$ in real coordinates x^μ becomes in the frame $\{e_i, \overline{e_i}\}$:

$$\gamma_i^2 = 0, \quad \gamma_i\gamma_j = -\gamma_j\gamma_i, \quad \gamma_i\gamma^{\bar{j}} = -\gamma^{\bar{j}}\gamma_i \tag{4.47}$$

and

$$\{\gamma_i, \gamma_{\bar{j}}\} = -2g_{i\bar{j}} I, \quad \{\gamma_i, \gamma^j\} = -2\delta^i{}_j, \tag{4.48}$$

since $g_{ij} = g_{\bar{i}\bar{j}} = 0$. These relations will be frequently used. We also note the following gamma matrix identities.

Lemma 3

$$[\gamma_{mn}, \gamma^r] = 2(\delta_m{}^r \gamma_n - \delta_n{}^r \gamma_m) \tag{4.49}$$

$$\{\gamma_{mnp}, \gamma^r\} = -2(\delta^r{}_m \gamma_{np} + \delta^r{}_p \gamma_{mn} + \delta^r{}_n \gamma_{pm}). \tag{4.50}$$

Proof: This follows from repeated application of

$$\gamma_m\gamma^n + \gamma^n\gamma_m = -2\delta_m{}^n. \tag{4.51}$$

Indeed

$$\begin{aligned}\gamma_{mn}\gamma^r &= \frac{1}{2}(\gamma_m\gamma_n - \gamma_n\gamma_m)\gamma^r \\ &= \frac{1}{2}(-\gamma_m\gamma^r\gamma_n + \gamma_n\gamma^r\gamma_m - 2\delta^r{}_n\gamma_m + 2\gamma_n\delta^r{}_m),\end{aligned} \tag{4.52}$$

and applying it again gives

$$[\gamma_{mn}, \gamma^r] = \frac{1}{2}(2\delta^r{}_m\gamma_n - 2\delta^r{}_n\gamma_m - 2\delta^r{}_n\gamma_m + 2\gamma_n\delta^r{}_m) \tag{4.53}$$

which gives the first identity. For the second, we start with

$$\begin{aligned}\gamma_m\gamma_n\gamma_p\gamma^r &= -\gamma_m\gamma_n\gamma^r\gamma_p - 2\gamma_m\gamma_n\delta^r{}_p \\ &= \gamma_m\gamma^r\gamma_n\gamma_p + 2\gamma_m\delta^r{}_n\gamma_p - 2\gamma_m\gamma_n\delta^r{}_p \\ &= -\gamma^r\gamma_m\gamma_n\gamma_p - 2\delta^r{}_m\gamma_n\gamma_p + 2\gamma_m\delta^r{}_n\gamma_p - 2\gamma_m\gamma_n\delta^r{}_p.\end{aligned} \tag{4.54}$$

Therefore

$$\{\gamma_m\gamma_n\gamma_p, \gamma^r\} = 2(-\delta^r{}_m\gamma_n\gamma_p + \delta^r{}_n\gamma_m\gamma_p - \delta^r{}_p\gamma_m\gamma_n). \tag{4.55}$$

Skew-symmetrizing gives

$$\{\gamma_{mnp}, \gamma^r\} = \frac{1}{3}(-\delta^r{}_{[m}\gamma_n\gamma_{p]} + \delta^r{}_{[n}\gamma_m\gamma_{p]} - \delta^r{}_{[p}\gamma_m\gamma_{n]}) \tag{4.56}$$

and so

$$\{\gamma_{mnp}, \gamma^r\} = -\delta^r{}_{[m}\gamma_n\gamma_{p]}. \tag{4.57}$$

This is

$$\{\gamma_{mnp}, \gamma^r\} = -\delta^r{}_m\gamma_n\gamma_p - \delta^r{}_p\gamma_m\gamma_n - \delta^r{}_n\gamma_p\gamma_m + \delta^r{}_n\gamma_m\gamma_p + \delta^r{}_p\gamma_n\gamma_m + \delta^r{}_m\gamma_p\gamma_n. \tag{4.58}$$

and so

$$\{\gamma_{mnp}, \gamma^r\} = -2(\delta^r{}_m\gamma_{np} + \delta^r{}_p\gamma_{mn} + \delta^r{}_n\gamma_{pm}). \tag{4.59}$$

□

The main result of this section is:

Corollary 2 *Let (M, g, H, ϕ) be as before, solving*

$$\eta^\dagger \eta = 1, \quad \hat{\nabla}\eta = 0, \quad (H + 2\phi)\cdot\eta = 0. \tag{4.60}$$

The almost-complex structure $J^k{}_j = i\eta^\dagger \gamma^k{}_j \eta$ is integrable: M admits the structure of a complex manifold. The 3-form field H satisfies $H^{0,3} = 0$ and is given by $H = i(\partial - \bar{\partial})\omega$. In holomorphic coordinates, H satisfies the constraint

$$g^{p\bar{q}} H_{\bar{q}pk} = 2\nabla_k \phi. \tag{4.61}$$

Remark: we note the similarity with the Hermitian-Yang-Mills equations for a 2-form field strength F on a complex manifold, which is $g^{p\bar{q}} F_{p\bar{q}} = 0$ and $F^{0,2} = 0$. This is because the HYM equation can be derived from a similar supersymmetric equation $F \cdot \eta = 0$.

Proof: In this calculation, indices i, j, k etc represent a local frame $\{e_i\}$ for $T^{1,0}M$. The full tangent bundle $T_{\mathbb{C}}M$ is then generated by the frame $\{e_i, \overline{e_i}\}$. In this frame, the dilatino equation $(H + 2\phi)\cdot\eta = 0$ becomes

$$\frac{1}{6}H_{ijk}\gamma^{ijk}\eta + \frac{1}{6}H_{\bar{i}\bar{j}\bar{k}}\gamma^{\bar{i}\bar{j}\bar{k}} + \frac{3}{6}(H_{\bar{i}jk}\gamma^{\bar{i}jk} + H_{\bar{i}\bar{j}k}\gamma^{\bar{i}\bar{j}k})\eta = -2(\partial_i\phi\gamma^i + \partial_{\bar{i}}\phi\gamma^{\bar{i}})\eta. \tag{4.62}$$

