ON KRONECKER PRODUCTS OF CHARACTERS OF THE SYMMETRIC GROUPS WITH FEW COMPONENTS

C. BESSENRODT AND S. VAN WILLIGENBURG

Abstract. Confirming a conjecture made by Bessenrodt and Kleshchev in 1999, we classify all Kronecker products of characters of the symmetric groups with only three or four components. On the way towards this result, we obtain new information about constituents in Kronecker products.

1. Introduction

The decomposition of the tensor product of two representations of a group is an ubiquitous and notoriously difficult problem which has been investigated for a long time. For complex representations of a finite group this is equivalent to decomposing the Kronecker product of their characters into irreducible characters. An equivalent way of phrasing this problem for the symmetric groups is to expand the inner product of the corresponding Schur functions in the basis of Schur functions. Examples for such computations were done already a long time ago by Murnaghan and Littlewood (cf. [10, 8]). While the answer in specific cases may be achieved by computing the scalar product of the characters, for the important family of the symmetric groups no reasonable general combinatorial formula is known.

Over many decades, a number of partial results have been obtained by a number of authors. To name just a few important cases that come up in the present article, the products of characters labelled by hook partitions or by two-row partitions have been computed (see [5, 11, 14]), and special constituents, in particular of tensor squares, have been considered [16, 17]. For general products, the largest part and the maximal number of parts in a constituent of the product have been determined (see [3, 4]); in fact, this is a special case of Dvir's recursion result [4, 2.3] that will be crucial in this paper.

In general, Kronecker products of irreducible representations have many irreducible constituents (see e.g. [7, 2.9]). In [1], situations are considered where the Kronecker product of two irreducible S_n-characters has few different constituents. It was shown there that such products are inhomogeneous (i.e., they contain at least two different irreducible constituents) except for the trivial situation where one of the characters is of degree 1. Investigating the

Date: Version of May 14, 2011.

2010 *Mathematics Subject Classification.* Primary 05E05; Secondary 05E10, 20C30.

Key words and phrases. Kronecker coefficients, Kronecker products, Schur functions.

The authors thank the Alexander von Humboldt Foundation and the National Sciences and Engineering Research Council of Canada for the support of their collaboration.

question on homogeneous products for the representations of the alternating group A_n motivated the study of products of S_n-characters with two different constituents; all Kronecker products of irreducible S_n-characters with two homogeneous components were then classified in [1]. Also, some partial results for products with up to four homogeneous components were obtained, and we will use these here. Moreover, a complete classification of the pairs (χ, ψ) of irreducible complex S_n-characters such that the Kronecker product $\chi \cdot \psi$ has three or four homogeneous components was conjectured [1, Conjecture]. In this article we verify these conjectures in the following theorem. Note that $[\mu]$ denotes the irreducible complex character of S_n labelled by the partition μ of n; further details on notation and background can be found in Section 2.

Theorem 1.1. *Let μ, ν be partitions of n. Then the following holds for the Kronecker product of the characters $[\mu]$ and $[\nu]$.*

(i) The product $[\mu] \cdot [\nu]$ has three homogeneous components if and only if $n = 3$ and $\mu = \nu = (2, 1)$ or $n = 4$ and $\mu = \nu = (2, 2)$.

(ii) The product $[\mu] \cdot [\nu]$ has four homogeneous components if and only if $n \geq 4$ and one of the following holds:

(1) $\mu, \nu \in \{(n-1, 1), (2, 1^{n-2})\}$.

(2) $n = 2k + 1$ and one of μ, ν is in $\{(n-1, 1), (2, 1^{n-2})\}$ while the other one is in $\{(k+1, k), (2^k, 1)\}$.

(3) $\mu, \nu \in \{(2^3), (3^2)\}$.

In Section 2 we introduce some notation and recall some results from [1] and [4] that will be used in the following sections. In Section 3, we collect results on Kronecker products and skew characters with at most two components; on the way, we slightly generalize some of the results in [1] from irreducible characters to arbitrary characters. Then Section 4 deals with special products such as products with the natural character, squares, products of characters to 2-part partitions or products of hooks and products of characters to partitions which differ only by one box; this is to some extent based on already available work. In Section 5, some key results are obtained that help to produce components in a product which are of almost maximal width. Here we exclude the situations dealt with in the previous sections, and focus on pairs of characters that are important in the proof of the classification theorem, to which Section 6 is devoted.

2. Preliminaries

We denote by $\mathbb{N}$ the set $\{1, 2, \dots\}$ of natural numbers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let G be a finite group. We denote by $\mathrm{Irr}(G)$ the set of irreducible (complex) characters of G. Let ψ be a character of G. Then we consider the decomposition of ψ into irreducible characters, i.e., $\psi = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$, with $a_\chi \in \mathbb{N}_0$. If $a_\chi > 0$, then χ is called a *constituent* of ψ and $a_\chi \chi$ is the corresponding *homogeneous component* of ψ. The character ψ is called *homogeneous* if it has only one homogeneous component. For any character ψ, we let $\mathcal{X}(\psi)$ be the set of irreducible constituents of ψ; so $c(\psi) = |\mathcal{X}(\psi)|$ is the number of homogeneous components of ψ. If ψ is a virtual character, i.e., $\psi = \sum_{\chi\in\mathrm{Irr}(G)} a_\chi \chi$ with $a_\chi \in \mathbb{Z}$, then we denote by $c(\psi)$ the number of components with positive coefficient a_χ.

For the group $G = S_n$, we use the usual notions and notation of the representation theory of symmetric groups and the related combinatorics and refer the reader to [7] or [15] for the relevant background. In particular, we write $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ if λ is a *partition* of n; in this case we write $|\lambda|$ for the *size* n of λ, $\max \lambda$ for its largest part or *width*, and $\ell(\lambda)$ for the *length* of λ, i.e., the number of its (positive) parts. We often gather together equal parts of a partition and write i^m for m occurrences of the number i as a part of the partition λ. The partition *conjugate* to λ is denoted by $\lambda' = (\lambda'_1, \dots, \lambda'_{\lambda_1})$. If $\lambda = \lambda'$ we say that λ is *symmetric*. We do not distinguish between a partition λ and its *Young diagram* $\lambda = \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid j \leq \lambda_i\}$. Pictorially, we will draw the diagram using matrix conventions, i.e., starting with row 1 of length λ_1 at the top. Elements $(i,j) \in \mathbb{N} \times \mathbb{N}$ are called *nodes*. If $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ are two partitions we write $\lambda \cap \mu$ for the partition $(\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \dots)$ whose Young diagram is the intersection of the diagrams for λ and μ. A node $(i, \lambda_i) \in \lambda$ is called a *removable* λ-node if $\lambda_i > \lambda_{i+1}$. A node $(i, \lambda_i + 1)$ is called *addable* (for λ) if $i = 1$ or $i > 1$ and $\lambda_i < \lambda_{i-1}$. We denote by

$$\lambda_A = \lambda \setminus \{A\} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots)$$

the partition obtained by removing a removable node $A = (i, \lambda_i)$ from λ. Similarly

$$\lambda^B = \lambda \cup \{B\} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots)$$

is the partition obtained by adding an addable node $B = (i, \lambda_i + 1)$ to λ.

For brevity, when we sum over all removable λ-nodes A, we will sometimes just write $\sum_{A\ \lambda\text{-node}}$ or $\sum_A$ to indicate this summation.

For a node $(i,j) \in \lambda$, we denote by

$$h_{ij} = h_{ij}^\lambda = \lambda_i - j + \lambda'_j - i + 1$$

the corresponding (i,j)-*hook length*.

For a partition λ of n, we write $[\lambda]$ (or $[\lambda_1, \lambda_2, \dots]$) for the (complex) character of S_n associated to λ. Thus, $\{[\lambda] \mid \lambda \vdash n\}$ is the set $\mathrm{Irr}(S_n)$ of all (complex) irreducible characters of S_n.

The standard inner product on the class functions on a group G is denoted by $\langle \cdot, \cdot \rangle$. If χ and ψ are two class functions of G we write $\chi \cdot \psi$ for the class function $(g \mapsto \chi(g)\psi(g))$ of G. For characters $[\lambda]$, $[\mu]$ of S_n, the class function $[\lambda] \cdot [\mu]$ is again a character of S_n, the *Kronecker product* of $[\lambda]$, $[\mu]$. We define the numbers $d(\mu, \nu; \lambda)$, for $\lambda, \mu, \nu \vdash n$, via

$$[\mu] \cdot [\nu] = \sum_{\lambda \vdash n} d(\mu, \nu; \lambda)[\lambda] \,.$$

If $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ are two partitions then we write $\beta \subseteq \alpha$ if $\beta_i \leq \alpha_i$ for all i. In this case we also consider the skew partition α/β; again, we do not distinguish between α/β and its skew Young diagram, which is the set of nodes $\alpha \setminus \beta$ belonging to α

but not β. If this diagram has the shape of the Young diagram for a partition, we will also speak of a *partition diagram*.

If α/β is a skew Young diagram and $A = (i, j)$ is a node we say A is *connected* to α/β if at least one of the nodes $(i \pm 1, j), (i, j \pm 1)$ belongs to α/β. Otherwise A is *disconnected* from α/β.

Let β and γ be two partitions. Taking the outer product of the corresponding characters, the *Littlewood-Richardson coefficients* are defined as the coefficients in the decomposition

$$[\beta] \otimes [\gamma] = \sum_{\alpha} c^{\alpha}_{\beta\gamma}[\alpha] \, .$$

For any partition α, the *skew character* $[\alpha/\beta]$ is then defined to be the sum

$$[\alpha/\beta] = \sum_{\gamma} c^{\alpha}_{\beta\gamma}[\gamma] \, .$$

The Littlewood-Richardson coefficients can be computed via the *Littlewood–Richardson rule* [7, 9] that says it is the number of ways of filling the nodes of α/β with positive integers such that the rows weakly increase left to right, the columns strictly increase top to bottom, and when the entries are read from right to left along the rows starting at the top, the numbers of i's read is always weakly greater than the number of $(i+1)$'s. Note that $[\alpha/\beta] = 0$ unless $\beta \subseteq \alpha$.

As in [1], we will repeatedly use the following results. The first describes the rectangular hull of the partition labels of the constituents of a Kronecker product.

Theorem 2.1. [4, 1.6], [3, 1.1]. *Let μ, ν be partitions of n. Then*

$$\max\{\max \lambda \mid [\lambda] \in \mathcal{X}([\mu] \cdot [\nu])\} = |\mu \cap \nu|$$

and

$$\max\{\ell(\lambda) \mid [\lambda] \in \mathcal{X}([\mu] \cdot [\nu])\} = |\mu \cap \nu'| \, .$$

Since skew characters of S_n can be decomposed into irreducible characters using the Littlewood-Richardson rule, the following theorem provides a recursive formula for the coefficients $d(\mu, \nu; \lambda)$. In the following, if $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition of n, we set $\hat{\lambda} = (\lambda_2, \lambda_3, \dots)$.

Theorem 2.2. [4, 2.3]. *Let μ, ν and $\lambda = (\lambda_1, \lambda_2, \dots)$ be partitions of n. Define*

$$Y(\lambda) = \{\eta = (\eta_1, \dots) \vdash n \mid \eta_i \geq \lambda_{i+1} \geq \eta_{i+1} \textit{ for all } i \geq 1\} \, .$$

Then

$$d(\mu, \nu; \lambda) = \sum_{\substack{\alpha \vdash \lambda_1 \\ \alpha \subseteq \mu \cap \nu}} \langle [\mu/\alpha] \cdot [\nu/\alpha], [\hat{\lambda}] \rangle - \sum_{\substack{\eta \in Y(\lambda) \\ \eta \neq \lambda \\ \eta_1 \leq |\mu \cap \nu|}} d(\mu, \nu; \eta) \, .$$

Corollary 2.3. [4, 2.4], [3, 2.1(d)]. *Let μ, ν, λ be partitions of n, $\gamma = \mu \cap \nu$, and assume* $\max \lambda = |\mu \cap \nu|$. *Then*

$$d(\mu, \nu; \lambda) = \langle [\mu/\gamma] \cdot [\nu/\gamma], [\hat{\lambda}] \rangle \ .$$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of length m, set $\bar{\lambda} = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_m - 1)$.

Corollary 2.4. [4, 2.4']. *Let μ, ν and λ be partitions of n, and assume $\ell(\lambda) = |\mu \cap \nu'|$. Then*

$$d(\mu, \nu; \lambda) = \langle [\mu/(\mu \cap \nu')] \cdot [\nu/(\mu' \cap \nu)], [\bar{\lambda}] \rangle \ .$$

We will later use these results for finding *extreme constituents* in a product, i.e., those with partition labels of maximal width or length; explicitly, we state this in the following lemma. Here, for a partition $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $m \in \mathbb{N}$, we set $\alpha + (1^m) = (\alpha_1 + 1, \ldots, \alpha_m + 1, \alpha_{m+1}, \ldots)$.

Lemma 2.5. *Let μ, ν, λ be partitions of n, $\gamma = \mu \cap \nu \vdash m$, $\tilde{\gamma} = \mu \cap \nu' \vdash \tilde{m}$.*

(i) If $[\alpha] = [\alpha_1, \ldots]$ appears in $[\mu/\gamma] \cdot [\nu/\gamma]$, then $[m, \alpha_1, \ldots]$ appears in $[\mu] \cdot [\nu]$.

(ii) If $[\alpha] = [\alpha_1, \ldots]$ appears in $[\mu/\tilde{\gamma}] \cdot [\nu'/\tilde{\gamma}]$, then $[\alpha' + (1^{\tilde{m}})]$ appears in $[\mu] \cdot [\nu]$.

Proof. (i) If $[\alpha]$ appears in $[\mu/\gamma] \cdot [\nu/\gamma]$, then it is a constituent of some $[\rho] \cdot [\tau]$ with $\rho \subset \mu$, $\tau \subset \nu$ by the Littlewood-Richardson rule and hence by Theorem 2.1 $\alpha_1 \leq |\rho \cap \tau| \leq |\mu \cap \nu| = m$. Thus $(m, \alpha_1, \ldots)$ is a partition and the claim follows by Corollary 2.3.

(ii) If $[\alpha]$ appears in $[\mu/\tilde{\gamma}] \cdot [\nu'/\tilde{\gamma}]$, then by Theorem 2.1 $\alpha_1 \leq |\mu \cap \nu'| = \tilde{m}$. Hence $\alpha' + (1^{\tilde{m}})$ is a partition and the claim follows by Corollary 2.4. □

It will also be important to note that in *nontrivial* Kronecker products $[\mu] \cdot [\nu]$, i.e., products where both characters are of degree > 1, there is no constituent which is both of maximal width and maximal length:

Theorem 2.6. [1, Theorem 3.3] *Let μ, ν be partitions of n, both different from (n) and (1^n). If $[\lambda]$ is a constituent of $[\mu] \cdot [\nu]$, then $h_{11}^{\lambda} < |\mu \cap \nu| + |\mu \cap \nu'| - 1$.*

The constituents of maximal width (or length) in a Kronecker product may be handled by the results above. In [1], information on constituents of almost maximal width is obtained in the special situation described below; this will be used here as well, and we will later also provide further information of this type for modified situations as needed.

