GENERALIZED CHROMATIC FUNCTIONS

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Abstract. We define vertex-colourings for edge-coloured digraphs, which unify the theory of \( P \)-partitions and proper vertex-colourings of graphs. We use our vertex-colourings to define generalized chromatic functions, which merge the chromatic symmetric and quasisymmetric functions and generating functions of \( P \)-partitions. Moreover, numerous classical bases of symmetric and quasisymmetric functions, both in commuting and noncommuting variables, can be realized as special cases of our generalized chromatic functions. We also establish product and coproduct formulas for our functions. Additionally, we construct the new Hopf algebra of \( r \)-quasisymmetric functions in noncommuting variables, and apply our functions to confirm its Hopf structure, and establish natural bases for it.

Contents

1. Introduction 2
2. Symmetric functions and generalizations 4
2.1. Partitions, compositions, and \( r \)-compositions 5
2.2. Quasisymmetric functions 5
2.3. Symmetric functions 6
2.4. \( r \)-quasisymmetric functions 6
3. Proper colourings 7
4. Generalized chromatic functions 9
5. Other chromatic functions 10
6. Product and coproduct formulas for generalized chromatic functions 12
7. Bases for \( \text{QSym}^r(x) \) using generalized chromatic functions 15
8. Symmetric functions and generalizations in noncommuting variables 18
8.1. Set partitions, set compositions, and \( r \)-set-compositions 18
8.2. Quasisymmetric functions in noncommuting variables 19
8.3. Symmetric functions in noncommuting variables 20
8.4. \( r \)-quasisymmetric functions in noncommuting variables 20
9. Generalized chromatic functions in noncommuting variables 20
10. Product and coproduct formulas for generalized chromatic functions in noncommuting variables 23
11. Bases for \( \text{NCSym}(x) \) and \( \text{NCQSym}(x) \) using generalized chromatic functions in noncommuting variables 23

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1. Introduction

MacMahon [19] discovered the primary idea of the theory of $P$-partitions (for a more complete history, see [12]) at the start of the 20th century, and 60 years later Stanley, in his Ph.D. thesis [25], studied the relation between plane partitions and $P$-partitions. The problem that MacMahon considered in his work on plane partitions was the same as counting the number of fillings of Ferrers diagrams with nonnegative integers with a given sum such that the entries are weakly decreasing in each row and column.

\[
\begin{array}{ccc}
4 & 3 & 3 \\
4 & 3 & 2 \\
2 & 1 & 1
\end{array}
\]

Stanley, in his work on the theory of $P$-partitions [25], generalized MacMahon’s idea and replaced Ferrers diagrams with posets and the weakly decreasing relation with weakly and strictly decreasing relations. In this paper, we generalize Stanley’s $P$-partitions to certain vertex-colourings of digraphs whose edges are coloured with three colours. Roughly speaking, we replace posets in the theory of $P$-partitions with digraphs, and in addition to the weakly and strictly decreasing relations, we have another relation related to the proper colourings of digraphs. More precisely, we define a proper vertex-colouring of a digraph whose edges are coloured by three colours, identified by $\rightarrow, \rightarrow$, and $\Rightarrow$, to be a function $\kappa : V(G) \rightarrow \mathbb{P}$, where $\mathbb{P}$ is the set of positive integers and

1. If $a \rightarrow b$ in $G$, then $\kappa(a) \neq \kappa(b)$.
2. If $a \rightarrow b$ in $G$, then $\kappa(a) < \kappa(b)$.
3. If $a \Rightarrow b$ in $G$, then $\kappa(a) \leq \kappa(b)$.

Let $C(G)$ be the set of all proper vertex-colourings of the edge-coloured digraph $G$.

![Figure 1. A proper vertex-colouring of an edge-coloured digraph](image)

In the above definition, by replacing the infinite paintbox of colours $\mathbb{P}$ with a finite paintbox of colours $[p] = \{1, 2, \ldots, p\}$, we generalize the classic proper vertex-colouring and weak/strong proper colouring in [3]. The generalized chromatic number of an edge-coloured digraph $G$ is the smallest number of colours we need to make a proper vertex-colouring of $G$. Let $P(G, p)$ be the number of ways that one can properly colour the vertices of $G$ with $p$ colours. We show that $P(G, p)$ is a polynomial in $p$ (see Theorem...
7.1) and call it the *generalized chromatic polynomial* of $G$.