Since $\gamma^{\bar{i}}\eta = \gamma_i \eta = 0$, we have

$$H_{\bar{i}\bar{j}\bar{k}}\gamma^{\bar{i}\bar{j}\bar{k}}\eta = 0. \tag{4.63}$$

We also have

$$\begin{aligned} \gamma_r H_{\bar{i}\bar{j}k}\gamma^{\bar{i}\bar{j}k}\eta &= H_{\bar{i}\bar{j}k}\{\gamma_r, \gamma^{\bar{i}\bar{j}k}\}\eta \\ &= H_{\bar{i}\bar{j}k}(-2)(\delta^{\bar{i}}{}_r \gamma^{\bar{j}k} + \delta^k{}_r \gamma^{\bar{i}\bar{j}} + \delta^{\bar{j}}{}_r \gamma^{k\bar{i}})\eta \\ &= 0. \end{aligned} \tag{4.64}$$

Therefore

$$H_{\bar{i}\bar{j}k}\gamma^{\bar{i}\bar{j}k}\eta = 0. \tag{4.65}$$

The dilatino equation then reduces to

$$\frac{1}{6}H_{ijk}\gamma^{ijk}\eta + \frac{3}{6}H_{\bar{i}jk}\gamma^{\bar{i}jk}\eta = -2(\partial_i\phi)\gamma^i\eta. \tag{4.66}$$

We multiply through by γ_r to obtain

$$\frac{1}{6}H_{ijk}\{\gamma_r, \gamma^{ijk}\}\eta + \frac{1}{2}(H_{\bar{i}jk}\{\gamma_r, \gamma^{\bar{i}jk}\})\eta = -2\partial_i\phi\gamma_r\gamma^i\eta. \tag{4.67}$$

Applying (4.50)

$$
\begin{aligned}
& \frac{1}{6}\left[H_{ijk}(-2)(\delta^i{}_r\gamma^{jk}+\delta^k{}_r\gamma^{ij}+\delta^j{}_r\gamma^{ki})\right]\eta \\
& +\frac{1}{2}\left[H_{\bar{i}jk}(-2)(\delta^{\bar{i}}{}_r\gamma^{jk}+\delta^k{}_r\gamma^{\bar{i}j}+\delta^j{}_r\gamma^{k\bar{i}})\right]\eta \\
= \quad & -2(\partial_i\phi)\gamma_r\gamma^i\eta \qquad (4.68)
\end{aligned}
$$

Simplifying and using $\gamma^{\bar{i}}\eta=\gamma_i\eta=0$, we obtain

$$-H_{rjk}\gamma^{jk}\eta+(-H_{\bar{i}jr}\gamma^{\bar{i}j}-H_{\bar{i}rk}\gamma^{k\bar{i}})\eta=-2(\partial_i\phi)(0-2\delta^i{}_r)\eta. \qquad (4.69)$$

This becomes

$$-H_{rjk}\gamma^{jk}\eta-H_{\bar{i}jr}\gamma^{\bar{i}}\gamma^j\eta=(4\partial_r\phi)\eta \qquad (4.70)$$

which, by $\gamma^j\gamma^{\bar{i}}+\gamma^{\bar{i}}\gamma^j=-2g^{j\bar{i}}$ is

$$-H_{rjk}\gamma^{jk}\eta+2H_{\bar{i}jr}g^{j\bar{i}}\eta=(4\partial_r\phi)\eta. \qquad (4.71)$$

We now multiply through by γ_s, and use $\gamma_s\eta=0$ to obtain

$$-H_{rjk}[\gamma_s,\gamma^{jk}]\eta=0. \qquad (4.72)$$

By (4.49),

$$H_{rjk}(\delta^j_s\gamma^k-\delta^k_s\gamma^j)\eta=0 \qquad (4.73)$$

which implies

$$H_{rsk}\gamma^k\eta=0. \qquad (4.74)$$

Multiplying by γ_q implies

$$0=H_{rsk}\gamma_q\gamma^k\eta=-2H_{rsk}\delta^k{}_q\eta, \qquad (4.75)$$

hence $H_{rsq}=0$. Therefore the dilatino equation (4.71) implies

$$g^{k\bar{i}}H_{\bar{i}kr}=2\partial_r\phi, \quad H_{ijk}=0. \qquad (4.76)$$

We note that taking the conjugate gives the barred version $H_{\bar{i}\bar{j}\bar{k}}=0$ and

$$2\partial_{\bar{\ell}}\phi=\overline{g^{p\bar{q}}H_{\bar{q}p\ell}}=g^{q\bar{p}}H_{q\bar{p}\bar{\ell}}=-H^p{}_{p\bar{\ell}}. \qquad (4.77)$$

Next, we use $\hat{\nabla}\eta=0$, which implies that $\hat{\nabla}J=0$. The relevant discussion is already contained in §4.2: by Lemma 2 we have that $H_{ijk}=0$ implies $H=i(\partial-\bar{\partial})\omega$ and $N_{ijk}=0$. $\square$

4.4 Holomorphic volume form

The references for this section are [1, 11].

In the previous section, we entered the realm of complex geometry starting from equations on spinors. In this section, we will show that the complex manifold M^6 is also equipped with a holomorphic volume form. We say that $\Omega \in \Omega^{(n,0)}(M)$ on a complex manifold of complex dimension n is a holomorphic volume form if locally

$$\Omega = f(z) dz^1 \wedge \cdots \wedge dz^n \tag{4.78}$$

for a non-vanishing holomorphic local function $f(z)$. If M admits a holomorphic volume form, we say that M has trivial canonical bundle.

Theorem 3 *Let (M, g) be a compact spin manifold of dimension $n = 6$. Let η be a positive chirality nowhere vanishing spinor satisfying*

$$\eta^\dagger \eta = 1, \quad \hat{\nabla}\eta = 0, \tag{4.79}$$

$$(H + 2d\phi) \cdot \eta = 0, \tag{4.80}$$

where H is a 3-form and ϕ is a function, and $\hat{\nabla} = \nabla + \frac{1}{2} g^{-1} H$. Then M admits the structure of a complex manifold with holomorphic volume form Ω and non-Kähler hermitian metric g satisfying

$$d(|\Omega|_g \omega^2) = 0. \tag{4.81}$$

The structure (M, g, Ω) on the complex manifold M can be viewed as a non-Kähler Calabi-Yau structure. The standard definition in the literature of a Calabi-Yau manifold requires a complex manifold with trivial canonical bundle and Kähler metric g satisfying $d\omega = 0$. Here we obtain a complex manifold with trivial canonical bundle with $d(|\Omega|_g \omega^2) = 0$.