Lemma 2.7. [1, Lemma 4.6] *Let $\mu \neq \nu$ be partitions of n, both different from (n), (1^n), $(n-1, 1)$ and $(2, 1^{n-2})$. Put $\gamma = \mu \cap \nu$, $m = |\gamma|$. Assume that ν/γ is a row and that $[\mu/\gamma]$ is an irreducible character $[\alpha]$, $\alpha = (\alpha_1, \alpha_2, \ldots)$. Then $[m, \alpha_1, \alpha_2, \ldots] \in \mathcal{X}([\mu] \cdot [\nu])$.*

Moreover if an S_{n-m+1}-character $[\theta_1, \theta_2, \ldots]$ appears in

$$\sum_{A \text{ removable for } \gamma} [\mu/\gamma_A] \cdot [\nu/\gamma_A] - \sum_{B \text{ addable for } \alpha} [\alpha^B] \tag{2.1}$$

with a positive coefficient, then $[m-1, \theta_1, \theta_2, \ldots] \in \mathcal{X}([\mu] \cdot [\nu])$.

3. Products and skew characters with at most two components

In [1], the Kronecker products of S_n-characters with at most two components were classified. We recall this here as we will need this in the proof of our main result, Theorem 1.1.

Theorem 3.1. [1, Corollary 3.5] *Let μ, ν be partitions of n. Then $[\mu]\cdot[\nu]$ is homogeneous if and only if one of the partitions is (n) or (1^n). In this case $[\mu]\cdot[\nu]$ is irreducible.*

Theorem 3.2. [1, Theorem 4.8] *Let μ, ν be partitions of n. Then $[\mu]\cdot[\nu]$ has exactly two homogenous components if and only if one of the partitions μ,ν is a rectangle (a^b) with $a,b>1$, and the other is $(n-1,1)$ or $(2,1^{n-2})$. In these cases we have:*

$$[n-1,1]\cdot[a^b] = [a+1,a^{b-2},a-1]+[a^{b-1},a-1,1]$$
$$[2,1^{n-2}]\cdot[a^b] = [b+1,b^{a-2},b-1]+[b^{a-1},b-1,1]\,.$$

Further results from [1] on products with few components will be recalled as required later.

Next we consider skew characters with few components; from [1] we quote the following (see also [18]):

Lemma 3.3. [1, Lemma 4.4] *Let μ, γ be partitions, $\gamma\subset\mu$. Then the following assertions are equivalent:*

(a) $[\mu/\gamma]$ is homogeneous.
(b) $[\mu/\gamma]$ is irreducible.
(c) The skew diagram μ/γ is the diagram of a partition α or the rotation of a diagram of a partition α. In this case, $[\mu/\gamma]=[\alpha]$.

Note that throughout this paper, whenever we say rotation we mean *rotation by* 180°. In connection with the situation described above, we will also need the following from [1].

Lemma 3.4. *Let μ, γ be partitions, $\gamma\subset\mu$, such that $[\mu/\gamma]$ is irreducible, say $[\mu/\gamma]=[\alpha]$. Let A be a removable node of γ.*

(1) If A is disconnected from μ/γ then

$$[\mu/\gamma_A] = \sum_{B \text{ addable for } \alpha} [\alpha^B]\,.$$

(2) Let A be connected to μ/γ. Let B_0 and B_1 denote the top and bottom removable node of α respectively.

If μ/γ has partition shape α, then

$$[\mu/\gamma_A] = \begin{cases} \displaystyle\sum_{\substack{B \text{ addable for } \alpha\\ B\neq B_0}} [\alpha^B] & \text{if } A \text{ is connected to the top row of } \alpha \\ \displaystyle\sum_{\substack{B \text{ addable for } \alpha\\ B\neq B_1}} [\alpha^B] & \text{if } A \text{ is connected to the bottom row of } \alpha. \end{cases}$$

If μ/γ is the rotation of α, then

$$[\mu/\gamma_A] = [\alpha^B]\,, \text{ where } B \text{ is an addable node of } \alpha.$$

We will also need a result classifying skew characters with two components; this has recently been done by Gutschwager [6].

We denote an outer product of two characters χ, ψ by $\chi \otimes \psi$; recall that an outer product of two irreducible characters corresponds to a character associated to a skew diagram decomposing into two disconnected partition diagrams. In the classification list below, r, s, a, b are arbitrary nonnegative integers such that all characters appearing on the left hand side correspond to partitions. For example, $[(r+1)^{a+1}, r^b, s] + [(r+1)^a, r^{b+1}, s+1] = [((r+1)^{a+1}, r^{b+1})/(r-s)]$ also incorporates that $r \geq s+1$ since otherwise $((r+1)^a, r^{b+1}, s+1)$ is not a partition; thus here $s, a, b \geq 0$ and $r \geq s+1$. Choosing the minimal values $r = 1, s = a = b = 0$ gives $[2] + [1^2] = [(2,1)/(1)]$. Note that any proper skew character $[\mu/\gamma]$, i.e., one that is not irreducible, always has a constituent obtained by sorting the row lengths of μ/γ as well as a (different) constituent obtained by sorting the column lengths and conjugating. These appear on the right hand side below, which we write out explicitly for the convenience of the reader. Note also that these constituents are of multiplicity 1; in view of the previous lemma, the classification list for skew characters with exactly two components coincides with the corresponding list of skew characters with exactly two constituents.

Proposition 3.5. [6] *The following is a complete list of skew characters of symmetric groups with exactly two homogeneous components; up to rotation, ordering and translation all corresponding skew diagrams are given.*

(i) $[1] \otimes [r^{a+1}] = [r+1, r^a] + [r^{a+1}, 1]$.
(ii) $[1^{a+1}] \otimes [r] = [r, 1^{a+1}] + [r+1, 1^a]$.
(iii) $[((r+1)^{a+1}, r^{b+1})/(r-s)] = [((r+1)^{a+b+1}, s+1)/(1^{b+1})]$
$= [(r+1)^{a+1}, r^b, s] + [(r+1)^a, r^{b+1}, s+1]$.
(iv) $[((r+1)^{a+1}, (s+1)^{b+1})/(1)] = [(r+1)^a, r, (s+1)^{b+1}] + [(r+1)^{a+1}, (s+1)^b, s]$.
(v) $[((r+1)^{a+1}, (s+1)^{b+1})/(1^{a+b+1})] = [r^{a+1}, s+1, s^b] + [r+1, r^a, s^{b+1}]$.
(vi) $[(r^{a+1}, s^{b+1})/(r-1)] = [r^a, s^{b+1}, 1] + [r^a, s+1, s^b]$.
(vii) $[(r+1, (s+1)^{a+b+1})/(1^{b+1})] = [r, (s+1)^{a+1}, s^b] + [r+1, (s+1)^a, s^{b+1}]$.
(viii) $[(r^{a+1}, 1^{b+1})/(r-s)] = [r^a, s, 1^{b+1}] + [r^a, s+1, 1^b]$.

For later purposes we note the following slight generalization of Theorem 3.1; recall that $c(\chi)$ denotes the number of homogeneous components of a character χ.

Proposition 3.6. *Let χ, ψ be characters of S_n. Then $c(\chi \cdot \psi) = 1$ if and only if we are in one of the following situations (up to ordering the characters):*

(1) $c(\chi) = 1$ and $\psi = a[n]$ or $\psi = b[1^n]$, with $a, b \in \mathbb{N}$.
(2) $\chi = k[\alpha]$ with $\alpha = \alpha'$, $k \in \mathbb{N}$, and $\psi = a[n] + b[1^n]$, with $a, b \in \mathbb{N}_0$.

Proof. This follows easily using Theorem 3.1. □

Corollary 3.7. *Let $D, \tilde{D}$ be skew diagrams of size n. Then $c([D] \cdot [\tilde{D}]) = 1$ if and only if one of the diagrams is a partition or rotated partition diagram, and the other one is a row or column.*

In particular, the product of two skew characters is a homogeneous character if and only if it is an irreducible character.

Proof. Use Proposition 3.6 and Lemma 3.3, and note that a skew character can be of the form $a[n] + b[1^n]$ with $a, b > 0$ only for $n = 2$, where there is no symmetric partition. □

It takes a bit more work to generalize Theorem 3.2 and characterize the products of characters with exactly two homogeneous components:

Theorem 3.8. *Let χ, ψ be characters of S_n, $n > 1$. Then $c(\chi \cdot \psi) = 2$ exactly in the following situations (up to ordering the characters):*

(1) $\chi = a[\lambda]$ *with* $\lambda \neq \lambda'$, $\psi = b[n] + c[1^n]$, $a, b, c \in \mathbb{N}$.
(2) $\chi = a[r^s]$ *with* $r \neq s$, $r, s > 1$, $\psi = b[n-1, 1]$ *or* $b[2, 1^{n-2}]$, $a, b \in \mathbb{N}$.
(3) $\chi = a[r^r]$, $\psi = b[n-1, 1] + c[2, 1^{n-2}]$, $a, b, c \in \mathbb{N}_0$, $a, b + c > 0$.
(4) $\chi = a[n]$ *or* $a[1^n]$, $\psi = b[\lambda] + c[\mu]$, $\lambda \neq \mu$ *arbitrary*, $a, b, c \in \mathbb{N}$.
(5) $\chi = a[n] + b[1^n]$, $\psi = c[\lambda] + d[\lambda']$ *with* $\lambda \neq \lambda'$, $a, b, c, d \in \mathbb{N}$.
(6) $\chi = a[n] + b[1^n]$, $\psi = c[\alpha] + d[\beta]$ *with* $\alpha = \alpha' \neq \beta = \beta'$, $a, b, c, d \in \mathbb{N}$.
(7) $\chi = a[2^2] + b[4] + c[1^4]$, $\psi = d[3, 1] + e[2, 1^2]$, $a, b, c, d, e \in \mathbb{N}_0$, $a, b + c, d + e > 0$.

Proof. Use Theorem 3.2. □

In the following, by a *nontrivial rectangle* we mean a rectangle with at least two rows and at least two columns.

Corollary 3.9. *Let $D, \tilde{D}$ be skew diagrams of size n. Then $c([D] \cdot [\tilde{D}]) = 2$ if and only if we are in one of the following situations (up to ordering of the diagrams):*

(1) $n = 2$, $[D] = [\tilde{D}] = [2] + [1^2]$*; here, D and $\tilde{D}$ both consist of two disconnected nodes.*
(2) D is a row or column, $[\tilde{D}]$ is one of the skew characters in Proposition 3.5; the diagram $\tilde{D}$ corresponds to one of the diagrams in Proposition 3.5 up to rotation, translation and reordering disconnected parts.
(3) D is a nontrivial rectangle, $\tilde{D}$ has shape $(n-1, 1)$ up to rotation and conjugation.
(4) $n = 4$, $D = (2^2)$*, $\tilde{D}$ consists of a disconnected row and column of size 2 each; here,* $[\tilde{D}] = [3, 1] + [2, 1^2]$.

Proof. Theorem 3.8 strongly restricts the number of cases to be considered. Then, note that a skew character $[D]$ can be of the form $a[n] + b[1^n]$ with $a, b > 0$ only for $n = 2$, when D consists of two disconnected nodes. In fact, if both $[n]$ and $[1^n]$ appear in the skew character, then all irreducible characters appear as constituents. The only case where both $[D]$ and $[\tilde{D}]$ are inhomogeneous is the one described for $n = 2$ in (1) above. Homogeneous skew characters are labelled by partition diagrams, up to rotation, and are indeed irreducible (by Lemma 3.3). Furthermore, we use Proposition 3.5 to identify the diagrams for the skew characters with two components appearing above, and we note that $a[2^2]+b[4]$ and $a[2^2]+c[1^4]$ are not skew characters. □

4. Special products

The products with the character $[n-1,1]$ are easy to compute, and then the classification of such products with few components is not hard to deduce (see [1]).

Lemma 4.1. [1, Lemma 4.1] *Let $n \geq 3$ and μ be a partition of n. Then*

$$[\mu]\cdot[n-1,1] = \sum_A \sum_B [(\mu_A)^B] - [\mu]$$

where the first sum is over all removable nodes A for μ, and the second sum runs over all addable nodes B for μ_A.

Corollary 4.2. [1, Cor. 4.2] *Let $n \geq 3$ and μ be a partition of n. Then*

(i) $c([\mu]\cdot[n-1,1]) = 1$ if and only if μ is (n) or (1^n).

(ii) $c([\mu]\cdot[n-1,1]) = 2$ if and only if μ is a rectangle (a^b) for some $a, b > 1$. In this case we have

$$[a^b]\cdot[n-1,1] = [a+1, a^{b-2}, a-1] + [a^{b-1}, a-1, 1]\,.$$

(iii) $c([\mu]\cdot[n-1,1]) = 3$ if and only if $n = 3$ and $\mu = (2,1)$. In this case we have

$$[2,1]\cdot[2,1] = [3] + [2,1] + [1^3]\,.$$

(iv) $c([\mu]\cdot[n-1,1]) = 4$ if and only if one of the following happens:

(a) $n \geq 4$ and $\mu = (n-1,1)$ or $(2,1^{n-2})$.

(b) $\mu = (k+1,k)$ or $(2^k,1)$ for $k \geq 2$.

We then have:

$$\begin{aligned} [n-1,1]\cdot[n-1,1] &= [n]+[n-1,1]+[n-2,2]+[n-2,1^2] \\ [k+1,k]\cdot[2k,1] &= [k+2,k-1]+[k+1,k]+[k+1,k-1,1]+[k^2,1] \end{aligned}$$

and the remaining products are obtained by conjugation.

As a further special family of products the Kronecker squares were considered in [1]:

Proposition 4.3. [1, Lemma 4.3] *Let λ be a partition of n. Then $c([\lambda]^2) \leq 4$ if and only if one of the following holds:*

(1) $\lambda = (n)$ or (1^n), when $[\lambda]^2 = [n]$.
(2) $n \geq 4$, $\lambda = (n-1,1)$ or $(2,1^{n-2})$, when $[\lambda]^2 = [n]+[n-1,1]+[n-2,2]+[n-2,1^2]$.
(3) $n = 3$, $\lambda = (2,1)$, when $[\lambda]^2 = [3]+[2,1]+[1^3]$.
(4) $n = 4$, $\lambda = (2^2)$, when $[\lambda]^2 = [4]+[2^2]+[1^4]$.
(5) $n = 6$, $\lambda = (3^2)$ or (2^3), when $[\lambda]^2 = [6]+[4,2]+[3,1^3]+[2^3]$.

We now turn to further families of Kronecker products where we can classify the products with few components. While the products of characters to 2-part partitions and hooks have been determined in work by Remmel et al. [11, 12, 13] and Rosas [14], here we do not use these intricate results but prove the following weaker facts for the sake of a self-contained presentation.

By a *2-part partition* we mean a partition of length at most 2; we say it is *proper* if it has length exactly 2.

Proposition 4.4. *Let $\mu, \nu \vdash n$ be different proper 2-part partitions, say $\mu = (n-k,k)$, $\nu = (n-l,l)$ with $1 < l < k$. Let $\gamma = \mu \cap \nu$, a partition of $m = n-k+l$. Then we have the following constituents in $[\mu]\cdot[\nu]$:*

(1) $[m, n-m]$.
(2) $[m-1, n-m+1]$, *except when $\mu = (k,k)$.*
(3) $[m-1, n-m, 1]$.
(4) At least 2 different constituents of length 4, except when $l = 2$, where the product only has one constituent $[m-3, n-m+1, 1^2] = [n-k-1, k-1, 1^2]$ of length 4.
(5) For $l = 2$, $[\mu]\cdot[\nu]$ always has the constituent $[m-2, n-m+1, 1] = [n-k, k-1, 1]$; for $k > 3$ it also contains $[m-2, n-m, 2] = [n-k, k-2, 2]$, and for $k = 3$, $n \geq 7$, it contains $[n-4, 2^2]$.
(6) When $\mu = (k,k)$ and $l > 3$, $[\mu]\cdot[\nu]$ has at least 3 different constituents of length 4.
(7) When $\mu = (k,k)$ and $l = 3$, $[\mu]\cdot[\nu]$ has a constituent $[k+1, k-1]$.
(8) $[3,3]\cdot[4,2]$ *contains* $[3,3]$.