Stanley in [24] defined the chromatic symmetric function of a finite simple graph. This symmetric function gives Birkhoff’s chromatic polynomial by setting the first $k$ variables to 1 and all others to 0. There are two main conjectures regarding chromatic symmetric functions that have been open for more than 25 years—the Tree Conjecture [24] and the $(3+1)$-free Conjecture [26]. Later on, refining the conjectures and seeking classical properties, other chromatic symmetric and quasisymmetric functions emerged, such as the chromatic quasisymmetric functions of graphs and digraphs [8, 22], the extended chromatic symmetric function of a graph [7], and the $k$-balanced chromatic quasisymmetric function of a graph [17].

Let $Q[[x_1, x_2, \ldots]]$ be the set of all power series in commuting variables $x = x_1, x_2, \ldots$. We define the *generalized chromatic function* of an edge-coloured digraph $G$ to be the bounded degree power series

$$
\mathcal{Y}_G(x) = \sum_{\kappa \in \mathcal{C}(G)} x_\kappa,
$$

where $x_\kappa = \prod_{a \in V(G)} x_{\kappa(a)}$. Note that the generalized chromatic polynomial $P(G, p)$ is equal to the generalized chromatic function of $G$ by setting the first $p$ variables to 1 and all others to 0. In the theory of $P$-partitions, a labelled poset $P$ corresponds to a quasisymmetric function $F_P$ [23]. Since each $P$-partition is a proper vertex-colouring of an edge-coloured digraph (see Section 3), we can view $F_P$ as a generalized chromatic function. Moreover, Stanley’s chromatic symmetric functions [24], extended chromatic symmetric functions [7], and chromatic quasisymmetric functions in [8, 17, 22] are the generalized chromatic functions of certain edge-coloured digraphs (see Section 5). Therefore, we can merge all of these chromatic functions and the generating functions of $P$-partitions, $F_P$, into one object.

As we will see, generalized chromatic functions are quasisymmetric functions. There are many well-known bases for the Hopf algebras of symmetric and quasisymmetric functions. One substantial strength of the theory of generalized chromatic functions is that any of these bases can be expressed as a family of generalized chromatic functions. Therefore, if one finds a product, coproduct, antipode, etc., formula for generalized chromatic functions, then this impacts knowledge about all these bases of the Hopf algebras of symmetric and quasisymmetric functions. For example, we give a generic coproduct formula for generalized chromatic functions.

We also study a chain of Hopf algebras starting from symmetric functions and ending with quasisymmetric functions. We present different bases for the Hopf algebras in the chain using generalized chromatic functions.

We then change gears to the noncommuting world and define $\mathcal{Y}(G, L)(x)$, the generalized chromatic function of a labelled edge-coloured digraph $(G, L)$ in noncommuting variables $x = x_1, x_2, \ldots$. We extend most of the results in commuting variables to noncommuting
variables.

Moreover, we consider a natural injection of the Malvenuto-Reutenauer Hopf algebra to the quasisymmetric functions in noncommuting variables that maps the fundamental basis elements of the Malvenuto-Reutenauer Hopf algebra to the fundamental basis elements of the quasisymmetric functions in noncommuting variables.

The Hopf algebra of $r$-quasisymmetric functions is defined in [15] by Hivert. In [10], Garsia and Wallach showed that the algebra of $r$-quasisymmetric functions is free over symmetric functions. In the last section, we introduce the Hopf algebra of $r$-quasisymmetric functions in noncommuting variables.