Recall that our conventions are $\nabla^+_\alpha \eta = \nabla_\alpha \eta + \frac{1}{8} H_{\mu\alpha\nu} \gamma^\nu \gamma^\mu \eta$, and by our work so far, we know that M is a complex manifold, and

$$H^p{}_{pk} = 2\nabla_k \phi, \quad H^p{}_{p\bar{k}} = -2\nabla_{\bar{k}} \phi, \tag{4.82}$$

$$H_{\bar{k}ij} = -\partial_i g_{\bar{k}j} + \partial_j g_{\bar{k}i}. \tag{4.83}$$

We now show that M has trivial canonical bundle by writing down the holomorphic volume form.

$$\Omega = (e^{-2\phi} \bar{\eta}^T \gamma_{ijk} \bar{\eta})\, dz^i \wedge dz^j \wedge dz^k. \tag{4.84}$$

This is well-defined by a similar argument to the one which shows J is well-defined.

Proposition 4 Ω *is a nowhere vanishing holomorphic* $(3,0)$ *form with norm*

$$|\Omega|_g = e^{-2(\phi+\phi_0)} \tag{4.85}$$

for some constant ϕ_0.

Proof: We compute in complex coordinates

$$\nabla_{\bar\ell}\Omega_{ijk} = \nabla_{\bar\ell}(e^{-2\phi}\bar\eta^T\gamma_{ijk}\bar\eta) = -2(\nabla_{\bar\ell}\phi)\Omega_{ijk} + 2e^{-2\phi}\bar\eta^T\gamma_{ijk}\nabla_{\bar\ell}\bar\eta. \tag{4.86}$$

This used $\gamma^T_{ijk} = \gamma_{ijk}$, $\nabla\eta^T = (\nabla\eta)^T$, and

$$((\nabla\eta)^T\gamma_{ijk}\eta)^T = \eta^T\gamma_{ijk}\nabla\eta. \tag{4.87}$$

The equation $\hat\nabla\eta = 0$ (4.14) gives

$$\nabla_{\bar\ell}\bar\eta = -\frac{1}{8}H_{\alpha\bar\ell\beta}\gamma^\beta\gamma^\alpha\bar\eta, \tag{4.88}$$

where α,β are real coordinates. Since $\gamma^{\bar i}\eta = 0$ and $H^{0,3} = 0$, in complex coordinates this is

$$\nabla_{\bar\ell}\bar\eta = -\frac{1}{8}H_{\bar p\bar\ell q}\gamma^q\gamma^{\bar p}\bar\eta. \tag{4.89}$$

Using $\gamma^q\gamma^{\bar p} + \gamma^{\bar p}\gamma^q = -2g^{q\bar p}$, we obtain

$$\nabla_{\bar\ell}\bar\eta = \frac{1}{4}H_{\bar p\bar\ell q}g^{q\bar p}\bar\eta, \tag{4.90}$$

which is by (4.77)

$$\nabla_{\bar\ell}\bar\eta = \frac{1}{2}(\nabla_{\bar\ell}\phi)\bar\eta. \tag{4.91}$$

Substituting into (4.86) gives

$$\nabla_{\bar\ell}\Omega_{ijk} = -(\nabla_{\bar\ell}\phi)\Omega_{ijk}. \tag{4.92}$$

On the other hand, by definition

$$\nabla_{\bar\ell}\Omega_{ijk} = \partial_{\bar\ell}\Omega_{ijk} - \Gamma^\alpha_{\bar\ell i}\Omega_{\alpha jk} - \Gamma^\alpha_{\bar\ell j}\Omega_{i\alpha k} - \Gamma^\alpha_{\bar\ell k}\Omega_{ij\alpha}, \tag{4.93}$$

where α denotes real coordinates. Since Ω is type $(3,0)$ in complex dimension 3, this implies

$$\nabla_{\bar\ell}\Omega_{ijk} = \partial_{\bar\ell}\Omega_{ijk} - \Gamma^p_{\bar\ell p}\Omega_{ijk}. \tag{4.94}$$

The Christoffel symbols are

$$\Gamma^p_{\bar{\ell}p} = \frac{g^{p\bar{r}}}{2}(-\partial_{\bar{r}} g_{\bar{\ell}p} + \partial_{\bar{\ell}} g_{\bar{r}p} + 0), \tag{4.95}$$

which is by (4.40) and (4.77)

$$\Gamma^p_{\bar{\ell}p} = \frac{g^{p\bar{r}}}{2} H_{p\bar{r}\bar{\ell}} = \nabla_{\bar{\ell}}\phi. \tag{4.96}$$

Therefore

$$\nabla_{\bar{\ell}}\Omega_{ijk} = \partial_{\bar{\ell}}\Omega_{ijk} - (\nabla_{\bar{\ell}}\phi)\Omega_{ijk} \tag{4.97}$$

Combining (4.92) and (4.97), we obtain

$$\partial_{\bar{\ell}}\Omega_{ijk} = 0. \tag{4.98}$$

This proves that $\bar{\partial}\Omega = 0$, and so Ω is holomorphic. We can also compute the unbarred derivative

$$\nabla_\ell \Omega_{ijk} = -2(\nabla_\ell\phi)\Omega_{ijk} + 2e^{-2\phi}\bar{\eta}^T\gamma_{ijk}\Big[-\frac{1}{8}H_{\alpha\ell\beta}\gamma^\beta\gamma^\alpha\bar{\eta}\Big], \tag{4.99}$$

using real α, β. In holomorphic coordinates, this is

$$\nabla_\ell \Omega_{ijk} = -2(\nabla_\ell\phi)\Omega_{ijk} + 2e^{-2\phi}\bar{\eta}^T\gamma_{ijk}\Big[-\frac{1}{8}H_{\bar{p}\ell q}\gamma^q\gamma^{\bar{p}}\bar{\eta} - \frac{1}{8}H_{\bar{p}\ell\bar{q}}\gamma^{\bar{q}}\gamma^{\bar{p}}\bar{\eta}\Big]. \tag{4.100}$$

The last term is zero. Indeed,

$$\gamma_{ijk}\gamma^{\bar{q}}\gamma^{\bar{p}}\bar{\eta} = \gamma^{\bar{q}}\gamma^{\bar{p}}\gamma_{ijk}\bar{\eta} = \Psi_{ijk}\gamma^{\bar{q}}\gamma^{\bar{p}}\eta = 0 \tag{4.101}$$

where $\Psi_{ijk} = \bar{\eta}^T\gamma_{ijk}\bar{\eta}$ are scalar 3-form components. The identity $\gamma_{ijk}\bar{\eta} = \Psi_{ijk}\eta$ will be shown later in (4.131). We use it directly for now, and conclude

$$\nabla_\ell \Omega_{ijk} = -2(\nabla_\ell\phi)\Omega_{ijk} + 2e^{-2\phi}\bar{\eta}^T\gamma_{ijk}\Big[\frac{1}{4}H_{\bar{p}\ell q}g^{\bar{p}q}\bar{\eta}\Big]. \tag{4.102}$$