In all cases, we have at least five different constituents.

Proof. Constituent (1) comes from Corollary 2.3; the constituents in (2) and (3) are obtained by applying Lemma 2.7. The constituents in (4) and (6) are obtained using Corollary 2.4 and Theorem 3.1 and Theorem 3.2, respectively. For (5), we apply Lemma 2.7 to $[\mu']\cdot[\nu]$, and then obtain after conjugation constituents of length 3 in $[\mu]\cdot[\nu]$ as given. The constituent for (7) may be obtained with the help of [17] or from [2]. Assertion (8) follows from Proposition 4.3(5).

The final assertion follows by collecting in each case suitable constituents found above. □

Corollary 4.5. *Let $\mu, \nu \vdash n$ be proper 2-part partitions. Then $c = c([\mu]\cdot[\nu]) \leq 4$ if and only if one of the following holds:*

(1) $c = 1$, when $n = 2$, and the product is irreducible.
(2) $c = 2$, when $n = 2k$ is even, and the product is $[n-1,1]\cdot[k,k]$.
(3) $c = 3$, when $n = 3$, $\mu = \nu = (2,1)$, or $n = 4$, $\mu = \nu = (2^2)$.
(4) $c = 4$, when $n \geq 4$, $\mu = \nu = (n-1,1)$, or $n = 2k+1$, $\{\mu,\nu\} = \{(n-1,1),(k+1,k)\}$, or $n = 6$, $\mu = \nu = (3^2)$.

A *hook* partition is of the form $(a, 1^b)$; we call this partition a *proper hook* when both $a > 1$ and $b > 0$ hold.

Proposition 4.6. *Let $\mu, \nu \vdash n$ be different proper hooks, say $\mu = (n-k, 1^k)$, $\nu = (n-l, 1^l)$, with $1 < l < k$ and $n-k > k$, $n-l > l$. Set $\gamma = \mu \cap \nu$; this is a partition of $m = n-k+l$. Then we have the following constituents in $[\mu] \cdot [\nu]$:*

(1) $[m, 1^{n-m}]$.
(2) $[m-1, 2, 1^{n-m-1}]$.
(3) $[m-1, 1^{n-m+1}]$.
(4) $[n-k-l, 1^{k+l}]$.
(5) $[n-k-l+1, 1^{k+l-1}]$.
(6) $[n-k-l, 2, 1^{k+l-2}]$.

In all cases we have at least six different constituents.

Proof. Again, the constituent in (1) comes from Corollary 2.3; the constituents in (2) and (3) are obtained using Lemma 2.7. The constituents in (4), (5) and (6) are obtained by considering the three "highest" constituents in the product $[\mu] \cdot [\nu']$, again via Corollary 2.3 and Lemma 2.7, and then conjugating.

Considering, for example, the length of the partitions in (1)-(6), one easily sees that they are all different. □

Corollary 4.7. *Let $\mu, \nu \vdash n$ be hooks. Then $c = c([\mu] \cdot [\nu]) \leq 4$ if and only if one of the following holds:*
(1) $c = 1$, when one of the partitions is a trivial hook (n) or (1^n).
(2) $c = 3$, when $\mu = \nu = (2, 1)$.
(3) $c = 4$, when $n \geq 4$, and $\mu, \nu \in \{(n-1, 1), (2, 1^{n-2})\}$.

Next, turning a small step away from the square case considered in [1], we consider products of characters to partitions which differ by at most one node. Note that the *depth* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n is defined to be $n - \lambda_1$.

Proposition 4.8. *Let $\mu, \nu \vdash n$, $\mu \cap \nu = \gamma \vdash n-1$, $\mu, \nu \notin \{(n), (n-1, 1), (1^n), (2, 1^{n-2})\}$, $\nu \neq \mu'$. Furthermore, we assume that μ, ν are not both 2-part partitions (or their conjugates). Then we have the following constituents in $[\mu] \cdot [\nu]$:*

(1) $[\lambda^{(1)}] = [n-1, 1]$ *(with multiplicity 1).*
(2) $[\lambda^{(2)}] = [n-2, 1^2]$.
(3) $[\lambda^{(3)}] = [n-2, 2]$.
(4) $[\lambda^{(4)}]$ *with* $\ell(\lambda^{(4)}) = |\mu \cap \nu'|$.
(5) $[\lambda^{(5)}]$ *with* $\ell(\lambda^{(5)}) \leq 5$ *and depth* $n - \lambda_1^{(5)} \geq 3$.

In all cases, the product has at least five components.

Proof. Since $\mu \cap \nu = \gamma \vdash n-1$, we immediately get the constituent $[n-1, 1]$ with multiplicity 1, by Corollary 2.3. By Lemma 2.7, we get both constituents $[n-2, 1^2]$ and $[n-2, 2]$ as soon as γ has a removable node A_0 disconnected from μ/γ and A_1 disconnected from ν/γ (possibly $A_0 = A_1$). If this is not the case, by our assumptions, γ has a removable node A_0 that is disconnected from μ/γ but connected to ν/γ (say) and a node A_1 connected to both; we then only get a constituent $[n-2,1^2]$; but this exceptional case occurs only for $\gamma=(k,k-1)$ (or conjugate), and corresponding to $\{\mu,\nu\}=\{(k^2),(k+1,k-1)\}$ (or conjugate), a case excluded by our assumptions. Also, Corollary 2.4 gives a constituent of length $|\mu\cap\nu'|$; by our assumptions, this length is at least 4, so this constituent is different from the ones obtained so far.

Now we turn to a constituent of type $\lambda^{(5)}$. By conjugating both μ and ν, if necessary, we may assume that $\gamma_1 \geq \ell(\gamma)$. If γ is rectangular, then $[\gamma]^2$ contains $[n-4,1^3]$, and if γ is non-rectangular and of depth ≥ 2, then $[\gamma]^2$ contains one of the characters $[n-4,3]$ or $[n-4,2,1]$ (see [16, 19]). Since the restriction of $[\mu]\cdot[\nu]$ to S_{n-1} contains the constituents of $[\gamma]^2$, in all these cases we find a constituent $[\lambda^{(5)}]$ in $[\mu]\cdot[\nu]$ as claimed; in fact, in the non-rectangular case we find such a constituent of length ≤ 4.

If $|\mu\cap\nu'|\geq 6$, then $[\lambda^{(5)}]$ is different from $[\lambda^{(4)}]$, and hence the product has at least five components. Note that $|\mu\cap\nu'|\geq 4$, so in the situation considered here it remains to deal with the cases $|\mu\cap\nu'|=4$ or 5.

If γ is rectangular, the case $|\mu\cap\nu'|=4$ is impossible. If γ is rectangular and $|\mu\cap\nu'|=5$, then μ, ν are $(k+1,k)$ and $(k,k,1)$, up to order, and $\mu\cap\nu'=(3,2)$ or $(2^2,1)$. Then $[\mu]\cdot[\nu']$ has at least two components of maximal width 5, so $[\mu]\cdot[\nu]$ has at least two components of length 5 and hence the product has again at least five components.

Now consider the case where γ is non-rectangular and of depth ≥ 2; since we have found a constituent $[\lambda^{(5)}]$ of length at most 4 and depth ≥ 3 above, we only need to discuss the case when $|\mu\cap\nu'|=4$. As (2^2) is contained in γ it is also contained in $\mu\cap\nu'$ hence $\mu\cap\nu'=(2^2)$. But since μ,ν are not both 2-part partitions, we immediately get a contradiction.

We still have to discuss the case where γ is non-rectangular and of depth 1, i.e., $\gamma=(n-2,1)$. But then, up to order, $\mu=(n-2,2)$, $\nu=(n-2,1^2)$. By Corollary 2.3 we always have the constituent $[4,1^{n-4}]$ in $[\mu]\cdot[\nu']$, and for $n\geq 6$ also the constituent $[4,2,1^{n-6}]$, from which we obtain the two constituents $[n-3,1^3]$ and $[n-4,2,1^2]$ in $[\mu]\cdot[\nu]$ of length ≤ 4 and depth ≥ 3. For $n=5$, $\nu=(3,1^2)$ is symmetric, and the constituent $[3,2]$ in $[\mu]\cdot[\nu']$ also gives $[2^2,1]$ in $[\mu]\cdot[\nu]$. So also in these cases we always find at least five constituents. $\square$

The classification of the corresponding products with few components may easily be deduced:

Corollary 4.9. *Let $\mu,\nu\vdash n$, $\mu\cap\nu=\gamma\vdash n-1$. Then $c([\mu]\cdot[\nu])\leq 4$ if and only if up to conjugation one of the following holds:*
(1) One of μ,ν is (n) or (1^n), and the product is irreducible.
(2) $n=4$, and the product is $[3,1]\cdot[2^2]$ or $[3,1]\cdot[2,1^2]$.
(3) $n=5$, and the product is $[4,1]\cdot[3,2]$.

5. Key technical lemmas and propositions

From now on we fix the following notation:
$$\mu,\nu\vdash n\,,\ \mu\cap\nu=\gamma\vdash m\,,\ d=n-m\,.$$

Our aim is to prove the classification of the products with few components. Thus, because of the results on special products in the previous sections we may and will assume the following properties referred to as *Hypothesis* $(*)$:

(1) $\mu, \nu \notin \{(n), (n-1,1), (1^n), (2,1^{n-2})\}$.
(2) $\mu \neq \nu$, $\mu \neq \nu'$.
(3) $m \leq n-2$, i.e., $d \geq 2$.
(4) μ, ν are not both 2-line partitions or both hooks.

Here, by a *2-line partition* we mean a partition which has at most two rows or at most two columns. Note that we cannot have $\gamma = (m)$, since otherwise one of μ, ν is (n). Also, the assumptions on μ, ν imply that $m \geq 4$ and $m < n$.

Besides the extreme components in the product, the almost extreme components play an important role. First we make the following useful observation which follows from Theorem 2.2.

Lemma 5.1. *Let* $\lambda = (\lambda_1, \ldots) \vdash n$ *with* $\lambda_1 = m-1$. *Set*

$$\tilde{Y}(\lambda) = \{\eta \vdash n \mid \eta_i \geq \lambda_{i+1} \geq \eta_{i+1} \text{ for all } i \geq 1, \eta_1 \leq m, \eta \neq \lambda\}\ ,\ \varepsilon = \sum_{\eta \in \tilde{Y}(\lambda)} d(\mu,\nu;\eta)\ .$$

Then we have:

(i) $\tilde{Y}(\lambda) = \{(m, \hat{\lambda}_A) \mid A \text{ removable } \hat{\lambda}\text{-node}\}$.
(ii) $\varepsilon = \langle ([\mu/\gamma]\cdot[\nu/\gamma])\uparrow^{S_{d+1}}, [\hat{\lambda}]\rangle$.
(iii) We define the virtual character

$$\chi = \left(\sum_{A\ \gamma-\text{node}} [\mu/\gamma_A]\cdot[\nu/\gamma_A] \right) - ([\mu/\gamma]\cdot[\nu/\gamma])\uparrow^{S_{d+1}}\ .$$

Then $\quad d(\mu,\nu;\lambda) = d(\chi, [\hat{\lambda}])$.

Proof. (i) Since $\eta \in \tilde{Y}(\lambda)$ arises from $\hat{\lambda}$ by putting on a row strip of size $\lambda_1 = m-1$, but with the restriction $\eta_1 \leq m$, it follows that we must have $\eta = (m, \hat{\lambda}_A)$ for some removable $\hat{\lambda}$-node A.

(ii) Let $\quad [\mu/\gamma]\cdot[\nu/\gamma] = \sum_\alpha c_\alpha [\alpha]$; then by Corollary 2.3

$$[\mu]\cdot[\nu] = \sum_\alpha c_\alpha [m,\alpha] + \sum_{\substack{\beta \vdash n\\ \beta_1 < m}} d(\mu,\nu;\beta)\ [\beta]\ .$$

Thus

$$\begin{aligned}\varepsilon &= \sum_{\eta\in\tilde{Y}(\lambda)} d(\mu,\nu;\eta) = \sum_{\eta\in\tilde{Y}(\lambda)} \sum_\alpha c_\alpha \langle [m,\alpha],[\eta]\rangle = \sum_{A\ \hat{\lambda}\text{-node}} \sum_\alpha c_\alpha \langle [\alpha],[\hat{\lambda}_A]\rangle \\ &= \langle [\mu/\gamma]\cdot[\nu/\gamma], \sum_{A\ \hat{\lambda}\text{-node}} [\hat{\lambda}_A]\rangle = \langle ([\mu/\gamma]\cdot[\nu/\gamma])\uparrow^{S_{d+1}}, [\hat{\lambda}]\rangle\ .\end{aligned}$$

(iii) As $\lambda_1 = m-1$ and by the definition of $\tilde{Y}(\lambda)$, the assertion on $d(\mu,\nu;\lambda)$ now follows from Theorem 2.2. □

Lemma 5.2. *With Hypothesis* $(*)$ *on* μ,ν *as above, assume that we have one of the following situations:*
(i) ν/γ *is a row.*
(ii) $[\mu/\gamma]$ *and* $[\nu/\gamma]$ *are irreducible and* $c([\mu/\gamma]\cdot[\nu/\gamma]) = 2$.
Let $\theta \vdash n-m+1$ *such that* $[\theta] = [\theta_1, \theta_2, \dots]$ *appears as a constituent in*

$$\sum_{A\ \gamma-\text{node}} [\mu/\gamma_A]\cdot[\nu/\gamma_A]\,. \tag{5.1}$$

Then in both situations above, we have $\theta_1 \le m-1$.

Proof. First note that since $[\theta]$ appears in $[\mu/\gamma_A]\cdot[\nu/\gamma_A]$ for some removable γ-node A, it is a constituent of $[\rho]\cdot[\tau]$ for some constituents $[\rho]$ of $[\mu/\gamma_A]$ and $[\tau]$ of $[\nu/\gamma_A]$. Then $\rho \subset \mu$, $\tau \subset \nu$ and hence $\theta_1 \le |\rho\cap\tau| \le |\mu\cap\nu| = m$. Now assume that $\theta_1 = m$. Since $\rho\cap\tau \subseteq \mu\cap\nu$, this implies $\rho\cap\tau = \gamma$.

In Case (i), ν/γ_A is a union of a row and a node, so $[\nu/\gamma_A] \subseteq [n-m+1]+[n-m,1]$ (where the inclusion here means a subcharacter). As $\gamma\subseteq\tau$ and $\gamma \ne (m)$, we get $\tau = (n-m,1)$ and then $\gamma = (m-1,1)$. Since ν/γ is a row of size $n-m\ge 2$, and $\nu \ne (n-1,1)$, we must have $\nu = (m-1, n-m+1)$. But then $|\theta| = n-m+1 \le m-1$, a contradiction.