More precisely, our paper is structured as follows. In Section 2, we recall the background of symmetric, quasisymmetric, and $r$-quasisymmetric functions. In Section 3, we present some basic definitions in graph theory and then define some operators between edge-coloured digraphs. The vertex-colouring of an edge-coloured digraph is defined in Definition 3.1. We conclude Section 3 by showing that every $P$-partition is the proper vertex-colouring of an edge-coloured digraph in Proposition 3.2. In Section 4, we introduce generalized chromatic functions in Definition 4.1 and show that they are quasisymmetric functions, and then in Section 5, show that other chromatic symmetric and quasisymmetric functions are special cases of generalized chromatic functions. In Section 6, the product and coproduct formulas for generalized chromatic functions are presented in Proposition 6.1 and Theorem 6.3. In Section 7, we show that many well-known bases of symmetric, quasisymmetric, and $r$-quasisymmetric functions can be realized as special cases of generalized chromatic functions of edge-coloured digraphs. Moreover, in Theorem 7.1, we show that the generalized chromatic polynomial of an edge-coloured digraph is indeed a polynomial. In Section 8, we recall the background of symmetric and quasisymmetric functions in noncommuting variables and then define $r$-quasisymmetric functions in noncommuting variables. In Section 9, we introduce generalized chromatic symmetric functions in noncommuting variables in Definition 9.1, and the product and coproduct formulas for them are presented in Propositions 10.1 and 10.2. In Sections 11 and 12, we show that several bases for symmetric functions in noncommuting variables are the symmetrizations of certain generalized chromatic functions and give several bases for quasisymmetric functions in noncommuting variables, including its fundamental basis, which contains the fundamental basis of the Malvenuto-Reutenauer Hopf algebra. We conclude by showing that the set of $r$-quasisymmetric functions in noncommuting variables is a Hopf algebra in Theorem 13.1 and constructing the $r$-dominant monomial and upper-fundamental bases of the Hopf algebra of $r$-quasisymmetric functions in noncommuting variables in Proposition 13.2.

2. Symmetric functions and generalizations

This section introduces the Hopf algebras of symmetric, quasisymmetric, and $r$-quasisymmetric functions. The bases of these Hopf algebras are indexed by partitions, compositions, and $r$-compositions, respectively. We begin by recalling the definitions and notation related to these combinatorial objects.
2.1. Partitions, compositions, and \( r \)-compositions. A composition \( \alpha \) of \( n \), denoted \( \alpha \vdash n \), is a list of positive integers whose sum is \( n \). Given a composition \( \alpha = (\alpha_1,\alpha_2,\ldots,\alpha_k) \), each \( \alpha_i \) is called a part of \( \alpha \), the size of \( \alpha \) is \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k \), and the length of \( \alpha \) is \( k \). For convenience we denote by \( \emptyset \) the unique composition of size and length zero. If \( \alpha = (\alpha_1,\alpha_2,\ldots,\alpha_k) \vdash n \), then we define

\[
\text{set}(\alpha) = \{ \alpha_1,\alpha_1 + \alpha_2,\ldots,\alpha_1 + \cdots + \alpha_{k-1} \} \subseteq [n-1].
\]

For example, \((2, 1, 2)\) is a composition of 5 with length 3 and \(\text{set}(\alpha) = \{2, 3\}\). For compositions \( \alpha \) and \( \beta \) of \( n \), we write \( \alpha \leq \beta \) and say \( \alpha \) coarsens \( \beta \) (or \( \beta \) refines \( \alpha \)) if \( \text{set}(\alpha) \subseteq \text{set}(\beta) \).

A partition \( \lambda = (\lambda_1,\lambda_2,\ldots,\lambda_k) \) of \( n \), denoted \( \lambda \vdash n \), is a weakly decreasing composition. Let \( m_i \) be the number of parts of \( \lambda \) that are equal to \( i \). Let \( \lambda' = m_1!m_2!\cdots m_n! \), and let \( \lambda! = \lambda_1!\lambda_2!\cdots \lambda_k! \). We sometimes write

\[
\lambda = (n^{m_n},(n-1)^{m_{n-1}},\ldots,1^{m_1}).
\]

For partitions \( \lambda = (\lambda_1,\lambda_2,\ldots,\lambda_k) \) and \( \mu = (\mu_1,\mu_2,\ldots,\mu_l) \) of \( n \), we write \( \mu \leq \lambda \) if \( k \leq l \) and for every \( 1 \leq i \leq k \),

\[
\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i.
\]

For example, \((3,1,1,1) \preceq (3,2,1)\).

Let \( r \) be a positive integer or infinity. A pair \((\beta,\mu)\) where

1. \( \beta \) is a composition whose parts are at least \( r \) and
2. \( \mu \) is a partition whose parts are strictly smaller than \( r \)

is called an \( r \)-composition of \(|\beta|+|\mu|\). For example, \(((7,4,5),(3,2))\) is a 4-composition of 21.