Thus

$$\nabla_\ell \Omega_{ijk} = \Big[-2\nabla_\ell\phi - \frac{1}{2}H^p{}_{p\ell}\Big]\Omega_{ijk} = [-3\nabla_\ell\phi]\Omega_{ijk}. \tag{4.103}$$

Therefore, using the induced metric on $(3,0)$ forms,

$$|\Omega|^2_g = g^{i\bar{p}}g^{j\bar{q}}g^{k\bar{r}}\Omega_{ijk}\bar{\Omega}_{\bar{p}\bar{q}\bar{r}} \tag{4.104}$$

$$\nabla_\ell |\Omega|_g^2 = \langle \nabla_\ell \Omega, \Omega\rangle + \langle \Omega, \nabla_{\bar\ell}\Omega\rangle = -4\nabla_\ell \phi \|\Omega\|^2 \tag{4.105}$$

and

$$\nabla_\ell \big(\log |\Omega|_g + 2\phi \big) = 0, \tag{4.106}$$

which on a compact manifold implies that $\log|\Omega|_g = -2\phi - 2\phi_0$ for a constant ϕ_0. $\square$

It remains to show $d(|\Omega|_g \omega^2) = 0$. This identity was observed by Li-Yau [10]. Expanding $d(|\Omega|_g\omega^2) = 0$, we must equivalently show that

$$\partial \log |\Omega|_g \wedge \omega^2 + 2\partial\omega \wedge \omega = 0. \tag{4.107}$$

A computation with $H_{ij\bar k} = -\partial_i g_{j\bar k} + \partial_j g_{i\bar k}$ shows

$$2\partial\omega \wedge \omega = \theta \wedge \omega^2, \quad \theta = \theta_i dz^i, \quad \theta_i = g^{i\bar k} H_{\bar k ij}. \tag{4.108}$$

Since $\partial_j \log|\Omega|_g = -2\partial_j \phi$ (4.106) and $H^p{}_{pj} = 2\partial_j\phi$ (4.61), then $\partial \log|\Omega|_g = -\theta$, which proves (4.107).

4.5 Holonomy in $SU(3)$

Let (M,g) be a spin manifold of dimension $n=6$ with 3-form H, function ϕ, and positive chirality spinor η. To summarize, we showed that the spinor constraints $\eta^\dagger \eta = 1$, $\hat\nabla \eta = 0$ and $(H + 2d\phi)\cdot \eta = 0$ imply the existence of (g, J, Ψ) such that

$$\hat\nabla g = 0, \quad \hat\nabla J = 0, \quad \hat\nabla \Psi = 0, \tag{4.109}$$

where

$$\Psi = \frac{\Omega}{|\Omega|_g}. \tag{4.110}$$

In fact, J and Ψ are explicitly constructed by

$$J^\alpha{}_\beta = i\eta^\dagger \gamma^\alpha{}_\beta \eta, \quad \Psi_{ijk} = e^{-2\phi_0} \bar\eta^T \gamma_{ijk} \bar\eta, \tag{4.111}$$

where ϕ_0 is a constant. It follows that

$$\mathrm{Hol}(\hat\nabla) \subseteq SU(3). \tag{4.112}$$

Note that for a general complex manifold (M,g,J), the condition $\mathrm{Hol}(\hat\nabla) \subseteq SU(3)$ does not imply that M has holomorphically trivial canonical bundle.

The holonomy constraint gives the existence of a parallel smooth section of K_M, but it need not be holomorphic.

If we set $H = 0$ and $\phi = const$, then

$$\mathrm{Hol}(\nabla^{LC}) \subset SU(3). \tag{4.113}$$

In this case, the structure (M, g, J, Ψ) is a Kähler Calabi-Yau structure: g is a Kähler Ricci-flat metric and Ψ is a holomorphic volume form with $|\Psi|_g = const$. The $SU(3)$ structure satisfying (4.109) is a generalization of this geometry to the non-Kähler setting.

4.6 Special Lagrangian submanifolds

In this section, we give an exposition of the calculation of Becker-Becker-Strominger [2] (see [1] for a textbook reference). The study of special Lagrangians in the non-Kähler setting can be found in joint work [4] with T. Collins, S. Gukov and S.-T. Yau.

Let (M, g, η_+) be a manifold of dimension 6 with nowhere vanishing positive chirality spinor η_+ with 3-form field H and scalar field ϕ. Suppose the supersymmetric equations of Theorem 3 are satisfied, so that M admits an integrable complex structure $J^p{}_q = i\eta_+^\dagger \gamma^p{}_q \eta_+$ and holomorphic volume form $\Omega_{ijk} = e^{-2\phi}\bar{\eta}^T \gamma_{ijk}\bar{\eta}$. We will also use the $\hat{\nabla}$ parallel 3-form Ψ:

$$\Psi = \frac{1}{|\Omega|_g}\Omega, \quad \Psi_{ijk} = \bar{\eta}^T \gamma_{ijk}\bar{\eta}, \quad |\Psi|_g = 1. \tag{4.114}$$

Let

$$X : L^3 \to M^6 \tag{4.115}$$

be a parametrized submanifold of real dimension 3. In local coordinates, $X = (X^1, \dots, X^6)$ with $X^i(u^1, u^2, u^3)$. The metric g induces a volume form on L denoted

$$\mu = d\mathrm{vol}_L \in \Lambda^3(L, \mathbb{R}). \tag{4.116}$$

From the submanifold $X : L^3 \to M^6$, we can construct an operator Γ on spinors given by

$$\Gamma\eta := \frac{1}{3!}\mu^{\alpha\beta\gamma}\partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P \gamma_{MNP}\eta. \tag{4.117}$$

Here α, β, γ are coordinate indices on L^3 and M, N, P are real coordinates on M^6. It turns out that $\Gamma^\dagger = \Gamma$ and $\Gamma^2 = I$; we will discuss some aspects of this calculation in Lemma 4. Then Γ breaks the spinor bundle into $+1$ and -1 eigenspaces. Thus the submanifold defines another notion of "chirality" of spinors, and we look for positive pairs (L^3, η):

$$\Gamma\eta = \eta. \tag{4.118}$$

This equation arises in string theory from supersymmetry, and it will lead to the equation for special Lagrangian cycles [1]. Using the projection $P_- = \frac{1}{2}(I - \Gamma)$ to the (-1) eigenspace, it can also be written as $P_-\eta = 0$.