In Case (ii), we are in the situation of Corollary 3.9(3). Then one of the skew diagrams, say μ/γ, is a nontrivial rectangle (r^s), and the other is $(n-m-1,1)$ up to rotation and conjugation. Then $[\rho]$ is one of $[r+1, r^{s-1}]$, $[r^s,1]$, and $[\tau]$ is one of $[n-m,1]$, $[n-m-1,2]$, $[n-m-1,1^2]$ or their conjugates. Assuming that $\gamma\subset\tau$ is a hook, the conditions on the shapes of μ/γ and ν/γ easily give a contradiction. Hence γ is $(m-2,2)$ or its conjugate. First let $m>4$; conjugating we may assume that $\gamma = (m-2,2)$. Then $\mu = (m-2, 2^{s+1})$ and $\nu = ((m-1)^2)$. Thus $|\theta| = n-m+1 = m-1$, again a contradiction. If $m=4$, then up to conjugation we have $\mu = (2^{s+2})$ and $\nu = (n-3,3)$, i.e., both partitions are 2-line partitions, a case we have excluded above. □

We use Lemma 5.1 and Lemma 5.2 to obtain the following result. We know that the second case in Lemma 5.2 only occurs when one of the skew characters corresponds to a nontrivial rectangle and the other one is $[d-1,1]$ or $[2,1^{d-2}]$; conjugating, if necessary, we may assume that we are in the first situation.

Lemma 5.3. *Assume Hypothesis* $(*)$. *Let* $\hat\lambda = (\lambda_2,\lambda_3,\dots) \vdash d+1$ *and set* $\lambda = (m-1,\lambda_2,\lambda_3,\dots)$.

(i) *Assume that* ν/γ *is a row.*
If $[\hat\lambda]$ *appears with positive coefficient in the virtual character*

$$\chi = \sum_{A\ \gamma-\text{node}} [\mu/\gamma_A]\cdot[\nu/\gamma_A] - [\mu/\gamma]\uparrow^{S_{d+1}}\,,$$

then $[\lambda]$ *appears in* $[\mu]\cdot[\nu]$*, more precisely*

$$d(\mu,\nu;\lambda) = \langle \chi, [\hat\lambda]\rangle \ .$$

(ii) Assume $[\mu/\gamma]=[\alpha]$ *with* $\alpha=(a^b)$, $a,b>1$, $[\nu/\gamma]=[d-1,1]$*. Set*

$$\overline{\alpha}=(a+1,a^{b-2},a-1), \underline{\alpha}=(a^{b-1},a-1,1), \overline{\underline{\alpha}}=(a+1,a^{b-2},a-1,1)$$

and let B_0, B_1 *denote the top and bottom addable nodes for* α*.*

If $[\hat\lambda]$ *appears with positive coefficient in the virtual character*

$$\chi=\sum_A [\mu/\gamma_A]\cdot[\nu/\gamma_A] - ([\overline{\alpha}]+[\underline{\alpha}])\uparrow^{S_{d+1}}$$

then $[\lambda]$ *appears in* $[\mu]\cdot[\nu]$*, more precisely*

$$d(\mu,\nu;\lambda) = \langle \chi, [\hat\lambda]\rangle \ .$$

Furthermore,

$$\varepsilon=\sum_{\eta\in\tilde{Y}(\lambda)} d(\mu,\nu;\eta)=\begin{cases} 2 & \text{if } \hat\lambda=\overline{\underline{\alpha}} \\ 1 & \text{if } \hat\lambda=\overline{\alpha}^B, \text{for some addable } B\neq B_1 \\ & \text{or } \hat\lambda=\underline{\alpha}^B, \text{for some addable } B\neq B_0 \\ 0 & \text{otherwise.}\end{cases}$$

Proof. By Lemma 5.2, in both cases $\lambda_2\le m-1$, so λ is a partition and we can then use Lemma 5.1. Hence, as $\lambda_1=m-1$ we obtain from Theorem 2.2 and Lemma 5.1:

$$\begin{aligned} d(\mu,\nu;\lambda) &= \sum_{A\,\gamma\text{-node}} \langle [\mu/\gamma_A]\cdot[\nu/\gamma_A],[\hat\lambda]\rangle - \varepsilon \\ &= \sum_{A\,\gamma\text{-node}} \langle [\mu/\gamma_A]\cdot[\nu/\gamma_A],[\hat\lambda]\rangle - \langle ([\mu/\gamma]\cdot[\nu/\gamma])\uparrow^{S_{d+1}},[\hat\lambda]\rangle \ . \end{aligned}$$

When ν/γ is a row, $[\mu/\gamma]\cdot[\nu/\gamma]=[\mu/\gamma]$, and we have the statement in (i). In Case (ii), $[\mu/\gamma]\cdot[\nu/\gamma]=[\alpha]\cdot[d-1,1]=[\underline{\alpha}]+[\overline{\alpha}]$. The assertion on ε can immediately be read off from this. □

We first want to get information on the number of (positive) components in the virtual character χ defined above when $[\mu/\gamma]\cdot[\nu/\gamma]$ is homogeneous. Using the results in Section 3 to set aside the cases excluded in Hypothesis $(*)$, this leads to the situation in the following crucial result.

Proposition 5.4. *Assume Hypothesis* $(*)$*. Assume* $[\mu/\gamma]=[\alpha]$ *is irreducible and* ν/γ *is a row. If there exists a removable* γ*-node* A_0 *disconnected from* ν/γ *then*

$$\chi=\sum_{A\ \gamma-\text{node}} [\mu/\gamma_A]\cdot[\nu/\gamma_A] - [\mu/\gamma]\uparrow^{S_{d+1}}$$

is a character and one of the following holds.

(1) $c(\chi)\ge 4$*.*

(2) $c(\chi) = 3$ *and one of the following holds:*

(a) $d = 2$*, and we are not in one of the cases in (3).*

(b) $d = 2k$ *for some* $k \in \mathbb{N}$*,* $k > 1$ *and we have one of the following:*
$\mu = ((a+k)^2)$*,* $\nu = (a^2, 2k)$ *for some* $a > d$*,*
$\chi = [k+2, k-1] + [k+1, k] + [k+1, k-1, 1]$*,*
or $\mu = (2^{a+k})$*,* $\nu = (2k+2, 2^{a-1})$ *for some* $a > 1$*,*
$\chi = [2^k, 1] + [3, 2^{k-2}, 1^2] + [2^{k-1}, 1^3]$*,*
or $\mu = ((k+1)^3)$*,* $\nu = (3k+1, 2)$*,*
$\chi = [k+1, k] + [k, k, 1] + [k+1, k-1, 1]$*,*
or $\mu = (2^{k+2})$*,* $\nu = (2k+2, 1^2)$*,*
$\chi = [3, 2^{k-1}] + [2^k, 1] + [2^{k-1}, 1^3]$*.*

(c) $d > 2$ *and we have one of the following:*
$\mu = ((2d)^{a+1})$*,* $\nu = ((2d)^a, d^2)$ *for some* $a \in \mathbb{N}$*,*
$\chi = 2[d, 1] + [d-1, 1^2] + [d-1, 2]$*,*
or $\mu = ((d+a+1)^d)$*,* $\nu = ((d+a)^d, d)$ *for some* $a \in \mathbb{N}$*,*
$\chi = [2, 1^{d-1}] + [2^2, 1^{d-3}] + [3, 1^{d-2}]$*,*
or $\mu = (d^{a+2})$*,* $\nu = (2d, d^a)$ *for some* $a \in \mathbb{N}$*,*
$\chi = [d, 1] + [d-1, 2] + [d-1, 1^2]$*.*

(3) $c(\chi) = 2$*, and* $d = 2$ *and* $\mu = (4^{a+1})$*,* $\nu = (4^a, 2^2)$*,* $\chi = 2[2, 1] + [1^3]$*,*
or $\mu = (2^{a+2})$*,* $\nu = (4, 2^a)$*,* $\chi = [2, 1] + [1^3]$*,*
or $\mu = ((a+3)^2)$*,* $\nu = ((a+2)^2, 2)$ *for some* $a \in \mathbb{N}$*,* $\chi = [2, 1] + [3]$*.*

Furthermore, any constituent $[\theta]$ *of* χ *gives a constituent* $[m-1, \theta]$ *in* $[\mu] \cdot [\nu]$*.*

Remark. The case $c(\chi) = 2$ may also occur when $n-m = 1$, or $\chi = [n-m+1]+[n-m, 1]$ and μ, ν are both hooks or both 2-part partitions. But we had explicitly assumed in $(*)$ that $n - m \geq 2$ and that μ, ν are not both hooks or both 2-part partitions.

Proof. We have already proved in Lemma 5.3 that every constituent appearing with positive coefficient in χ gives a constituent in $[\mu] \cdot [\nu]$.

Let A_0 be a removable γ-node, disconnected from ν/γ. By assumption, we have

$$[\nu/\gamma] = [n-m] \ , \ [\nu/\gamma_{A_0}] = [n-m+1] + [n-m, 1] \, .$$

Case 1. A_0 is disconnected from μ/γ.
Then

$$[\mu/\gamma_{A_0}] = [\mu/\gamma] \uparrow^{S_{n-m+1}} = [\alpha] \uparrow^{S_{n-m+1}} \ .$$

Consider

$$\chi_0 = [\mu/\gamma_{A_0}] \cdot [\nu/\gamma_{A_0}] - [\mu/\gamma] \uparrow^{S_{n-m+1}} = [\mu/\gamma_{A_0}] \cdot [n-m, 1] = \sum_{B \ \alpha\text{-node}} [\alpha^B] \cdot [n-m, 1] \, .$$

We may already note at this point that χ is then a character. Since $n-m \geq 2$, this character is not homogeneous; in fact, since one of the partitions α^B is not a rectangle, χ_0 has more than two components, by Corollary 4.2. It has three components exactly if $n-m=2$; in this case, both for $\alpha=(2)$ and $\alpha=(1^2)$ we get

$$\chi_0 = [3] + 2[2,1] + [1^3]\,.$$

Thus, χ_0 and hence

$$\chi = \chi_0 + \sum_{\substack{A \ \gamma-\text{node} \\ A \neq A_0}} [\mu/\gamma_A] \cdot [\nu/\gamma_A]$$

has at least four components, when $n-m>2$, and three components when $n-m=2$. Thus we are in situation (1) of the proposition for $n-m>2$ and in situation (2) for $n-m=2$.

Having dealt with Case 1, we may now assume that we are in the following situation:

Case 2. Every removable γ-node disconnected from ν/γ is connected to μ/γ.

Then $[\mu/\gamma_{A_0}]$ has a constituent $[\alpha^{B_1}]$ for some addable node B_1 of α by Lemma 3.4. Thus $[\mu/\gamma_{A_0}] \cdot [\nu/\gamma_{A_0}]$ contains $[\alpha^{B_1}] \cdot ([n-m+1] + [n-m,1])$, and thus it contains $[\alpha]\uparrow^{S_{n-m+1}}$. Hence χ_0 (as defined above) is a character and we get constituents in the character χ from the character

$$\chi' = \sum_{\substack{A \ \gamma-\text{node} \\ A \neq A_0}} [\mu/\gamma_A] \cdot [\nu/\gamma_A]\,.$$

Case 2.1. Assume that there is a further γ-node $A_1 \neq A_0$ that is disconnected from ν/γ.

Then as above $\chi_1 = [\mu/\gamma_{A_1}] \cdot [\nu/\gamma_{A_1}]$ (and hence χ) contains $[\alpha^{B_2}] \cdot ([n-m+1]+[n-m,1])$, for some addable node $B_2 \neq B_1$, and hence $[\alpha]\uparrow^{S_{n-m+1}} = \sum_B [\alpha^B]$. This latter character is never homogeneous; it has two components exactly when α is a rectangle and three components exactly when α is a *fat hook*, i.e., it has exactly two different part sizes. Otherwise we already get four components and thus $c(\chi) \geq 4$.

Now when α is a nontrivial rectangle, then α^{B_2} is not a rectangle, and thus $[\alpha^{B_2}] \cdot ([n-m+1] + [n-m,1])$ (and hence χ) has at least four components by Corollary 4.2 (note that $|\alpha| \geq 4$).

When α is a trivial rectangle, we may still assume that α^{B_2} is not a rectangle, by interchanging A_0 and A_1, if necessary. Then $[\alpha^{B_2}] \cdot ([n-m+1] + [n-m,1])$ (and hence χ) has again at least four components except if $n-m=2$, when it has three components and then also $c(\chi)=3$.

Case 2.2. There is no removable γ-node $\neq A_0$ disconnected from ν/γ, but there is a removable γ-node A_1 connected to ν/γ that is disconnected from μ/γ.

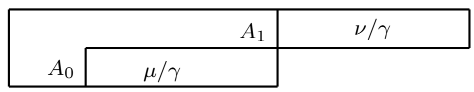
In this situation $[\nu/\gamma_{A_1}]$ is one of $[n-m+1]$ or $[n-m,1]$, and $[\mu/\gamma_{A_1}] = [\alpha]\uparrow^{S_{n-m+1}}$.

Clearly, $\chi_1 = [\mu/\gamma_{A_1}] \cdot [\nu/\gamma_{A_1}]$ is not homogeneous. If $c(\chi_1) \geq 4$, we are done. Now $c(\chi_1) = 3$ if and only if $n-m=2$ and $[\nu/\gamma_{A_1}] = [2,1]$, or α is a fat hook and $[\nu/\gamma_{A_1}] = [n-m+1]$; in the first case, we are again done. If $\mu/\gamma = \alpha$ is a fat hook, $[\mu/\gamma_{A_0}]$ has a further constituent $[\alpha^{B_2}]$, $B_2 \neq B_1$, and $c(\chi_0) \geq 4$. If μ/γ is a rotated fat hook, A_0 must be its inner addable node. As ν is not equal or conjugate to $(n-1,1)$, χ_0 has a constituent not appearing in

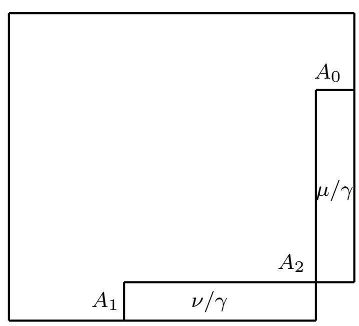
$\chi_1 = [\alpha]\uparrow^{S_{n-m+1}}$, and hence $\chi_0 + \chi_1$ has at least four components. Finally, $c(\chi_1) = 2$ if and only if $[\nu/\gamma_{A_1}] = [n-m+1]$ and α is a rectangle; in this case, $[\mu/\gamma_{A_0}] = [\alpha^{B_1}]$ and χ contains

$$\chi_0 + \chi_1 = [\alpha^{B_1}] \cdot ([n-m+1] + [n-m,1])\,.$$

If α^{B_1} is not a rectangle, then this has at least four components and thus $c(\chi) \geq 4$, except when $n-m=2$, when we get $c(\chi) = 3$. If α^{B_1} is a rectangle, it must be a row or column. If it is a row, then μ/γ is a row, and the roles of μ, ν can be interchanged. Thus we may assume that there is no further γ-node $\neq A_1$ disconnected from μ/γ. But then we are in the situation where μ, ν are both 2-part partitions:



If α^{B_1} is a column and there is a further γ-node $\neq A_1$ disconnected from μ/γ, then we conjugate the partitions and use the previous arguments to obtain at least four components in χ, when $n-m > 2$, and three when $n-m=2$. If there is no further γ-node $\neq A_1$ disconnected from μ/γ, then μ, ν are both hooks, or we are in the following situation:



In this latter case, we get a further contribution to χ from A_2:

$$\chi_2 = [\mu/\gamma_{A_2}] \cdot [\nu/\gamma_{A_2}] = [n-m,1] \cdot [n-m,1]$$

and thus $\chi = \chi_0 + \chi_1 + \chi_2$ has at least four components, except when $n-m=2$, where $c(\chi) = 3$.