2.2. Quasisymmetric functions. The Hopf algebra of quasisymmetric functions was formally introduced by Gessel [13] in 1984. From this concept, a whole research area emerged; a history can be found in [18, Introduction].

Recall that \( \mathbb{Q}[[x_1,x_2,\ldots]] \) is the algebra of formal power series in infinitely many commuting variables \( x = x_1, x_2, \ldots \) over \( \mathbb{Q} \). Let \( S_n \) be the group of all permutations of \([n]\). Let \( S_\infty = \cup_{n \geq 0} S_n \). We identify a permutation \( \sigma \in S_n \subseteq S_\infty \) with a bijection of the positive integers by defining \( \sigma(i) = i \) if \( i > n \).

Definition 2.1. A quasisymmetric function is a formal power series \( f \in \mathbb{Q}[[x_1,x_2,\ldots]] \) such that

1. The degrees of the monomials in \( f \) are bounded.
2. For every composition \((\alpha_1,\alpha_2,\ldots,\alpha_k)\), all monomials \( x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k} \) in \( f \) with indices \( i_1 < i_2 < \cdots < i_k \) have the same coefficient.

The set of all quasisymmetric functions is denoted by \( \text{QSym}(x) \).

The vector space \( \text{QSym}(x) \) is a Hopf algebra, where its product is the same as the product of the formal power series and its coproduct \( \Delta \) is defined as follows (for more details see [14, p. 142]). Consider the linear order on two sets of commuting variables
\((x, y) = (x_1 < x_2 < \cdots < y_1 < y_2 < \cdots)\), and inject \(\text{QSym}(x) \otimes \text{QSym}(y)\) into \(\mathbb{Q}[x, y]\) by identifying every \(f \otimes g \in \text{QSym}(x) \otimes \text{QSym}(y)\) with \(fg \in \mathbb{Q}[x, y]\). We then have that
\[
\text{QSym}(x, y) \subseteq \text{QSym}(x) \otimes \text{QSym}(y).
\]

We can define \(\Delta : \text{QSym}(x) \to \text{QSym}(x) \otimes \text{QSym}(x)\) as the composite of the following maps.
\[
\text{QSym}(x) \cong \text{QSym}(x, y) \to \text{QSym}(x) \otimes \text{QSym}(y) \cong \text{QSym}(x) \otimes \text{QSym}(x)
\]

\[f \mapsto f(x_1, x_2, \ldots, y_1, y_2, \ldots)\]

For more details about the Hopf algebra of quasisymmetric functions and its well-known bases, see [14, Section 5].

2.3. \textbf{Symmetric functions.} The Hopf algebra of symmetric functions is a Hopf subalgebra of \(\text{QSym}(x)\).

\textbf{Definition 2.2.} A symmetric function is a formal power series \(f \in \mathbb{Q}[[x_1, x_2, \ldots]]\) such that

1. The degrees of the monomials in \(f\) are bounded.
2. For any permutation \(\sigma \in S_\infty\),
   \[
   \sigma.f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots).
   \]

The set of all symmetric functions is denoted by \(\text{Sym}(x)\).

For more details about the Hopf algebra of symmetric functions and its well-known bases, see [14, Section 2].

2.4. \textbf{r-quasisymmetric functions.} Hivert introduced the Hopf algebra of \(r\)-quasisymmetric functions in [15], which is a Hopf subalgebra of \(\text{QSym}(x)\).

For each \(r\)-composition \((\beta, \mu)\) where \(\beta = (\beta_1, \beta_2, \ldots, \beta_k)\) and \(\mu = (\mu_1, \mu_2, \ldots, \mu_l)\), define the \(r\)-dominant monomial function to be
\[
M_{(\beta, \mu)} = \sum x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k} x_{k+1}^{\mu_1} x_{k+2}^{\mu_2} \cdots x_{k+l}^{\mu_l},
\]
where the sum is over all distinct positive integers \(i_1, i_2, \ldots, i_{k+l}\) such that \(i_1 < i_2 < \cdots < i_k\). Define
\[
\text{QSym}^r(x) = \bigoplus_{n \geq 0} \text{QSym}^r_n(x),
\]
where
\[
\text{QSym}^r_n(x) = \mathbb{Q}\text{-span}\{M_{(\beta, \mu)} : (\beta, \mu) \text{ is an } r\text{-composition of } n\}.
\]

We have that
\[
\text{QSym}(x) = \text{QSym}^1(x) \supset \text{QSym}^2(x) \supset \cdots \supset \text{QSym}^\infty(x) = \text{Sym}(x).
\]

In Section 7, we present different bases for the Hopf algebras in this chain.
3. Proper colourings

In graph theory, there are many families of graph colourings. These colourings are usually defined by setting specific constraints on the colours of vertices or edges of a graph. In this paper, we consider particular vertex-colourings of a graph to unify several combinatorial constructions. In our colouring, the constraints on the colours of the vertices are subject to an edge-colouring of the graph. We recall the definitions and notation in graph theory that we need throughout this paper.