We now fix the spinor η_+ inducing the complex structure as before, and look for special submanifolds solving

$$P_-(\eta_\theta) = 0, \quad \eta_\theta = e^{i\theta}\eta_+ + e^{-i\theta}\eta_- \tag{4.119}$$

on L, where $P_- \in \Gamma(X^*\mathrm{End}\, S)$ is given by

$$P_- = \frac{1}{2}\left(I - \frac{1}{3!}\mu^{\alpha\beta\gamma}\partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P \gamma_{MNP}\right). \tag{4.120}$$

We use the notation $\eta_- = \overline{\eta_+}$ for the corresponding spinor of negative chirality. We will denote

$$A^{MNP} = \mu^{\alpha\beta\gamma}\partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P, \tag{4.121}$$

which is anti-symmetric in MNP. Our goal is to understand the implications of the equation $P_-(\eta_\theta) = 0$ on the geometry of L. We start by simplifying the expression

$$P_-(\eta_\theta) = e^{i\theta}\eta_+ - e^{i\theta}\frac{A^{MNP}}{3!}\gamma_{MNP}\eta_+ + e^{-i\theta}\eta_- - e^{-i\theta}\frac{A^{MNP}}{3!}\gamma_{MNP}\eta_-. \tag{4.122}$$

Since $\gamma_i\eta_+ = 0$, if i, j, k denote indices for complex coordinates then

$$\frac{1}{3!}A^{MNP}\gamma_{MNP}\eta_+ = \left(\frac{1}{2}A^{ij\bar{k}}\gamma_{ij\bar{k}} + \frac{1}{2}A^{i\bar{j}\bar{k}}\gamma_{i\bar{j}\bar{k}} + \frac{1}{3!}A^{\bar{i}\bar{j}\bar{k}}\gamma_{\bar{i}\bar{j}\bar{k}}\right)\eta_+. \tag{4.123}$$

We compute each of these terms one by one.

- The term $\gamma_{ij\bar{k}}\eta_+$ contributes zero, since

$$\gamma_{ij\bar{k}}\eta_+ = \frac{1}{3!}(\gamma_i\gamma_j\gamma_{\bar{k}} - \gamma_j\gamma_i\gamma_{\bar{k}})\eta_+ = 0 \tag{4.124}$$

using $\{\gamma_j, \gamma_{\bar{k}}\} = -2g_{\bar{k}j}I$ and $\gamma_i\eta_+ = 0$.

• Next, we compute

$$\gamma_{i\bar{j}\bar{k}}\eta_+ = \frac{1}{6}(\gamma_i\gamma_{\bar{j}}\gamma_{\bar{k}} - \gamma_{\bar{j}}\gamma_i\gamma_{\bar{k}} - \gamma_i\gamma_{\bar{k}}\gamma_{\bar{j}} + \gamma_{\bar{k}}\gamma_i\gamma_{\bar{j}})\eta_+. \tag{4.125}$$

Commuting the γ_i to annihilate η_+ gives

$$\gamma_{i\bar{j}\bar{k}}\eta_+ = (-g_{\bar{j}i}\gamma_{\bar{k}} + g_{\bar{k}i}\gamma_{\bar{j}})\eta_+. \tag{4.126}$$

The symmetry $A^{i\bar{j}\bar{k}} = -A^{i\bar{k}\bar{j}}$ implies

$$A^{i\bar{j}\bar{k}}\gamma_{i\bar{j}\bar{k}}\eta_+ = -2A^{i\bar{j}\bar{k}}g_{\bar{j}i}\gamma_{\bar{k}}\eta_+. \tag{4.127}$$

• The last term is $\gamma_{\bar{i}\bar{j}\bar{k}}\eta_+$. We start by noting that

$$\text{span}\,\{\eta_-, \gamma_{\bar{1}}\eta_+, \gamma_{\bar{2}}\eta_+, \gamma_{\bar{3}}\eta_+\} = S_- \tag{4.128}$$

Indeed, we observed in (2.26) that these vectors all lie in S_-, and at a point where $g_{\bar{k}j} = \delta_{kj}$ they are orthogonal. For example $(\eta_-^\dagger\gamma_{\bar{i}}\eta_+)^T = -(\eta_-^\dagger\gamma_{\bar{i}}\eta_+)$ and $(\gamma_{\bar{i}}\eta_+)^\dagger\gamma_{\bar{j}}\eta_+ = -\eta_+^\dagger\gamma_i\gamma_{\bar{j}}\eta_+ = 2g_{\bar{j}i}$.

Commuting γ_7 with $\gamma_{\bar{i}}$, we see that $\gamma_{\bar{i}\bar{j}\bar{k}}\eta_+ \in S_-$. Taking the inner product with η_- gives

$$\eta_-^\dagger\gamma_{\bar{i}\bar{j}\bar{k}}\eta_+ = \Psi_{\bar{i}\bar{j}\bar{k}}. \tag{4.129}$$

by definition of the 3-form Ψ. Therefore

$$\gamma_{\bar{i}\bar{j}\bar{k}}\eta_+ = \Psi_{\bar{i}\bar{j}\bar{k}}\eta_- + \sum_{i=1}^{3} a_i\gamma_{\bar{i}}\eta_+. \tag{4.130}$$

Acting by $\gamma_{\bar{1}}$ gives zero since $\gamma_{\bar{i}\bar{j}\bar{k}}$ is the skew-symmetrization of $\gamma_{\bar{1}}\gamma_{\bar{2}}\gamma_{\bar{3}}$ and $\gamma_{\bar{1}}^2 = 0$. Therefore $0 = a_2\gamma_{\bar{1}}\gamma_{\bar{2}}\eta_+ + a_3\gamma_{\bar{1}}\gamma_{\bar{3}}\eta_+$, and acting by $\gamma_{\bar{2}}$ gives $a_3 = 0$. Similarly $a_1 = a_2 = 0$. We thus have

$$\gamma_{\bar{i}\bar{j}\bar{k}}\eta_+ = \Psi_{\bar{i}\bar{j}\bar{k}}\eta_-. \tag{4.131}$$

Altogether,

$$\frac{1}{3!}A^{MNP}\gamma_{MNP}\eta_+ = -A^{i\bar{j}\bar{k}}g_{\bar{j}i}\gamma_{\bar{k}}\eta_+ + \frac{1}{3!}A^{\bar{i}\bar{j}\bar{k}}\Psi_{\bar{i}\bar{j}\bar{k}}\eta_-, \tag{4.132}$$

and

$$2P_-(\eta_\theta) \;=\; e^{i\theta}\eta_+ + e^{i\theta}A^{i\bar{j}\bar{k}}g_{\bar{j}i}\gamma_{\bar{k}}\eta_+ - e^{i\theta}\frac{1}{3!}A^{\bar{i}\bar{j}\bar{k}}\Psi_{\bar{i}\bar{j}\bar{k}}\eta_-$$
$$+e^{-i\theta}\eta_- + e^{-i\theta}A^{\bar{i}jk}g_{\bar{i}j}\gamma_k\eta_- - e^{-i\theta}\frac{1}{3!}A^{ijk}\Psi_{ijk}\eta_+. \quad (4.133)$$

To understand the condition $P_-(\eta_\theta) = 0$, we now set the coefficient of each independent component in (4.133) to zero.