We now have to consider

Case 2.3. All removable γ-nodes $\neq A_0$ are connected with ν/γ, and all removable γ-nodes are connected with μ/γ.

Since $[\mu/\gamma]$ is irreducible, μ/γ is a partition or rotated partition.

Case 2.3.1 In the first case where μ/γ is a partition we have by Lemma 3.4

$$[\mu/\gamma_{A_0}] = \sum_{B \neq B_0} [\alpha^B]$$

where B_0 is the top or bottom addable node of α. Then

$$\begin{aligned}\chi_0 &= [\mu/\gamma_{A_0}]\cdot[\nu/\gamma_{A_0}] - [\mu/\gamma]\uparrow^{S_{d+1}} = \left(\sum_{B\neq B_0}[\alpha^B]\right)([d+1]+[d,1]) - \sum_B [\alpha^B] \\ &= [d,1]\cdot\left(\sum_{B\neq B_0}[\alpha^B]\right) - [\alpha^{B_0}]\,.\end{aligned}$$

As $d\geq 2$, α has a further addable node $B_1\neq B_0$; if α is not a row or column, we may choose B_1 such that α^{B_1} is not a rectangle. Let r be the number of removable nodes of α^{B_1}. Then by Lemma 4.1 we obtain

$$\chi_0 = (r-1)[\alpha^{B_1}] + ([n-m+1]+[n-m,1])(\sum_{B\neq B_0,B_1}[\alpha^B]) + \sum_{\substack{C\neq B_1\\ D\neq C}} [(\alpha^{B_1})_C{}^D]\,.$$

We note that χ is thus a character, and we consider the contributions coming from the character

$$\chi_0' = ([n-m+1]+[n-m,1])(\sum_{B\neq B_0,B_1}[\alpha^B])\,.$$

If α is not a fat hook or rectangle, then α has two further addable nodes $B_2,B_3\neq B_0,B_1$. Then χ_0' contains $[\alpha^{B_0}]+[\alpha^{B_1}]+[\alpha^{B_2}]+[\alpha^{B_3}]$, and thus $c(\chi)\geq 4$.

If α is a fat hook, then it has a further addable node $B_2\neq B_0,B_1$. Then χ_0' contains $[\alpha^{B_0}]+[\alpha^{B_1}]+[\alpha^{B_2}]$, and

$$\chi_0'' = \sum_{\substack{C\neq B_1\\ D\neq C}} [(\alpha^{B_1})_C{}^D]$$

contributes a further constituent as α^{B_1} has a removable node $C\neq B_1$, and there is then a suitable $D\neq C$ with $(\alpha^{B_1})_C{}^D\neq\alpha^{B_i}$, for $i=0,1,2$.

Now assume that α is a nontrivial rectangle, with corner node Z; this is then the only removable node $\neq B_1$ of α^{B_1}. Then

$$\chi_0 = \sum_D [(\alpha^{B_1})_Z{}^D]$$

has at least four different constituents, except if α is a 2-row rectangle and B_1 is the top node, or α is a 2-column rectangle and B_1 is the bottom node. In these exceptional cases, if B_1' is the top or bottom node of α^{B_1}, respectively, then $\chi_0=[\alpha^{B_1}]+[(\alpha^{B_1})_Z^{B_0}]+[(\alpha^{B_1})_Z^{B_1'}]$ has exactly three components.

If there exists a removable γ-node $A_1\neq A_0$, then this must be connected to both μ/γ and ν/γ, and we have $[\mu/\gamma_{A_1}]=[\alpha^{B_0}]$ and $[\nu/\gamma_{A_1}]$ is $[n-m+1]$ or $[n-m,1]$. In both cases, we find as a fourth new constituent $[\alpha^{B_0}]$ in χ.

Now it remains to consider the situation when there is no such A_1, i.e., γ is a rectangle, and we are in one of the exceptional cases where $d=2k$ for some $k>1$ and α is a 2-line rectangle. Because of the additional condition on B_1, we then have one of the following situations:

(i) $\gamma = (2^a)$, $a \geq 2$, $\mu = (2^{a+k})$, $\nu = (2k+2, 2^{a-1})$.

(ii) $\gamma = (a^2)$, $a > d$, $\mu = ((a+k)^2)$, $\nu = (a^2, 2k)$.

Here, $c(\chi) = 3$, and these situations appear in part (2)(b) of the proposition.

Finally, we have to deal with the case where α is a row or column.

First let α be a row. Assume that there is a removable γ-node $A_1 \neq A_0$. As μ, ν are not both 2-part partitions, then μ/γ_{A_0} cannot be a row, but μ/γ_{A_1} is a row, and hence we obtain

$$\begin{aligned}\chi &= [d,1]\cdot([d+1]+[d,1]) + [d+1]\cdot[d,1] - [d+1] - [d,1] \\ &= \begin{cases} 2[d,1]+[d-1,2]+[d-1,1^2] & \text{if } d>2 \\ 2[2,1]+[1^3] & \text{if } d=2 \end{cases}.\end{aligned}$$

We have here $\mu = ((2d)^{a+1})$, $\nu = ((2d)^a, d^2)$ for some $a \in \mathbb{N}$, cases described in the proposition.

If there is no such γ-node A_1, then γ is a rectangle. Since μ, ν are not both 2-part partitions, we must then have $\mu = (d^{a+2})$, $\nu = (2d, d^a)$ for some $a \in \mathbb{N}$, and we have

$$\begin{aligned}\chi &= [d,1]\cdot([d+1]+[d,1]) - [d+1] - [d,1] \\ &= \begin{cases} [d,1]+[d-1,2]+[d-1,1^2] & \text{if } d>2 \\ [2,1]+[1^3] & \text{if } d=2 \end{cases},\end{aligned}$$

cases appearing in the proposition.

Now let α be a column. Assume that there is a removable γ-node $A_1 \neq A_0$. If μ/γ_{A_0} is a column, we obtain

$$\begin{aligned}\chi &= [1^{d+1}]\cdot([d+1]+[d,1]) + [d,1]\cdot[2,1^{d-1}] - [1^{d+1}] - [2,1^{d-1}] \\ &= [d,1]\cdot[2,1^{d-1}] \\ &= \begin{cases} [1^{d+1}]+[2,1^{d-1}]+[2^2,1^{d-3}]+[3,1^{d-2}] & \text{if } d>2 \\ [3]+[2,1]+[1^3] & \text{if } d=2 \end{cases}.\end{aligned}$$

This fits with the cases (1) and (2) in the proposition.

If μ/γ_{A_0} is not a column, then $\mu = ((a+1)^{d+1})$, $\nu = (a+d+1, a^d)$ and we obtain

$$\begin{aligned}\chi &= [2,1^{d-1}]\cdot([d+1]+[d,1]) + [1^{d+1}] - [1^{d+1}] - [2,1^{d-1}] \\ &= \begin{cases} [1^{d+1}]+[2,1^{d-1}]+[2^2,1^{d-3}]+[3,1^{d-2}] & \text{if } d>2 \\ [3]+[2,1]+[1^3] & \text{if } d=2 \end{cases}.\end{aligned}$$

Again, this is in accordance with (1) and (2) of the proposition.

When there is no removable γ-node $A_1 \neq A_0$, then, since μ, ν are not both hooks, we have $\mu = ((d+a+1)^d)$, $\nu = ((d+a)^d, d)$, for some $a \in \mathbb{N}$, and

$$\begin{aligned} \chi &= [2,1^{d-1}] \cdot ([d+1]+[d,1]) - [1^{d+1}] - [2,1^{d-1}] \\ &= \begin{cases} [2,1^{d-1}]+[2^2,1^{d-3}]+[3,1^{d-2}] & \text{if } d>2 \\ [2,1]+[3] & \text{if } d=2 \end{cases} . \end{aligned}$$

Again, these cases appear as exceptional situations in (2) and (3) of the proposition.

Case 2.3.2. It remains to treat the case where μ/γ is a rotated partition which is not a partition.

Since only the removable γ-node A_0 is disconnected from ν/γ, μ/γ can only be a fat hook α, and A_0 is the middle addable node B_1 (say) for α. Then

$$\begin{aligned} \chi_0 &= [\mu/\gamma_{A_0}] \cdot [\nu/\gamma_{A_0}] - [\mu/\gamma] \uparrow^{S_{n-m+1}} = [\alpha^{B_1}] \downarrow_{S_d} \uparrow^{S_{d+1}} - [\alpha] \uparrow^{S_{d+1}} \\ &= \sum_{\substack{B \neq B_1 \\ D}} [(\alpha^{B_1})_B{}^D] , \end{aligned}$$

where B runs over the removable α^{B_1}-nodes and D over the addable $(\alpha^{B_1})_B$-nodes; in particular, we see here again that χ is a character. There has to be exactly one further removable γ-node A_1, which corresponds to the top or bottom addable node B_0 or B_2 of α, respectively; in these two cases we obtain as the second contribution to χ:

$$\chi_1 = [\mu/\gamma_{A_1}] \cdot [\nu/\gamma_{A_1}] = \begin{cases} [\alpha^{B_0}] \cdot [d,1] = [\alpha^{B_1}] + [\alpha^{B_2}] + \sum_{\substack{B \neq B_0 \\ D}} [(\alpha^{B_0})_B{}^D] \\ \\ [\alpha^{B_2}] \end{cases} .$$

If α^{B_1} has three removable nodes B_1, X, Y, then

$$\sum_{D \neq X} [(\alpha^{B_1})_X{}^D] + \sum_{D \neq Y} [(\alpha^{B_1})_Y{}^D]$$

already gives at least four different constituents in χ_0.

Now assume that α^{B_1} has only two removable nodes, B_1 and either the top removable node X or the bottom removable node Y of α. Assume first that the top node X is removable. If $(\alpha^{B_1})_X$ has four addable nodes, then we have already four different constituents in χ_0. If $(\alpha^{B_1})_X$ has three addable nodes, then we have three different constituents from χ_0 and a further fourth constituent $[\alpha^{B_2}]$ from χ_1 for χ. If $(\alpha^{B_1})_X$ has only two addable nodes, then besides two constituents from χ_0 we get at least two further constituents from χ_1 when $[\mu/\gamma_{A_1}] = [\alpha^{B_0}]$. When $[\mu/\gamma_{A_1}] = [\alpha^{B_2}] = \chi_1$, χ has only three components; in this situation we have $d = 2k$, $\mu = ((k+1)^3)$, $\nu = (3k+1, 2)$ for some $k \in \mathbb{N}$, $k > 1$, and $\chi = [k+1,k] + [k,k,1] + [k+1,k-1,1]$.

If the bottom node Y is the second removable node of α^{B_1}, then we can argue analogously; this gives a further situation where χ has three components, namely for $d = 2k$, $\mu = (2^{k+2})$, $\nu = (2k+2, 1^2)$, for some $k \in \mathbb{N}$, $k > 1$; here $\chi = [3, 2^{k-1}] + [2^k, 1] + [2^{k-1}, 1^3]$.

Finally, we assume that α^{B_1} has only the removable node B_1. In this case $\chi_0 = 0$. As by assumption $(*)$, $\nu \neq (n-1,1)$, the situation $\chi_1 = [\mu/\gamma_{A_1}] = [\alpha^{B_2}]$ cannot occur. Let B_0' be the top addable node of α^{B_0}. Then $\chi = \chi_1$ has at least the four different constituents $[\alpha^{B_i}]$, $i = 0,1,2$, and $[(\alpha^{B_0})_Y{}^{B_0'}]$. □

Now we want to deal with the second case in Lemma 5.2. We know that this only occurs when one of the skew characters corresponds to a nontrivial rectangle, and the other one is $[d-1,1]$ or $[2,1^{d-2}]$; conjugating, if necessary, we may assume that we are in the first situation.

Proposition 5.5. *Assume Hypothesis* $(*)$.
Assume $[\mu/\gamma] = [\alpha]$ *with* $\alpha = (a^b)$, $a, b > 1$, $[\nu/\gamma] = [d-1,1]$ *and*

$$\chi = \sum_{A\ \gamma-\text{node}} [\mu/\gamma_A] \cdot [\nu/\gamma_A] - ([\mu/\gamma]\cdot[\nu/\gamma]) \uparrow^{S_{d+1}} .$$

Then χ *is a character with* $c(\chi) \geq 5$.

Furthermore, any constituent $[\theta]$ *of* χ *gives a constituent* $[m-1,\theta]$ *in* $[\mu]\cdot[\nu]$.

Proof. Let B_0, B_1 denote the top and bottom addable nodes for α; let X be the removable α-node.

Let A be a removable γ-node. Then $[\mu/\gamma_A]$ contains a constituent $[\alpha^B]$, for $B = B_0$ or $B = B_1$ (for both if A is disconnected from μ/γ); let $\bar{B}$ be the other addable node for α. If A is disconnected from ν/γ or if ν/γ is a partition diagram, then $[\nu/\gamma_A]$ contains $[d-1,1^2]+[d-1,2]$ or $[d-1,2]+[d,1]$. If ν/γ is a rotated partition and there is no γ-node disconnected from ν/γ, then we have at least two γ-nodes A_0 and A_1 connected to ν/γ giving us a contribution $[\alpha^B]\cdot([d-1,1^2]+[d-1,2])$ or $[\alpha^B]\cdot([d-1,2]+[d,1])$ to the sum in χ.

Thus we will now investigate the expressions

$$\chi' = [\alpha^B]\cdot([d-1,1^2]+[d-1,2]) - ([\alpha]\cdot[d-1,1]) \uparrow^{S_{d+1}}$$

and

$$\chi' = [\alpha^B]\cdot([d-1,2]+[d,1]) - ([\alpha]\cdot[d-1,1]) \uparrow^{S_{d+1}} ,$$

respectively. If then χ' is a character, so is χ, and $c(\chi') \leq c(\chi)$.
In the following, instead of $\uparrow^{S_{d+1}}$ and similar inductions one step up, we will just write $\uparrow$.

In the first case we use the relation $[d-1,1^2]+[d-1,2] = [d-1,1]\uparrow -[d,1]$:

$$\begin{aligned}
\chi' &= [\alpha^B]\cdot([d-1,1^2]+[d-1,2]) - ([\alpha]\cdot[d-1,1])\uparrow \\
&= ([(\alpha_X)^B]\cdot[d-1,1])\uparrow -[\alpha^B]\cdot[d,1] \\
&= (r-1)[(\alpha_X)^B]\uparrow + \sum_C \sum_{D\neq C} [(((\alpha_X)^B)_C)^D]\uparrow -[\alpha]\uparrow -[(\alpha_X)^B]\uparrow +[\alpha^B]
\end{aligned}$$

where r is the number of removable nodes of $(\alpha_X)^B$. As $r \geq 2$, χ' contains the subcharacter

$$\sum_{C\neq B}\sum_{D\neq C}[(((\alpha_X)^B)_C)^D]\uparrow + \sum_{D\neq B,X} [(\alpha_X)^D]\uparrow +[\alpha^B] .$$

Now $(\alpha_X)^B$ has at least one removable node $Y \neq B$, and α_X has the addable node $\bar{B} \neq B, X$; hence the character above contains

$$\sum_{D\neq Y} [((\alpha_X)^B)_Y)^D] \uparrow + [(\alpha_X)^{\bar{B}}] \uparrow + [\alpha^B] \, ,$$

which has at least five components. Thus in this case we have $c(\chi) \geq 5$.