A simple digraph $G = (V(G), E(G))$ is a digraph with no loops, and for any distinct vertices $a$ and $b$, there can be at most one directed edge from $a$ to $b$. Throughout this paper all digraphs are simple, and we usually use $G$ to denote a simple digraph. The underlying graph of $G$ is the undirected graph $\overrightarrow{G}$ whose vertex set is the vertex set of $G$ and two vertices $a$ and $b$ are adjacent, denoted by $ab$, in $\overrightarrow{G}$ if $(a, b)$ or $(b, a)$ is an edge of $G$. A directed cycle is a digraph $G$ with the vertex set $V(G) = \{a_1, a_2, \ldots, a_n\}$ and the edge set $E(G) = \{(a_i, a_{i+1}) : i = 1, 2, \ldots, n-1\} \sqcup \{(a_n, a_1)\}$. A directed path is a digraph $G$ with the vertex set $V(G) = \{a_1, a_2, \ldots, a_n\}$ and the edge set $E(G) = \{(a_i, a_{i+1}) : i = 1, 2, \ldots, n-1\}$. A complete digraph is a digraph $G$ with the vertex set $V(G) = \{a_1, a_2, \ldots, a_n\}$ and the edge set $E(G) = \{(a_i, a_j) : 1 \leq i < j \leq n\}$.

Let $S$ and $S'$ be subsets of $\mathbb{P}$, the set of positive integers. We regard the elements in $S$ and $S'$ as colours. Let $G$ be a digraph. A $S$-vertex-colouring of $G$ is a function $\kappa$ that assigns a colour in $S$ to each vertex of the digraph $G$. By a vertex-colouring, without mentioning the set $S$, we mean a $\mathbb{P}$-vertex-colouring.

An $S'$-edge-colouring of $G$ is a function $\kappa'$ that assigns a colour in $S'$ to each edge of the digraph $G$. An $S'$-edge-coloured digraph is a digraph $G$ together with an $S'$-edge-colouring $\kappa'$ of $G$. Throughout this paper, we only consider $S'$-edge-colourings of digraphs where $|S'| = 3$, so we only have three types of coloured edges in a digraph, which are denoted by $\rightarrow, \rightarrow, \Rightarrow$.

Consider edge-coloured digraphs $G_1$ with the edge-colouring $\kappa'_1$ and $G_2$ with the edge-colouring $\kappa'_2$. The disjoint union of the edge-coloured digraphs $G_1$ and $G_2$, denoted $G_1 \uplus G_2$, is an edge-coloured digraph $G$ together with an edge-colouring $\kappa'$ such that

1. The vertex set of $G$ is the disjoint union of the vertex sets of $G_1$ and $G_2$.
2. The edge set of $G$ is the disjoint union of the edge sets of $G_1$ and $G_2$.
3. $\kappa'(e) = \kappa'_j(e)$ if $e \in E(G_j)$ for $j = 1, 2$.

The disjoint union of two edge-coloured digraphs
Also, the dashed (solid and double, respectively) sum of the edge-coloured digraphs $G_1$ and $G_2$ is an edge-coloured digraph $G$ together with an edge-colouring $\kappa'$ such that

1. The vertex set of $G$ is the disjoint union of the vertex sets of $G_1$ and $G_2$.
2. The edge set of $G$ is the disjoint union of the edge set of $G_1$, edge set of $G_2$, and $\{(a, b) : a$ is a vertex of $G_1$ and $b$ is a vertex of $G_2\}$.
3. For every edge $e = (a, b)$ in $E(G)$, $\kappa'(e) = \kappa'_j(e)$ if $e \in G_j$ for $j = 1, 2$, and if $a$ is a vertex of $G_1$ and $b$ is a vertex of $G_2$, then $e$ is a dashed (solid and double, respectively) edge.