- Setting the coefficient of each $\gamma_{\bar{k}}\eta_+$ to zero gives

$$A^{i\bar{j}\bar{k}}g_{i\bar{j}} = 0, \quad \text{for all } k. \quad (4.134)$$

This expands to

$$\mu^{\alpha\beta\gamma}\partial_\alpha X^i\partial_\beta X^{\bar{j}}\partial_\gamma X^{\bar{k}}g_{i\bar{j}} = 0, \quad (4.135)$$

which implies

$$\partial_\alpha X^i\partial_\beta X^{\bar{j}}g_{i\bar{j}} - \partial_\beta X^i\partial_\alpha X^{\bar{j}}g_{i\bar{j}} = 0. \quad (4.136)$$

The $(1,1)$ form associated to the metric is $\omega = ig_{k\bar{j}}dz^k\wedge d\bar{z}^j$. It follows that

$$(X^*\omega)_{\alpha\beta} = \frac{\partial X^i}{\partial u^\alpha}\omega_{i\bar{j}}\frac{\partial X^{\bar{j}}}{\partial u^\beta} + \frac{\partial X^{\bar{j}}}{\partial u^\alpha}\omega_{\bar{j}i}\frac{\partial X^i}{\partial u^\beta} = 0. \quad (4.137)$$

- Setting the part involving η_+ to zero gives

$$e^{i\theta} - e^{-i\theta}\frac{1}{3!}A^{ijk}\Psi_{ijk} = 0. \quad (4.138)$$

This implies

$$\frac{1}{3!}\mu^{\alpha\beta\gamma}\partial_\alpha X^i\partial_\beta X^j\partial_\gamma X^k\Psi_{ijk} = e^{2i\theta} \quad (4.139)$$

Therefore

$$(X^*\Psi)_{\alpha\beta\gamma} = e^{2i\theta}\mu_{\alpha\beta\gamma}. \quad (4.140)$$

Altogether, the condition $P_-(\eta_\theta) = 0$ is equivalent to

$$X^*\omega = 0, \quad X^*\Psi = e^{2i\theta}d\mathrm{vol}_L \quad (4.141)$$

which in the Kähler Calabi-Yau case ($d\omega = 0$, $d\Psi = 0$) is the special Lagrangian condition. In the non-Kähler case, in terms of the holomorphic volume form this condition is

$$\omega|_L = 0, \quad \Omega|_L = e^{2i\theta}|\Omega|_g\, d\mathrm{vol}_L \quad (4.142)$$

since $\Psi = |\Omega|_g^{-1}\Omega$ where Ω is the holomorphic volume form.

We end this section with a computation showing that P_- can be viewed as a projection.

Lemma 4 *The operator* $P_- = \frac{1}{2}(I - \frac{1}{3!}A^{MNP}\gamma_{MNP})$ *satisfies*

$$P_-^\dagger = P_-, \quad P_-^2\eta_\theta = P_-\eta_\theta. \tag{4.143}$$

Proof: Let

$$\Gamma = \frac{1}{3!}A^{MNP}\gamma_{MNP}. \tag{4.144}$$

Then it suffices to show $\Gamma^\dagger = \Gamma$ and $\Gamma^2\eta_\theta = \eta_\theta$. That $\Gamma^\dagger = \Gamma$ follows from $\gamma_a^\dagger = -\gamma_a$. We now compute the square. By (4.132),

$$\Gamma^2\eta_+ = \frac{1}{3!}A^{MNP}\gamma_{MNP}(-A^{i\bar{j}\bar{k}}g_{\bar{j}i}\gamma_{\bar{k}}\eta_+ + \frac{1}{3!}A^{\bar{i}\bar{j}\bar{k}}\Psi_{\bar{i}\bar{j}\bar{k}}\eta_-) = (\mathrm{I}) + (\mathrm{II}). \tag{4.145}$$

The first term, in dimension 3 with $\gamma_{\bar{1}\bar{2}\bar{3}}\gamma_{\bar{k}} = 0$, is

$$(\mathrm{I}) = -\frac{1}{2}A^{m\bar{n}\bar{p}}A^i{}_i{}^{\bar{k}}\gamma_{m\bar{n}\bar{p}}\gamma_{\bar{k}}\eta_+ - \frac{1}{2}A^{mn\bar{p}}A^i{}_i{}^{\bar{k}}\gamma_{mn\bar{p}}\gamma_{\bar{k}}\eta_+ = (Ia) + (Ib). \tag{4.146}$$

We compute using $\{\gamma_i, \gamma_{\bar{j}}\} = -2g_{i\bar{j}}$ to commute γ_m, and $\gamma_m\eta_+ = 0$,:

$$\begin{aligned}(Ia) &= -\frac{1}{3!}A^{m\bar{n}\bar{p}}A^i{}_i{}^{\bar{k}}(\gamma_m\gamma_{\bar{n}}\gamma_{\bar{p}}\gamma_{\bar{k}} - \gamma_{\bar{n}}\gamma_m\gamma_{\bar{p}}\gamma_{\bar{k}} + \gamma_{\bar{n}}\gamma_{\bar{p}}\gamma_m\gamma_{\bar{k}})\eta_+ \\ &= -\frac{1}{3!}A^{m\bar{n}\bar{p}}A^i{}_i{}^{\bar{k}}(-\gamma_{\bar{n}}\gamma_m\gamma_{\bar{p}}\gamma_{\bar{k}} - 2g_{\bar{n}m}\gamma_{\bar{p}}\gamma_{\bar{k}} + \gamma_{\bar{n}}\gamma_{\bar{p}}\gamma_m\gamma_{\bar{k}} \\ &\quad +2g_{\bar{p}m}\gamma_{\bar{n}}\gamma_{\bar{k}} - 2g_{m\bar{k}}\gamma_{\bar{n}}\gamma_{\bar{p}})\eta_+\end{aligned} \tag{4.147}$$