Now we look at the second case. Since we have already dealt with the previous case we may here assume that $B = B_1$. When ν/γ is a rotated partition, one of the two γ-nodes connected to ν/γ is not connected to μ/γ, so that from these nodes we get the subcharacter

$$\begin{aligned} \chi' &= [\alpha] \uparrow \cdot [d-1,2] + [\alpha^B] \cdot [d,1] - ([\alpha] \cdot [d-1,1]) \uparrow \\ &= ([\alpha] \cdot [d-2,2]) \uparrow + [\alpha^B] \cdot [d,1] \end{aligned}$$

of χ. The second summand has at least four constituents, all of width $\leq a+1$, and the first summand has one of width $a+3$, so χ is a character with at least five constituents in this case. Now it only remains to consider the case where $\mu = (a^{b+2})$ and $\nu = (a+d-1, a+1)$. Since μ, ν are not both 2-line partitions, we have $a > 2$. We now want to show that the following is a subcharacter in χ with at least five components:

$$\begin{aligned} \chi' &= [\alpha^B] \cdot ([d-1,2] + [d,1]) - ([\alpha] \cdot [d-1,1]) \uparrow \\ &= [\alpha^B] \cdot [d-1,2] + [\alpha^B] \cdot [d] \uparrow - [\alpha^B] - ([(\alpha_X)^B] + [(\alpha_X)^{\bar{B}}]) \uparrow \\ &= [\alpha^B] \cdot [d-1,2] + [\alpha] \uparrow - [\alpha^B] - [(\alpha_X)^{\bar{B}}] \uparrow \\ &= [\alpha^B] \cdot [d-1,2] + [\alpha^{\bar{B}}] - [(\alpha_X)^{\bar{B}}] \uparrow \\ &= [\alpha^B] \cdot [d-1,2] - \textstyle\sum_{C\neq X} [((\alpha_X)^{\bar{B}})^C] \, . \end{aligned}$$

Note that the sum that is subtracted above has at most three terms, namely $[a+2, a^{b-2}, a-1]$ and $[a+1, a^{b-2}, a-1, 1]$, and for $b \geq 3$ also $[(a+1)^2, a^{b-3}, a-1]$.

We now investigate the product $[\alpha^B] \cdot [d-1,2]$. As $B = B_1$, $\alpha^B \cap (d-1,2) = (a,2) = \tau$ and thus

$$[\alpha^B/\tau] \cdot [(d-1,2)/\tau] = [(a^{b-2},1)/(2)] = [a^{b-2}, a-2, 1] + [a^{b-2}, a-1] \, ,$$

giving the components $[a+2, a^{b-2}, a-2, 1]$, $[a+2, a^{b-2}, a-1]$ in $[\alpha^B] \cdot [d-1,2]$.
We compute the terms of width $a+1$ in $[\alpha^B] \cdot [d-1,2]$ to see that χ' is a character. Since $[(d-1,2)/\tau]$ is a row, we can use Lemma 5.3, so we now compute the constituents of

$$\begin{aligned} \psi &= [(a^b,1)/(a-1,2)] + [(a^{b-1},1)/(1)] \cdot [d-2-a] \uparrow - ([(a^{b-2}, a-2, 1)] + [a^{b-2}, a-1]) \uparrow \\ &= [(a^b,1)/(a-1,2)] + [(a^{b-1},1)/(1)] \downarrow\uparrow - ([(a^{b-2}, a-2, 1)] + [a^{b-2}, a-1]) \uparrow \ . \end{aligned}$$

Now for the first term in ψ we have for $a \geq 4$ (see [6])

$$[(a^b,1)/(a-1,2)] = [(a^{b-1}, a-2, 1)/(a-1)] = [a^{b-2}, a-2, 1^2] + [a^{b-2}, a-2, 2] + [a^{b-2}, a-1, 1]$$

while for $a = 3$ the second summand does not appear, i.e.,

$$[(3^b, 1)/(2^2)] = [(3^{b-1}, 1^2)/(2)] = [3^{b-2}, 1^3] + [3^{b-2}, 2, 1] \,.$$

For the second term in ψ, we first get

$$[(a^{b-1}, 1)/(1)] = [a^{b-2}, a-1, 1] + [a^{b-1}] \,.$$

Now we notice that the restriction of the first summand already contains the two constituents in the third term subtracted in the expression for ψ. Then from the second and third term in ψ together we obtain the contribution

$$([a^{b-3}, (a-1)^2, 1] + [a^{b-2}, a-1]) \uparrow ,$$

where the first constituent only appears for $b \geq 3$.

Hence ψ is a character, and taking into regard the contribution from the first term, it contains for all $a \geq 3$, $b \geq 2$ the character

$$\psi' \;=\; [a^{b-2}, a-2, 1^2] + 2[a^{b-2}, a-1, 1] + [a+1, a^{b-3}, a-1] + [a^{b-2}, a] \,.$$

All these constituents in ψ' give constituents of width $a+1$ in $[\alpha^B] \cdot [d-1, 2]$, and thus the subtracted terms in the expression for χ' are all taken care of, i.e., χ' is a character and it contains the character

$$\chi'' = [a+2, a^{b-2}, a-2, 1] + [a+1, a^{b-2}, a-2, 1^2] + [a+1, a^{b-2}, a-1, 1] + [a+1, a^{b-2}, a] \,.$$

Furthermore, as $d = ab \geq b+2$ and hence $d-1 \geq b+1$, $\alpha^B \cap (2^2, 1^{d-3}) = (2^2, 1^{b-1}) = \rho$ and thus

$$[\alpha^B/\rho] \cdot [(2^2, 1^{d-3})/\rho] = [(a-1)^{b-2}, (a-2)^2] \cdot [1^{d-b-2}] = [b^{a-2}, b-2]$$

producing the only component of maximal length $b+3$ in $[\alpha^B] \cdot [d-1, 2]$: $[a^{b-2}, (a-1)^2, 1^3]$, then also appearing in χ'. Thus we have proved that χ is a character and $c(\chi) \geq c(\chi') \geq 5$. □

6. Proof of the classification theorem

We are now in a position to prove Theorem 1.1, confirming the conjectured classification of Kronecker products with only three or four homogeneous components. We recall the classification we want to prove below; of course, we know that all the products appearing in the statements below are correct. For the proof that the classification is complete we will need an involved analysis based on the results of the previous sections.

Theorem 6.1. *Let μ, $\nu \vdash n$.*

(i) We have $c([\mu] \cdot [\nu]) = 3$ if and only if $n = 3$ and $\mu = \nu = (2, 1)$ or $n = 4$ and $\mu = \nu = (2, 2)$.

The product is then one of

$$[2, 1]^2 = [3] + [2, 1] + [1^3] \,,\; [2^2]^2 = [4] + [2^2] + [1^4] \,.$$

(ii) We have $c([\mu] \cdot [\nu]) = 4$ if and only if one of the following holds:

(1) $n \geq 4$ *and* $\mu, \nu \in \{(n-1,1), (2,1^{n-2})\}$*; here the products are*

$$\begin{aligned} [n-1,1]^2 = [2,1^{n-2}]^2 &= [n] + [n-1,1] + [n-2,2] + [n-2,1^2] \\ [n-1,1] \cdot [2,1^{n-2}] &= [1^n] + [2,1^{n-2}] + [2^2,1^{n-4}] + [3,1^{n-3}] \ . \end{aligned}$$

(2) $n = 2k+1$ *for some* $k \geq 2$*, and one of* μ*,* ν *is in* $\{(2k,1), (2,1^{2k-1})\}$ *while the other one is in* $\{(k+1,k), (2^k,1)\}$*; here the products are*

$$\begin{aligned} [2k,1] \cdot [k+1,k] &= [k+2,k-1] + [k+1,k] + [k+1,k-1,1] + [k^2,1] \\ [2,1^{2k-1}] \cdot [k+1,k] &= [2^{k-1},1^3] + [2^k,1] + [3,2^{k-2},1^2] + [3,2^{k-1}] \, . \end{aligned}$$

(3) $n = 6$ *and* $\mu, \nu \in \{(2^3), (3^2)\}$*; here we have*

$$[3^2]^2 = [6] + [4,2] + [3,1^3] + [2^3] \, , \ [3^2] \cdot [2^3] = [1^6] + [2^2,1^2] + [4,1^2] + [3^2] \, .$$

Proof. Proposition 4.3 has already dealt with squares; thus we may now assume that $\mu \neq \nu$; by conjugating, we may also assume that $\mu \neq \nu'$. We keep the notation used in earlier sections, i.e., we set $\gamma = \mu \cap \nu \vdash m$, $d = n - m$. We may also assume that Hypothesis $(*)$ from Section 5 is satisfied since we know the result for the cases excluded in $(*)$ by [1] and Proposition 4.4, Proposition 4.6 and Proposition 4.8. Note that for all the products $[\mu] \cdot [\nu]$ with three or four components on the classification list we have $\mu = \nu$ or $\mu = \nu'$ or one of the partitions is $(n-1,1)$ or $(2,1^{n-2})$. Thus we are now in the situation that we also assume $c([\mu] \cdot [\nu])$ is 3 or 4 and we want to reach a contradiction.

Thus we assume now all of the following properties:

- $\mu, \nu \notin \{(n), (1^n), (n-1,1), (2,1^{n-2})\}$.
- $\mu \neq \nu$, $\mu \neq \nu'$.
- $d \geq 2$.
- μ, ν are not both hooks, and not both 2-line partitions.

Our aim is to show that the additional property

- $c([\mu] \cdot [\nu]) = 3$ or 4.

then leads to a contradiction.

We note at this point that already the general properties above imply that $|\mu \cap \nu'| \geq 4$. In particular, $[\mu] \cdot [\nu']$ has a component of width ≥ 4.

6.1. **One component of one extreme type.** We consider in the first case the situation that for one of the two extreme types (maximal width or maximal length) there is only one component in the product. Of course, when the product has only three components, this is always satisfied.

Replacing, if necessary, one of the partitions by its conjugate, we may assume that there is only one component of maximal width m. Then, by Corollary 2.3, we know that $[\mu/\gamma] \cdot [\nu/\gamma]$ must be homogeneous. By Corollary 3.7, both skew characters have to be irreducible, and one of them is of degree 1. Conjugating both partitions and renaming, if necessary, we may then assume that $[\mu/\gamma]$ is irreducible and that ν/γ is a row.

If there is a removable γ-node A_0 disconnected from ν/γ, we can use Proposition 5.4 to obtain constituents of (almost maximal) width $m-1$ in the product. Let

$$\chi = \sum_{A\ \gamma-\text{node}} [\mu/\gamma_A]\cdot[\nu/\gamma_A] - [\mu/\gamma]\uparrow^{S_{d+1}}$$

be as before. By Proposition 5.4, we obtain almost always at least four components of width $m-1$ in the product coming from constituents of χ unless we are in one of the exceptional cases described explicitly in Proposition 5.4. We now go through these in detail.

Assume first that $c(\chi)=3$.

First we consider the case $d=2$, where we have already found one of the constituents $[n-2,2]$ or $[n-2,1^2]$, and the constituents $[n-3,3]$, $[n-3,2,1]$, $[n-3,1^3]$ in the product. If $|\mu\cap\nu'|>4$, then we also have a constituent of length >4. Thus we may now assume that $|\mu\cap\nu'|=4$. In this case $(3,1^2)$ and (2^2) cannot be contained in γ, hence $\gamma=(2,1^{m-2})$ or $\gamma=(m-1,1)$. Hypothesis $(*)$ then gives a contradiction except for the case $\mu=(2^3)$, $\nu=(4,1^2)$ or the (doubly conjugate) case $\mu=(3,1^3)$, $\nu=(3^2)$. Since $[\mu/\mu\cap\nu']\cdot[\nu'/\mu\cap\nu']=[2]+[1^2]$, we get here a second constituent $[2^2,1^2]$ of length 4 in $[\mu]\cdot[\nu]$.

Next suppose $d=2k$ for some $k>1$. If $\mu=((a+k)^2)$, $\nu=(a^2,2k)$ for some $a>d$, then $\chi=[k+2,k-1]+[k+1,k]+[k+1,k-1,1]$, and we have four components (of width m and $m-1$) of length ≤ 4. Since $|\mu\cap\nu'|=6$, we must also have a component of length 6. Similarly, when $\mu=((k+1)^3)$, $\nu=(3k+1,2)$ for some $k\in\mathbb{N}$, $k>1$, we have $\chi=[k+1,k]+[k,k,1]+[k+1,k-1,1]$, and thus we have again four components of length ≤ 4 and a further component of length $|\mu\cap\nu'|=5$ in the product. If $\mu=(2^{a+k})$, $\nu=(2k+2,2^{a-1})$ for some $a>1$, then $\chi=[2^k,1]+[3,2^{k-2},1^2]+[2^{k-1},1^3]$, and we have four components of length $\leq k+3$ and a fifth component of length $|\mu\cap\nu'|\geq k+4$. When $\mu=(2^{k+2})$, $\nu=(2k+2,1^2)$, for some $k\in\mathbb{N}$, $k>1$, we have $\chi=[3,2^{k-1}]+[2^k,1]+[2^{k-1},1^3]$, giving three components of length $\leq k+2$ and the component $[3,2^{k-1},1^3]$ of length $k+3$. But here $|\mu\cap\nu'|=k+3$, so we have to look more closely at the components of maximal width in $[\mu]\cdot[\nu']$. Since

$$[\mu/(\mu\cap\nu')]\cdot[\nu/(\mu'\cap\nu)]=[1^{k+1}]+[2,1^{k-1}]$$

by Corollary 2.4, the product $[\mu]\cdot[\nu]$ has $[2^{k+1},1^2]$ and $[3,2^{k-1},1^3]$ as components of maximal length. Hence we have found five components in the product.

Now consider the cases for $d>2$ in Proposition 5.4(2)(c). When $\mu=((2d)^{a+1})$, $\nu=((2d)^a,d^2)$ for some $a\in\mathbb{N}$, we have $\chi=2[d,1]+[d-1,1^2]+[d-1,2]$, hence there are already four components of length ≤ 4, and because $|\mu\cap\nu'|\geq 6$, we also have a constituent of length ≥ 6. Similarly, if $\mu=(d^{a+2})$, $\nu=(2d,d^a)$ for some $a\in\mathbb{N}$, we have $\chi=[d,1]+[d-1,2]+[d-1,1^2]$. Thus we have four components of length ≤ 4, and a further component of length $|\mu\cap\nu'|\geq 6$. When $\mu=((d+a+1)^d)$, $\nu=((d+a)^d,d)$ for some $a\in\mathbb{N}$, we have $\chi=[2,1^{d-1}]+[2^2,1^{d-3}]+[3,1^{d-2}]$. Thus we have four components of length $\leq d+1$, and a further component of length $|\mu\cap\nu'|=d(d+1)>d+1$.