The dashed, solid, and double sums of the edge-coloured digraphs $G_1$ and $G_2$ are denoted by

$$G_1 \ominus G_2, \hspace{1cm} G_1 \oslash G_2, \hspace{1cm} \text{and} \hspace{1cm} G_1 \oslash G_2,$$

respectively.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{dashed_sum.png}
\caption{The dashed sum of two edge-coloured digraphs}
\end{figure}

We frequently use the edge-coloured digraphs in the following table.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
Notation & Expression \\
\hline
$C_n$ & The directed cycle with $n$ vertices and double edges \\
$P_n$ & The directed path with $n$ vertices and solid edges \\
$Q_n$ & The directed path with $n$ vertices and double edges \\
$K_n$ & The complete digraph with $n$ vertices and dashed edges \\
\hline
\end{tabular}
\caption{Useful edge-coloured digraphs}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{useful_digraphs.png}
\caption{The edge-coloured digraphs $C_3, P_3, Q_3, K_3$}
\end{figure}

Let $(P, \prec_P)$ be a poset. The digraph associated to $P$, denoted $G_P$, is a digraph whose vertex set is the elements of $P$ and $(a, b)$ is an edge of $G_P$ if and only if $b$ covers $a$ in $P$; that is, $a \prec_P b$ and there is no $c \in P$ such that $a \prec_P c \prec_P b$.

We now define our vertex-colouring of an edge-coloured digraph $G$, which plays an essential role in this paper.
Definition 3.1. A proper vertex-colouring of an edge-coloured digraph $G$ is a function $\kappa$ from $V(G)$ to $\mathbb{P}$ such that for any edge $(a,b)$ in $E(G)$,

1. If $a \rightarrow b$, then $\kappa(a) \neq \kappa(b)$.
2. If $a \rightarrow b$, then $\kappa(a) < \kappa(b)$.
3. If $a \Rightarrow b$, then $\kappa(a) \leq \kappa(b)$.

Let $\mathcal{C}(G)$ be the set of all proper vertex-colourings of the edge-coloured digraph $G$.

Remark. Note that for some edge-coloured digraphs $G$, $\mathcal{C}(G)$ is empty.

![Figure 6. An edge-coloured digraph $G$ with $\mathcal{C}(G) = \emptyset$](image)

We next see that each $P$-partition corresponds to a proper vertex-colouring of $G_P$ where its edges are solid or double.

Proper colourings and $P$-partitions. A labelled poset is a partially ordered set $P$ whose underlying set is some finite subset of positive integers. A $P$-partition is a function $f : P \rightarrow \mathbb{P}$ that satisfies

1. If $a \in P$ and $b \in P$ with $a <_P b$ and $a <_Z b$, then $f(a) \leq f(b)$.
2. If $a \in P$ and $b \in P$ with $a <_P b$ and $a >_Z b$, then $f(a) < f(b)$.

The definitions of $P$-partitions and proper vertex-colourings of edge-coloured digraphs yield the following proposition.

Proposition 3.2. Let $P$ be a poset, and let $G_P$ be the digraph associated to $P$. Then there is a bijection between the set of all $P$-partitions and all proper colourings of the edge-coloured digraph $G$ where the digraph $G$ is isomorphic to $G_P$ and

1. $a \Rightarrow b$ in $G$ if $a <_P b$ and $a <_Z b$.
2. $a \rightarrow b$ in $G$ if $a <_P b$ and $a >_Z b$.

4. Generalized chromatic functions

We now introduce our main object of study.

Definition 4.1. (The generalized chromatic function of an edge-coloured digraph) Let $G$ be an edge-coloured digraph. Define

$$X_G(x,t) = \sum_{\kappa \in \mathcal{C}(G)} t^{\text{asc}(\kappa)} x_{\kappa},$$

where

$$x_{\kappa} = \prod_{a \in V(G)} x_{\kappa(a)} \quad \text{and} \quad \text{asc}(\kappa) = |\{(a,b) \in E(G) : \kappa(a) < \kappa(b)\}|.$$

The power series

$$X_G(x) = X_G(x,1)$$

is called the generalized chromatic function of $G$.