Commuting γ_m again:

$$\begin{aligned}(Ia) &= -\frac{1}{3!}A^{m\bar{n}\bar{p}}A^i{}_i{}^{\bar{k}}(-2g_{m\bar{k}}\gamma_{\bar{n}}\gamma_{\bar{p}} + 2g_{\bar{p}m}\gamma_{\bar{n}}\gamma_{\bar{k}} - 2g_{\bar{n}m}\gamma_{\bar{p}}\gamma_{\bar{k}} - 2g_{\bar{k}m}\gamma_{\bar{n}}\gamma_{\bar{p}} \\ &\quad +2g_{\bar{p}m}\gamma_{\bar{n}}\gamma_{\bar{k}} - 2g_{m\bar{k}}\gamma_{\bar{n}}\gamma_{\bar{p}})\eta_+.\end{aligned} \tag{4.148}$$

Regrouping and relabeling, this is

$$(Ia) = A^{m\bar{n}\bar{p}}A^i{}_{im}\gamma_{\bar{n}}\gamma_{\bar{p}}\eta_+ + A^m{}_m{}^{\bar{p}}A^i{}_i{}^{\bar{k}}\gamma_{\bar{p}}\gamma_{\bar{k}}\eta_+. \tag{4.149}$$

The second term is zero by symmetry $\gamma_{\bar{p}}\gamma_{\bar{k}} = -\gamma_{\bar{k}}\gamma_{\bar{p}}$, so

$$(Ia) = A^{m\bar{n}\bar{p}}A^i{}_{im}\gamma_{\bar{n}}\gamma_{\bar{p}}\eta_+. \tag{4.150}$$

We move on to the next term. We compute using $\gamma_i \eta_+ = 0$ and $\{\gamma_i, \gamma_{\bar{j}}\} = -2g_{i\bar{j}}$,

$$\begin{aligned}
(Ib) &= -\frac{1}{(2)(3)} A^{mn\bar{p}} A^{i}{}_{i}{}^{\bar{k}} (\gamma_m \gamma_n \gamma_{\bar{p}} \gamma_{\bar{k}} - \gamma_m \gamma_{\bar{p}} \gamma_n \gamma_{\bar{k}} + \gamma_{\bar{p}} \gamma_m \gamma_n \gamma_{\bar{k}}) \eta_+ \\
&= -\frac{1}{(2)(3)} A^{mn\bar{p}} A^{i}{}_{i}{}^{\bar{k}} (-4g_{m\bar{p}} g_{n\bar{k}} + 4g_{n\bar{p}} g_{m\bar{k}} - 4g_{m\bar{p}} g_{n\bar{k}}) \eta_+ \\
&= -2A^{j}{}_{j}{}^{n} A^{i}{}_{in} \eta_+. \qquad (4.151)
\end{aligned}$$

This term is zero by symmetry. Indeed, by the definition of A and the definition of the pullback (4.137), and the identification $\omega_{j\bar{k}} = ig_{j\bar{k}}$, $\omega_{\bar{k}j} = -ig_{j\bar{k}}$, it is

$$\begin{aligned}
A^{j}{}_{j}{}^{n} A^{i}{}_{in} &= \mu^{\alpha\beta\gamma} \partial_\alpha X^j \partial_\beta X^{\bar{k}} g_{j\bar{k}} \partial_\gamma X^n \mu^{\mu\nu\rho} \partial_\mu X^i \partial_\nu X^{\bar{l}} g_{i\bar{l}} \partial_\rho X^{\bar{m}} g_{n\bar{m}} \\
&= \frac{i^{-3}}{2^3} \mu^{\alpha\beta\gamma} \mu^{\mu\nu\rho} (X^*\omega)_{\alpha\beta} (X^*\omega)_{\gamma\rho} (X^*\omega)_{\mu\nu} \qquad (4.152)
\end{aligned}$$

Since $X^*\omega$ is a 2-form on a manifold L^3 of dimension 3, and μ^{123} and its signed permutations are the only non-zero contributions of μ, all contributions to the term (Ib) cancel. Thus

$$(I) = A^{m\bar{n}\bar{p}} A^{i}{}_{im} \gamma_{\bar{n}} \gamma_{\bar{p}} \eta_+. \qquad (4.153)$$

Next, we analyse the second group of terms given by

$$(II) = \frac{1}{(3!)^2} A^{mnp} A^{\bar{i}\bar{j}\bar{k}} \gamma_{mnp} \Psi_{\bar{i}\bar{j}\bar{k}} \eta_- + \frac{1}{(2)(3!)} A^{\bar{m}np} A^{\bar{i}\bar{j}\bar{k}} \gamma_{\bar{m}np} \Psi_{\bar{i}\bar{j}\bar{k}} \eta_- \qquad (4.154)$$

Since $\gamma_{ijk} \eta_- = \Psi_{ijk} \eta_+$, the first term is

$$\begin{aligned}
(IIa) &= \frac{1}{(3!)^2} A^{mnp} A^{\bar{i}\bar{j}\bar{k}} \Psi_{mnp} \Psi_{\bar{i}\bar{j}\bar{k}} \eta_+ \\
&= \frac{\mu^{\alpha\beta\gamma}}{3!} (X^*\Psi)_{\alpha\beta\gamma} \frac{\mu^{\rho\sigma\tau}}{3!} \overline{(X^*\Psi)_{\rho\sigma\tau}} \eta_+ \\
&= \frac{X^*\Psi}{\mu} \frac{\overline{X^*\Psi}}{\mu} \eta_+ \\
&= |X^*\Psi|^2_{X^*g} \eta_+. \qquad (4.155)
\end{aligned}$$

We normalized such that $|\Psi|_g = 1$, so

$$(IIa) = \eta_+. \qquad (4.156)$$

The next term is

$$(IIb) = \frac{1}{(2)(3!)} A^{\bar{m}np} A^{\bar{i}\bar{j}\bar{k}} \gamma_{\bar{m}np} \Psi_{\bar{i}\bar{j}\bar{k}} \eta_-. \qquad (4.157)$$

We use (4.127) to obtain

$$\begin{aligned} A^{\bar{m}np}\gamma_{\bar{m}np}\eta_- &= \overline{A^{m\bar{n}\bar{p}}\gamma_{m\bar{n}\bar{p}}\eta_+} \\ &= -2\overline{A^{m\bar{n}\bar{p}}g_{m\bar{n}}\gamma_{\bar{p}}\eta_+} \\ &= -2A^{\bar{m}np}g_{\bar{m}n}\gamma_p\eta_- \\ &= +2A^{n}{}_{n}{}^{p}\gamma_p\eta_-. \end{aligned} \qquad (4.158)$$