Next we consider the cases where $c(\chi)=2$. We know that this can only occur for $d=2$. First, consider the case $\mu=(4^{a+1})$, $\nu=(4^a,2^2)$ where $\chi=2[2,1]+[1^3]$, and thus $[\mu]\cdot[\nu]$ has three components $[4a+2,2]$, $[4a+1,2,1]$ and $[4a+1,1^3]$ of length ≤ 4. For $a=1$, μ' and ν satisfy the assumptions of Proposition 5.4, hence the product $[\mu']\cdot[\nu]$ has three components of width $\geq |\mu'\cap\nu|-1=5$, giving three components of length ≥ 5 in $[\mu]\cdot[\nu]$. For $a=2$, $\nu=\nu'$, hence $[\mu]\cdot[\nu]=[\mu]\cdot[\nu']$ has three components of length $\geq 4a+1=9$; similarly, for $a=3$, $\mu=\mu'$ and thus the product has three components of length $\geq 4a+1=13$. For $a=4$, the pair μ, ν' satisfies the assumption of Proposition 5.4 (with the row diagram $\mu/(\mu\cap\nu')$); hence $[\mu]\cdot[\nu']$ has three components of width $\geq |\mu\cap\nu'|-1=15$, giving three components of length ≥ 15 in $[\mu]\cdot[\nu]$. Finally, for $a>4$, $[\mu]\cdot[\nu']$ has at least two components of maximal width 16, hence $[\mu]\cdot[\nu]$ has corresponding components of length 16.

Now suppose $\mu=(2^{a+2})$, $\nu=(4,2^a)$; here $\chi=[2,1]+[1^3]$, so we have again three components $[2a+2,2]$, $[2a+1,2,1]$ and $[2a+1,1^3]$ of length ≤ 4 in the product. For $a=1$, we have a further component of length $|\mu\cap\nu'|=5$; also, we have a further component $[2^3]$ (see Proposition 4.3(5)). For $a=2$, we may use Proposition 5.4 to get three components of width ≥ 5 in $[\mu']\cdot[\nu]$, and hence three components of length ≥ 5 in $[\mu]\cdot[\nu]$. For $a\geq 3$, $[\mu']\cdot[\nu]$ has at least two components of maximal width 6, giving corresponding components of length 6 in $[\mu]\cdot[\nu]$.

Finally, consider the case $\mu=((a+3)^2)$, $\nu=((a+2)^2,2)$ for some $a\in\mathbb{N}$; here $\chi=[2,1]+[3]$, so we have three components $[2a+4,2]$, $[2a+3,2,1]$, $[2a+3,3]$ of length ≤ 3 in the product $[\mu]\cdot[\nu]$. For $a=1$, $[\mu]\cdot[\nu']$ has ≥ 3 components of width ≥ 5 by Proposition 5.4, hence $[\mu]\cdot[\nu]$ has ≥ 3 components of length ≥ 5. For $a>1$, $[\mu/(\mu\cap\nu')]\cdot[\nu'/(\mu\cap\nu')]=[a^2]^2$ has ≥ 3 components by Proposition 4.3, hence $[\mu]\cdot[\nu]$ has ≥ 3 components of length $|\mu\cap\nu'|=6$.

The critical situation to be discussed now is the one where all removable γ-nodes are connected to ν/γ. In this case, ν must be a rectangle, since ν/γ is a row; since $\nu\neq(n)$, γ must have a removable node A_0 such that $[\nu/\gamma_{A_0}]=[d,1]$.

Let us first assume that μ/γ is disconnected from A_0. Then

$$\begin{aligned}\chi_0 &= [\nu/\gamma_{A_0}]\cdot[\mu/\gamma_{A_0}]-\sum_B[\alpha^B]=[d,1]\cdot\Big(\sum_B[\alpha^B]\Big)-\sum_B[\alpha^B]\\ &= \sum_B\sum_C\sum_D[{(\alpha^B)_C}^D]-2\sum_B[\alpha^B]\\ &= \sum_B(r_B-2)[\alpha^B]+\sum_B\sum_C\sum_{D\neq C}[{(\alpha^B)_C}^D]\end{aligned}$$

where B runs over the addable nodes of α, C runs over the removable nodes of α^B (for the respective node B), D runs over the addable nodes of $(\alpha^B)_C$ and r_B denotes the number of removable nodes of α^B.

We know that α has at least two addable nodes, say B_0 at the top and B_1 at the bottom. Assume first that α is not a row or column. Then $r_{B_0},r_{B_1}\geq 2$. Let X_1 be the top and X_0 the bottom removable node of α (we may have $X_0=X_1$); then X_i is also a removable node for α^{B_i}, $i=0,1$. Let B_0' be the top addable node for $(\alpha^{B_0})_{X_0}$ and B_1' be the bottom addable node for $(\alpha^{B_1})_{X_1}$. We then have at least the following contribution to χ_0:

$$(r_{B_0}-1)[\alpha^{B_0}]+[{(\alpha^{B_0})_{X_0}}^{B_0'}]+(r_{B_1}-1)[\alpha^{B_1}]+[{(\alpha^{B_1})_{X_1}}^{B_1'}]\,.$$

Thus we have found at least four components in χ_0 and hence four components of width $m-1$ in the product.

If α is a row, we may interchange μ and ν, and then we are in the situation discussed in the first part of the proof. When α is a column, we conjugate and then interchange both partitions; again, this is dealt with by the first part of the proof.

Thus now we treat the situation where μ/γ is connected to A_0. Then by Lemma 3.4

$$[\mu/\gamma_{A_0}]=\sum_{B\neq B_1}[\alpha^B]$$

where B_1 is the bottom addable node of α. Let B_0 be the top addable node of α. Then

$$\begin{aligned}\chi_0 &= [\nu/\gamma_{A_0}]\cdot[\mu/\gamma_{A_0}]-\sum_B[\alpha^B]=[d,1]\cdot\sum_{B\neq B_1}[\alpha^B]-\sum_B[\alpha^B]\\ &= \sum_{B\neq B_1}\sum_C\sum_D[{(\alpha^B)_C}^D]-\sum_{B\neq B_1}[\alpha^B]-\sum_B[\alpha^B]\\ &= \sum_{B\neq B_0,B_1}\sum_C\sum_D[{(\alpha^B)_C}^D]+\sum_{C\neq B_0}\sum_D[{(\alpha^{B_0})_C}^D]-\sum_{B\neq B_1}[\alpha^B]\end{aligned}$$

where B runs through the addable nodes of α, C runs through the removable nodes of α^B (for the respective node B) and D runs through the addable nodes of $(\alpha^B)_C$.

If α is not a rectangle, there is a third addable node, say B_2, not in the first row or column. Taking this contribution into account, χ_0 is a character containing

$$\chi_0'=[\alpha^{B_0}]+\sum_{C\neq B_0}\sum_{D\neq C}[{(\alpha^{B_0})_C}^D]+\sum_{C\neq B_2}\sum_D[{(\alpha^{B_2})_C}^D]+[\alpha^{B_1}]\,.$$

If α^{B_2} is not a rectangle, then the top or bottom removable node of α will be $\neq B_2$ and will also be removable from α^{B_2}; let this α-node be X. Depending on X being at the top or bottom of α, the node $Y=B_1$ or $Y=B_0$ will be addable for $(\alpha^{B_2})_X$. Then χ_0' contains

$$\chi_0''=[\alpha^{B_0}]+[\alpha^{B_2}]+[{(\alpha^{B_2})_X}^Y]+[\alpha^{B_1}]$$

and hence we have at least four components in the product of width $m-1$.

We are now in the situation where α^{B_2} is a rectangle. The bottom removable node X_0 of α is also removable from α^{B_0}. The top addable node B_0' for α^{B_0} is also addable for $(\alpha^{B_0})_{X_0}$. If $\alpha\neq(2,1)$, then $(\alpha^{B_0})_{X_0}$ has a second addable node $Y=B_1$ or $Y=B_0''$ (in the second row). Thus, in this situation χ_0' contains

$$\chi_0''=[\alpha^{B_0}]+[{(\alpha^{B_0})_{X_0}}^{B_0'}]+[{(\alpha^{B_0})_{X_0}}^Y]+[\alpha^{B_1}]\,,$$

giving us again four components of width $m-1$ in the product.

When $\alpha = (2,1)$, we have

$$\chi_0 = [\alpha^{B_0}] + [{(\alpha^{B_0})_{X_0}}^{B_0'}] + [\alpha^{B_1}] = [3,1] + [4] + [2,1^2]\,,$$

so that up to this point we have found four components of length ≤ 4 in $[\mu]\cdot[\nu]$.

If there is a γ-node $A_1 \neq A_0$ connected to ν/γ, then we also get a contribution to χ from

$$\chi_1 = [\mu/\gamma_{A_1}]\cdot[\nu/\gamma_{A_1}] = [\alpha]\uparrow^{S_4} = [3,1] + [2^2] + [2,1^2]\,,$$

and thus we have again four components of width $m-1$ in the product.

If there is no such γ-node A_1, then $\mu = (5,4)$, $\nu = (3^3)$, and $|\mu\cap\nu'| = |\mu\cap\nu| = 6$ yields a component of length 6 in the product.

Next we have to consider the case when α is a rectangle; let B_0 be the top and B_1 the bottom addable node of α. Let X be the corner node of α. Since μ,ν are not both 2-part partitions, α is not a row. Then α^{B_0} also has the removable node X and we have

$$\chi_0 = \sum_D [{(\alpha^{B_0})_X}^D] - [\alpha^{B_0}] = \sum_{D\neq X} [{(\alpha^{B_0})_X}^D]\,,$$

which gives three constituents of width $m-1$, except in the cases where α has only two rows or only one column. When $\alpha = (1^2)$, we have $\chi_0 = [3]$, and otherwise, when $\alpha \neq (1^2)$ has only two rows or one column, χ_0 has two constituents.

Now if γ has a further removable node A_1, then we also get two constituents of width $m-1$ from

$$\chi_1 = [\mu/\gamma_{A_1}]\cdot[\nu/\gamma_{A_1}] = \sum_B [\alpha^B] = [\alpha^{B_0}] + [\alpha^{B_1}]\,,$$

and these are different from the ones appearing in χ_0. Thus for $\alpha \neq (1^2)$, we have then found at least four components of width $m-1$ in the product. For $\alpha = (1^2)$ we have found at this stage four components of length ≤ 4; but here $|\mu\cap\nu'| \geq 6$, giving us also a component of length ≥ 6 in the product.

Thus we are now in the situation where γ is a rectangle and α is a rectangle, say $\alpha = (a^b)$, and we need to find further components. We already know that α is not a row, and it also cannot be a column because this would contradict $\mu \neq \nu'$, so $1 < a, b < d$, $d \geq 4$.

First assume $b \geq 3$. By the above, we already have four components in the product which are of length $\leq b+2$. Here $\mu\cap\nu' = ((b+1)^b)$, hence we also get a component of length $b(b+1) > b+2$.

It remains to consider the case $b = 2$, i.e., $\alpha = (a^2)$ and $\mu = ((3a)^2)$, $\nu = ((2a)^3)$, where $a > 1$. By the considerations so far, we have found the constituents $[4a, a^2]$, $[4a-1, a+2, a-1]$ and $[4a-1, a+1, a-1, 1]$ in the product $[\mu]\cdot[\nu]$ which are of length ≤ 4. Now $\mu\cap\nu' = (3^2)$ and

$$[\mu/(\mu\cap\nu')]\cdot[\nu'/(\mu\cap\nu')] = [(3a-3)^2]\cdot[3^{2a-2}]\,.$$

By Theorem 3.1 and Theorem 3.2, this product has at least three components, hence by Corollary 2.4 $[\mu]\cdot[\nu]$ also has at least three components of length 6.

This final contradiction proves our claim for the case that there is only one component for one of the two extreme types. Note in particular, that at this point we have proved the classification for products with exactly three components.

6.2. **Two components of each extreme type.** We now have to study the case of products with four components. We may assume that we are not in the case described above, hence we have exactly two components of maximal width m and two components of maximal length $\tilde{m} = |\mu \cap \nu'|$ and no other components, as no constituent is both of maximal width and length by Theorem 2.6.

We set $\tilde{\gamma} = \mu \cap \nu'$. Since we have two components of maximal width and length, respectively, both products $[\mu/\gamma] \cdot [\nu/\gamma]$ and $[\mu/\tilde{\gamma}] \cdot [\nu/\tilde{\gamma}]$ have two components.

We focus on the first product. This situation splits into the following cases.

(1) $[\mu/\gamma] = [1^2] + [2]$ and $[\nu/\gamma] = [1^2] + [2]$.
(2) $[\mu/\gamma]$ and $[\nu/\gamma]$ are both irreducible, and the product has two components.
(3) $[\mu/\gamma]$ has two components and $[\nu/\gamma]$ is of degree 1.

In Case (2), we use Theorem 3.2, in Case (3) we employ Proposition 3.5.

6.2.1. First we treat Case (1), where both skew diagrams decompose into two disconnected nodes. Note that the assumptions imply that $n \geq 6$ and that γ has at least three removable nodes.

By Corollary 2.3 we obtain in this case from $[\mu/\gamma] \cdot [\nu/\gamma] = 2[2] + 2[1^2]$ as components of maximal width $[n-2, 2]$ and $[n-2, 1^2]$ in $[\mu] \cdot [\nu]$.

We will show that all three possible components of width $n-3$ appear in the product, using Lemma 5.1. Assume there is a γ-node A_0 which is disconnected from μ/γ; then $[\mu/\gamma_{A_0}] = [3] + 2[2,1] + [1^3]$. We want to compute

$$\chi_0 = [\mu/\gamma_{A_0}] \cdot [\nu/\gamma_{A_0}] - ([\mu/\gamma] \cdot [\nu/\gamma]) \uparrow = [\mu/\gamma_{A_0}] \cdot [\nu/\gamma_{A_0}] - (2[3] + 4[2,1] + 2[1^3]) \,.$$

If A_0 is connected to at most one of the nodes of ν/γ, then $[\nu/\gamma_{A_0}]$ contains $[3] + [2,1]$ or $[2,1] + [1^3]$, and thus

$$\chi_0' = (3[3] + 6[2,1] + 3[1^3]) - (2[3] + 4[2,1] + 2[1^3]) = [3] + 2[2,1] + [1^3]$$

is a character contained in the character χ_0. Since $n \geq 6$, we get three constituents $[n-3, 3]$, $[n-3, 2, 1]$, $[n-3, 1^3]$ in the product $[\mu] \cdot [\nu]$.

The same argument can be used with μ, ν interchanged. Hence we may now assume that every removable γ-node is connected to two of the four nodes of μ/γ and ν/γ. Arguing from the top removable γ-node down, one easily sees that then we must have $\gamma = (3, 2, 1)$. If every γ-node is connected to both a μ/γ and a ν/γ-node, then

$$\begin{aligned} \chi &= \sum_{A \ \gamma\text{-node}} [\mu/\gamma_A] \cdot [\nu/\gamma_A] - ([\mu/\gamma] \cdot [\nu/\gamma]) \uparrow \\ &= 3([2,1] + [3]) \cdot ([2,1] + [1^3]) - (2[3] + 4[2,1] + 2[1^3]) \\ &= [3] + 5[2,1] + 4[1^3] \,. \end{aligned}$$

If some γ-node A_0 is connected to two nodes of the same skew diagram μ/γ or ν/γ, then there is also at least one γ-node A_1 that is connected to both a μ/γ and a ν/γ-node, and we have

$$\begin{aligned}\chi' &= \sum_{A\in\{A_0,A_1\}} [\mu/\gamma_A]\cdot[\nu/\gamma_A] - ([\mu/\gamma]\cdot[\nu/\gamma])\uparrow \\ &= [2,1]\cdot([3]+2[2,1]+[1^3]) + ([2,1]+[3])\cdot([2,1]+[1^3]) - (2[3]+4[2,1]+2[1^3]) \\ &= [3]+3[2,1]+2[1^3]\,,\end{aligned}$$

a subcharacter of the character χ. Hence in both cases we get all three constituents $[n-3,3]$, $[n-3,2,1]$, $[n-3,1^3]$ in the product $[\mu]\cdot[\nu]$.