Remark. Note that $X_G(x,t) = 0$ when $\mathcal{C}(G) = \emptyset$. 
If $G$ is an edge-coloured digraph and $a \rightarrow b$ is an edge of $G$, then the generalized chromatic function of $G$ is the sum of the generalized chromatic functions of two edge-coloured digraphs, one is obtained by deleting $a \rightarrow b$ in $G$ and replacing it with $a \rightarrow b$ and the other is obtained by deleting $a \rightarrow b$ in $G$ and replacing it with $b \rightarrow a$; that is

$$X_G(x) = X_{G-\rightarrow_{a\rightarrow b}+\rightarrow_{b\rightarrow a}}(x) + X_{G-\rightarrow_{a\rightarrow b}+\rightarrow_{b\rightarrow a}}(x).$$

For vertices $a$ and $b$ of $G$, write $a \sim b$ if $a = b$ or there is a directed cycle $C$ with double edges in $G$ and $a, b \in V(C)$. Consider that the transitive closure of this relation gives an equivalence relation on the vertices of $G$. Let $[a_1], [a_2], \ldots, [a_s]$ be all equivalence classes. We have

$$X_G(x) = \sum_{\kappa \in \mathcal{C}(G)} x_{\kappa(a_1)} x_{\kappa(a_2)} \cdots x_{\kappa(a_s)}.$$}

Thus we can realize each generalized chromatic function as a sum of generating functions of weighted $P$-partitions [2, Section 3], and so the generalized chromatic function of an edge-coloured digraph is a quasisymmetric function, which we now state as a proposition.

**Proposition 4.2.** Let $G$ be an edge-coloured digraph. Then $X_G(x) \in \text{QSym}$.  

For example, if $G$ is the following edge-coloured digraph

```
    a1 -- a2 -- a3
       \   /    \
        \ /     \\
      a4 -- a5 -- a6 -- a7
```

then $[a_1] = \{a_1, a_2\}$, $[a_3] = \{a_3\}$, and $[a_4] = \{a_4, a_5, a_6, a_7\}$ are the equivalence classes, and

$$X_G(x) = X_{G-\rightarrow_{a_2\rightarrow a_3}+\rightarrow_{a_2\rightarrow a_3}}(x) + X_{G-\rightarrow_{a_2\rightarrow a_3}+\rightarrow_{a_3\rightarrow a_2}}(x).$$

**5. Other chromatic functions**

Let $H = (V(H), E(H))$ be a finite simple graph. Throughout this paper, all graphs are simple, and we usually use $H$ to denote a simple graph. A proper vertex-colouring of $H$ is a function $\kappa$ from $V(H)$ to $\mathbb{P}$ such that if the vertices $a$ and $b$ are adjacent, then $\kappa(a) \neq \kappa(b)$. The set of all proper vertex-colourings of $H$ is denoted by $\mathcal{C}(H)$.

**Stanley’s chromatic symmetric function** [24]: For a graph $H$, the chromatic symmetric function of $H$ is

$$X_H(x) = \sum_{\kappa \in \mathcal{C}(H)} x_{\kappa},$$

where

$$x_{\kappa} = \prod_{a \in V(H)} x_{\kappa(a)}.$$

By definition we have the following.

**Proposition 5.1.** Let $H$ be a graph. Then

$$X_H(x) = X_G(x)$$

where the underlying graph of $G$, $\overline{G}$, is isomorphic to $H$, and $G$ only has dashed edges.
Crew-Spirkl’s extended chromatic symmetric function [7]: A weighted graph is a pair \((H, \text{wt})\) where \(H\) is a graph and \(\text{wt} : V(H) \rightarrow \mathbb{P}\) is a vertex-weight function. The extended chromatic symmetric function of \((H, \text{wt})\) is

\[
X_{(H, \text{wt})}(x) = \sum_{\kappa \in C(H)} x_{\kappa}^{\text{wt}},
\]

where

\[
x_{\kappa}^{\text{wt}} = \prod_{a \in V(H)} x_{\kappa(a)}^{\text{wt}(a)}.\]

By definition we have the following.