Thus

$$(IIb) = \frac{1}{3!} A^{n}{}_{n}{}^{p} A^{\bar{i}\bar{j}\bar{k}} \Psi_{\bar{i}\bar{j}\bar{k}} \gamma_p \eta_- = \frac{1}{3!} A^{n}{}_{n}{}^{p} A^{\bar{i}\bar{j}\bar{k}} \gamma_p \gamma_{\bar{i}\bar{j}\bar{k}} \eta_+. \qquad (4.159)$$

By skewsymmetry, and relabeling p,n, this is

$$(IIb) = \frac{1}{3!} A^{p}{}_{p}{}^{n} A^{\bar{i}\bar{j}\bar{k}} \gamma_n \gamma_{\bar{i}} \gamma_{\bar{j}} \gamma_{\bar{k}} \eta_+. \qquad (4.160)$$

Commuting γ_n gives

$$\begin{aligned} (IIb) &= \frac{1}{3!} A^{p}{}_{p}{}^{n} A^{\bar{i}\bar{j}\bar{k}} \Big[-\gamma_{\bar{i}}\gamma_n\gamma_{\bar{j}}\gamma_{\bar{k}}\eta_+ - 2g_{n\bar{i}}\gamma_{\bar{j}}\gamma_{\bar{k}}\eta_+ \Big] \\ &= \frac{1}{3!} A^{p}{}_{p}{}^{n} A^{\bar{i}\bar{j}\bar{k}} \Big[-2g_{n\bar{k}}\gamma_{\bar{i}}\gamma_{\bar{j}}\eta_+ + 2g_{n\bar{j}}\gamma_{\bar{i}}\gamma_{\bar{k}}\eta_+ - 2g_{n\bar{i}}\gamma_{\bar{j}}\gamma_{\bar{k}}\eta_+ \Big] \end{aligned}$$

This adds up to

$$(IIb) = -A^{p}{}_{p}{}^{n} A_{n}{}^{\bar{i}\bar{j}} \gamma_{\bar{i}} \gamma_{\bar{j}} \eta_+. \qquad (4.161)$$

Therefore

$$(II) = \eta_+ - A^{p}{}_{p}{}^{n} A_{n}{}^{\bar{i}\bar{j}} \gamma_{\bar{i}} \gamma_{\bar{j}} \eta_+ \qquad (4.162)$$

Adding (4.153) and (4.162) together, we obtain

$$\Gamma^2\eta_+ = \eta_+ + (A^{p}{}_{p}{}^{\bar{n}} A_{\bar{n}}{}^{\bar{i}\bar{j}} - A^{p}{}_{p}{}^{n} A_{n}{}^{\bar{i}\bar{j}}) \gamma_{\bar{i}} \gamma_{\bar{j}} \eta_+. \qquad (4.163)$$

We will show below that the second term vanishes by symmetry, and so $\Gamma^2\eta_+ = \eta_+$. The identity $\Gamma^2\eta_- = \eta_-$ follows by taking the conjugate, since the gamma matrices from §1.2 are real.

It remains to understand the vanishing of

$$(A^p{}_p{}^{\bar{n}} A_{\bar{n}}{}^{\bar{i}\bar{j}} - A^p{}_p{}^n A_n{}^{\bar{i}\bar{j}}) = \frac{i^{-2}}{2} \mu^{\alpha\beta\gamma} \mu^{\sigma\nu\rho} (X^*\omega)_{\alpha\beta} (X^*\omega)_{\gamma\sigma} \partial_\nu X^{\bar{i}} \partial_\rho X^{\bar{j}}. \quad (4.164)$$

The pullback of ω appears by definition of A and (4.137). That this quantity vanishes can be seen by direct computation using that $X^*\omega$ is a 2-form on a manifold L^3 of dimension 3 and the only nonzero components are $(X^*\omega)_{12}$, $(X^*\omega)_{13}$, $(X^*\omega)_{23}$, μ^{123} and signed permutations. □

4.6.1 Extremizing property

We now look for extremizing properties of $X : L^3 \to M^6$. The square term which will give us the inequality is

$$|P_- \eta_\theta|^2 = \eta_\theta^\dagger P_-^\dagger P_- \eta_\theta = \eta_\theta^\dagger P_- \eta_\theta. \quad (4.165)$$

Here we used properties of P_- derived in the previous section. Substituting (4.133) and using orthogonality of the basis of S_+, S_- gives

$$|P_- \eta_\theta|^2 = 1 - e^{-2i\theta} \frac{1}{3!} A^{ijk} \Psi_{ijk} - e^{2i\theta} \frac{1}{3!} A^{\bar{i}\bar{j}\bar{k}} \Psi_{\bar{i}\bar{j}\bar{k}}. \quad (4.166)$$

Multiplying through by $|\Omega|_g \mu$ and using the definition of A and $\Psi = |\Omega|_g^{-1} \Omega$ implies

$$(|P_- \eta_\theta|^2 |\Omega|_g \mu)|_L = (|\Omega|_g \mu)|_L - e^{-2i\theta} \Omega|_L - e^{2i\theta} \overline{\Omega}|_L. \quad (4.167)$$

Integrating over L gives

$$\int_L (e^{-2i\theta} \Omega + e^{2i\theta} \overline{\Omega}) = \int_L |\Omega|_g \mu - \int_L |P_- \eta_\theta|^2 |\Omega|_g \mu. \quad (4.168)$$

Therefore, for any submanifold L^3, we have

$$\int_L (e^{-2i\theta} \Omega + e^{2i\theta} \overline{\Omega}) \leq \int_L |\Omega|_g \mu \quad (4.169)$$

with equality if and only if (4.142) is satisfied. Since $d\Omega = 0$ defines a cohomology class $[\Omega] \in H^3(M, \mathbb{C})$, we can rewrite this inequality as

$$2\mathrm{Re}\left\{ e^{-2i\theta} [\Omega] \cdot [L] \right\} \leq \int_L |\Omega|_g d\mathrm{vol}_L. \quad (4.170)$$

We conclude that a 3-cycle L minimizes

$$L \mapsto \int_L |\Omega|_g d\mathrm{vol}_L \tag{4.171}$$

in its homology class $[L] \in H_3(M, \mathbb{R})$ if and only if

$$\omega|_L = 0, \quad e^{-i\hat{\varphi}}\Omega|_L = |\Omega|_g d\mathrm{vol}_L \tag{4.172}$$

for an optimal phase $e^{i\hat{\varphi}}$ satisfying $e^{-i\hat{\varphi}}[\Omega] \cdot [L] \in \mathbb{R}$. Thus special Lagrangian submanifolds are optimal representatives of their homology class $[L] \in H_3(M, \mathbb{R})$.

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