Thus, in any case we arrive at a contradiction to the assumption that we only have four components in the product.

6.2.2. We now deal with Case (2). By Theorem 3.2 we know that up to conjugation and renaming we are in the following situation:

$[\mu/\gamma]=[\alpha]$, $[\nu/\gamma]=[d-1,1]$, where $\alpha=(a^b)$ is a nontrivial rectangle, i.e., $a,b>1$.

Now for this case, Proposition 5.5 was tailor-made; it tells us that we obtain in fact at least five components of width $m-1$ in $[\mu]\cdot[\nu]$.

6.2.3. Now to Case (3), where $[\mu/\gamma]$ has two components and $[\nu/\gamma]$ is of degree 1. We may assume that ν/γ is a row.

From Proposition 3.5 we know that the two components of $[\mu/\gamma]$ are close to each other, i.e., this skew character has the form

$$[\mu/\gamma]=[\alpha^X]+[\alpha^Y]$$

for some partition α and two distinct addable nodes X, Y for α.

Thus $[\mu]\cdot[\nu]$ has $[m,\alpha^X]$ and $[m,\alpha^Y]$ as its components of maximal width.

Using Lemma 5.3, we want to produce components of width $m-1$ in the product. Thus we now consider

$$\chi = \sum_{A\ \gamma-\text{node}} [\mu/\gamma_A]\cdot[\nu/\gamma_A] - [\mu/\gamma]\uparrow\ .$$

First we assume that there is some γ-node A_0 disconnected from μ/γ. Then

$$[\mu/\gamma_{A_0}] = [\mu/\gamma]\uparrow = [\alpha^X]\uparrow + [\alpha^Y]\uparrow\ .$$

If $[d,1]$ is a constituent of $[\nu/\gamma_{A_0}]$, then $\chi_0=[\mu/\gamma_{A_0}]\cdot[\nu/\gamma_{A_0}]-[\mu/\gamma]\uparrow$ contains

$$\chi_0' = ([\mu/\gamma]\uparrow)\cdot[d,1] - [\mu/\gamma]\uparrow = ([\mu/\gamma]\cdot[d-1,1])\uparrow\ .$$

Now $([\alpha^X]+[\alpha^Y])\cdot[d-1,1]$ clearly contains $[\alpha^Y]+[\alpha^X]=[\mu/\gamma]$ as a subcharacter, hence the character χ_0 contains

$$\chi_0'' = [\mu/\gamma]\uparrow = [\alpha^X]\uparrow + [\alpha^Y]\uparrow\ .$$

If one of α^X and α^Y is not a rectangle, we clearly have $c(\chi) \geq c(\chi_0'') \geq 3$. That α^X and α^Y are both rectangles can only occur when $\alpha = (1)$, i.e., when $[\mu/\gamma] = [2] + [1^2]$. In this case we have

$$\chi_0'' = ([2] + [1^2]) \uparrow = [3] + 2[2,1] + [1^3] \,,$$

hence we obtain again $c(\chi) \geq 3$. Thus in any case we have three components of width $m-1$ in $[\mu] \cdot [\nu]$.

We may now assume that ν/γ_{A_0} is a row, and furthermore, that any removable γ-node $A_1 \neq A_0$ is connected to μ/γ. Then

$$\chi_0 = [\mu/\gamma_{A_0}] \cdot [\nu/\gamma_{A_0}] - [\mu/\gamma] \uparrow = 0 \,.$$

Since μ/γ is a proper skew diagram, there must be such a γ-node A_1 which is an inner node for μ/γ, i.e., it is connected to a node of μ/γ but it is not above the highest row nor to the left of the leftmost column of μ/γ. In fact, considering the list in Proposition 3.5 we see that then

$$[\mu/\gamma_{A_1}] = \begin{cases} [\alpha^{XY}] & \text{if } \mu/\gamma_{A_1} \text{ is a partition} \\ [\alpha^{XX'}] + [\alpha^{YY'}] & \text{if } \mu/\gamma_{A_1} \text{ is not a partition but on the list in Prop. 3.5} \end{cases}$$

where X', Y' are addable nodes for α^X, α^Y, respectively (possibly $X' = Y$ or $Y' = X$, but not both); note that X, Y are nodes that we can add in an independent way, whereas X, X' and Y, Y' may only be added in this order. When we are not in one of the two situations above, $[\mu/\gamma_{A_1}]$ is a skew character with at least three components, including $[\alpha^{XX'}] + [\alpha^{YY'}]$ as above.

Now whenever A_1 is disconnected from ν/γ we have

$$\chi_1 = [\mu/\gamma_{A_1}] \cdot [\nu/\gamma_{A_1}] = [\mu/\gamma_{A_1}] \cdot ([d] \uparrow) = [\mu/\gamma_{A_1}] \downarrow\uparrow \,.$$

From the description above, we see immediately that $[\mu/\gamma_{A_1}] \downarrow$ contains $[\alpha^X] + [\alpha^Y] = [\mu/\gamma]$, hence arguing as above, χ_1 (and thus χ) has at least three components.

Thus we assume now that all γ-nodes that are inner nodes for μ/γ are connected to ν/γ; this can only happen if μ/γ is a disconnected skew diagram with two parts and ν/γ between them. There can only be one such γ-node $A_1 \neq A_0$. We then have

$$\chi_1 = [\mu/\gamma_{A_1}] \cdot [\nu/\gamma_{A_1}] = [\mu/\gamma_{A_1}] \cdot [d,1] \,.$$

From the description above, we deduce that χ_1 always has $[\alpha^{XY}]$ as a constituent, giving a component $[m-1, \alpha^{XY}]$ in $[\mu] \cdot [\nu]$ which is neither of maximal width nor length as $\ell(\alpha^{XY}) = \max(\ell(\alpha^X), \ell(\alpha^Y))$ and there is no constituent which is maximal in both respects.

We may now assume that all removable γ-nodes are connected to μ/γ.

Let A_0 be a removable γ-node, and assume that this is not connected to ν/γ. In any case, $[\mu/\gamma_{A_0}] \downarrow$ contains $[\alpha^X] + [\alpha^Y] = [\mu/\gamma]$, so that

$$\chi_0 = [\mu/\gamma_{A_0}] \cdot [\nu/\gamma_{A_0}] - [\mu/\gamma] \uparrow = [\mu/\gamma_{A_0}] \downarrow\uparrow - [\mu/\gamma] \uparrow$$

is a character (or 0).

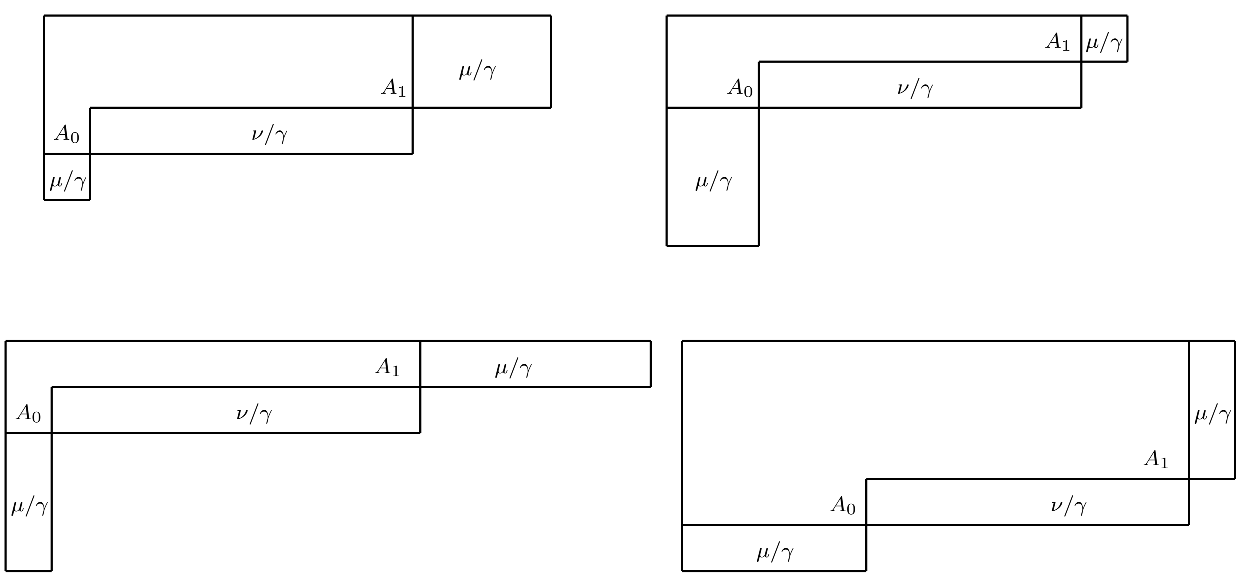
Since μ/γ is a proper skew character, there has to be a second γ-node $A_1 \neq A_0$. If $[\nu/\gamma_{A_1}]$ contains $[d,1]$, then as before, $\chi_1 = [\mu/\gamma_{A_1}] \cdot [\nu/\gamma_{A_1}]$ contains $[\alpha^{XY}]$, and this provides a fifth constituent $[m-1, \alpha^{XY}]$ in $[\mu] \cdot [\nu]$ as we have seen above. Thus we may now assume that $[\nu/\gamma_{A_1}] = [d+1]$.

Now if A_1 is an inner node for μ/γ, then $[\mu/\gamma_{A_1}]$ always has a constituent of length $\max(\ell(\alpha^X), \ell(\alpha^Y))$, namely the one coming from sorting the rows, and this would also provide a (fifth) constituent which neither has maximal width nor length.

Thus A_1 can only be an outer node for μ/γ. Considering the list in Proposition 3.5, we see that $[\mu/\gamma_{A_1}]$ has at least three components, except in the case where $[\mu/\gamma] = [1^{a+1}] \otimes [r]$ and we obtain $[\mu/\gamma_{A_1}] = [1^{b+1}] \otimes [s]$ with either $b = a+1$ or $s = r+1$. But in this case, $[\mu/\gamma_{A_1}]$ has $[\alpha^{XY}]$ as a constituent, giving a fifth component $[m-1, \alpha^{XY}]$ in $[\mu] \cdot [\nu]$ as before.

Hence we conclude that there is no A_0 as assumed above, i.e., any γ-node must be connected to ν/γ. Since μ/γ is a proper skew diagram and we always have a removable γ-node that is inner with respect to μ/γ, this can only be true when μ/γ is disconnected, i.e., we are in one of the cases (i) or (ii) of Proposition 3.5 (as before, up to translation and order of the two connected parts). Furthermore, ν/γ sits between the two parts of μ/γ and we must have two γ-nodes A_0, A_1 connected to both μ/γ and ν/γ. Altogether, there are now only four cases we have to consider.

Here are the corresponding pictures (we mark both parts of μ/γ by μ/γ):



In all cases, we let A_0 and A_1 be the γ-nodes such that ν/γ_{A_0} is a row and $[\nu/\gamma_{A_1}] = [d, 1]$. We know that $[\mu/\gamma] = [\alpha^X] + [\alpha^Y]$, with X being the top and Y the bottom addable node in our situation, where now α is a nontrivial rectangle or a hook. Let X' be the top addable node of α^X. One easily checks that then in all cases, $[\mu/\gamma_{A_0}]$ has a constituent $[\alpha^{XY}]$ and $[\mu/\gamma_{A_1}]$ has at least the constituents $[\alpha^{XY}]$ and $[\alpha^{XX'}]$. Hence, similarly as before, $[\mu/\gamma_{A_1}] \cdot [d, 1]$ contains $[\alpha^Y]\uparrow + [\alpha^X]\uparrow = [\mu/\gamma]\uparrow$. Thus in all cases, we have the following:

$$\chi = [\mu/\gamma_{A_0}] + [\mu/\gamma_{A_1}] \cdot [d, 1] - [\mu/\gamma]\uparrow$$

is a character containing $[\alpha^{XY}]$, and this produces, as before, a constituent $[m-1, \alpha^{XY}]$ in $[\mu] \cdot [\nu]$ which is neither of maximal width nor length.

Hence we have now reached the final contradiction, and thus also the case of a product with exactly two components of maximal width and two of maximal length cannot occur under Hypothesis $(*)$. □

Remark. Note that in fact we have not used in this final case that Hypothesis $(*)$ excluded certain partitions, and indeed, even in the exceptional families where we have a product with exactly four components it never occurs that there are two components of maximal width as well as two components of maximal length.

References

[1] C. Bessenrodt, A. Kleshchev, On Kronecker products of complex representations of the symmetric and alternating groups, *Pacific J. Math.* 190 (1999), 201–223.

[2] A. A. H. Brown, S. van Willigenburg, M. Zabrocki, Expressions for Catalan Kronecker products, *Pacific J. Math.* 248 (2010), 31–48.

[3] M. Clausen, H. Meier, Extreme irreduzible Konstituenten in Tensordarstellungen symmetrischer Gruppen, *Bayreuther Math. Schr.* 45 (1993), 1–17.

[4] Y. Dvir, On the Kronecker product of S_n characters, *J. Algebra* 154 (1993), 125–140.

[5] A. M. Garsia, J. Remmel, Shuffles of permutations and the Kronecker product, *Graphs Combin.* 1 (1985), 217–263.

[6] C. Gutschwager, Skew characters which contain only few components, preprint 2010.

[7] G. James, A. Kerber, *The representation theory of the symmetric group*, Addison-Wesley, London, 1981.

[8] D. E. Littlewood, The Kronecker product of symmetric group representations, *J. London Math. Soc.* 31 (1956), 89–93.

[9] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford Univ. Press, Oxford, 1995.

[10] F. D. Murnaghan, The analysis of the Kronecker product of irreducible representations of the symmetric group, *Amer. J. Math.* 60 (1938), 761–784.

[11] J. Remmel, A formula for the Kronecker products of Schur functions of hook shapes, *J. Algebra* 120 (1989), 100–118.

[12] J. Remmel, Formulas for the expansion of the Kronecker products $S_{(m,n)} \otimes S_{(1^{p-r},r)}$ and $S_{(1^k 2^l)} \otimes S_{(1^{p-r},r)}$, *Discrete Math.* 99 (1992), 265–287.

[13] J. Remmel, T. Whitehead, On the Kronecker product of Schur functions of two row shapes, *Bull. Belg. Math. Soc.* 1 (1994), 649–683.

[14] M. H. Rosas, The Kronecker product of Schur functions indexed by two-row shapes or hook shapes, *J. Algebraic Combin.* 14 (2001), 153–173.
[15] B. Sagan, *The Symmetric Group - Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd ed., Springer GTM 203, 2000.
[16] J. Saxl, The complex characters of the symmetric groups that remain irreducible in subgroups, *J. Algebra* 111 (1987), 210–219.
[17] E. Vallejo, On the Kronecker product of irreducible characters of the symmetric group, preprint 1997.
[18] S. van Willigenburg, Equality of Schur and skew Schur functions, *Ann. Comb.* 9 (2005), 355–362.
[19] I. Zisser, The character covering numbers of the alternating groups, *J. Algebra* 153 (1992), 357–372.

Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Hannover, D-30167, Germany
E-mail address: bessen@math.uni-hannover.de

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
E-mail address: steph@math.ubc.ca