**Proposition 5.2.** Let \((H, \text{wt})\) be a weighted graph. Fix a vertex \(v(a) \in V(C_{\text{wt}(a)})\). Define

\[
G = \bigsqcup_{a \in V(H)} C_{\text{wt}(a)} + \{v(a) \rightarrow v(b) : ab \in E(H)\}.
\]

Then

\[
X_{(H, \text{wt})}(x) = \mathcal{R}_G(x).
\]

Shareshian-Wachs’ chromatic quasisymmetric function [22]: For a graph \(H\) with \(V(H)\) a subset of \(\mathbb{P}\), the chromatic quasisymmetric function of \(H\) is

\[
X_H(x, t) = \sum_{\kappa \in C(H)} t^{\text{asc}(\kappa)} x_{\kappa},
\]

where

\[
x_{\kappa} = \prod_{a \in V(H)} x_{\kappa(a)} \quad \text{and} \quad \text{asc}(\kappa) = |\{ab \in E(H) : a < b \text{ and } \kappa(a) < \kappa(b)\}|.
\]

By definition we have the following.

**Proposition 5.3.** Let \(H\) be a graph such that \(V(H)\) is a subset of \(\mathbb{P}\). Then

\[
X_H(x, t) = \mathcal{Z}_G(x, t)
\]

where the underlying graph of \(G, \overline{G}\), is isomorphic to \(H\), and \(G\) has only dashed edges where \(E(G) = \{(a, b) : ab \in E(H), a < b\}\).

Eltzey’s chromatic quasisymmetric function [8]: Let \(G\) be a digraph. A proper vertex-colouring of \(G\) is a vertex-colouring of \(G\) such that the colours of adjacent vertices are different. Given a proper vertex-colouring \(\kappa\) of \(G\), one defines an ascent of \(\kappa\) to be a directed edge \((a, b) \in E(G)\) with \(\kappa(a) < \kappa(b)\), and let \(\text{asc}(\kappa)\) denote the number of ascents of \(\kappa\). Then the chromatic quasisymmetric function of \(G\) is

\[
Z_G(x, t) = \sum_{\kappa \in C(G)} t^{\text{asc}(\kappa)} x_{\kappa},
\]

where

\[
x_{\kappa} = \prod_{a \in V(G)} x_{\kappa(a)}.
\]

By definition we have the following.
Proposition 5.4. Let $G$ be a digraph. Then
$$Z_G(x,t) = \mathcal{X}_G(x,t)$$
where $G$ has only dashed edges.

Remark. It is known that Shareshian-Wachs’ chromatic quasisymmetric functions of graphs and Ellzey’s chromatic quasisymmetric functions of digraphs are symmetric functions when $t = 1$. However, the generalized chromatic function $\mathcal{X}_G(x)$ is a quasisymmetric function (not necessarily a symmetric function) as seen in Proposition 4.2. For instance, in Section 7, we show that many bases of $QSym$ can be realized by the generalized chromatic functions of edge-coloured digraphs.

Recall that an orientation of an undirected graph $H$ is a digraph $G$ with the same vertices, so that for every edge $ab$ of $H$, exactly one of $(a,b)$ and $(b,a)$ is an edge of $G$. A weak cycle of an orientation $G$ of $H$ is a subgraph $C$ of $G$ such that the underlying graph of $C$, $\overline{C}$, is a cycle in $H$. For $k \geq 1$, $G$ is $k$-balanced if for any weak cycle $C$ of $G$ with $E(\overline{C}) = \{a_ia_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{a_1a_n\}$, there are at least $k$ edges in $G$ of the form $(a_i, a_{i+1})$ and at least $k$ edges of the form $(a_{i+1}, a_i)$ (subscripts are taken modulo $n$).

Let $\kappa : V(H) \to \mathbb{P}$ be a proper vertex-colouring of $H$. Then the orientation induced by $\kappa$ is the orientation $G_\kappa$ where each edge is directed towards the vertex with the greater colour. If $G_\kappa$ is $k$-balanced, then $\kappa$ is called a $k$-balanced colouring.

**Humpert’s $k$-balanced chromatic quasisymmetric function** [17]: Given a graph $H$ with $n$ vertices and any positive integer $k$, the $k$-balanced chromatic quasisymmetric function of $H$ is
$$X_H^k(x) = \sum x_\kappa,$$
where the sum runs over all $k$-balanced colourings $\kappa : V(H) \to \mathbb{P}$ and
$$x_\kappa = \prod_{a \in V(H)} x_{\kappa(a)}.$$

By definition we have the following.

**Proposition 5.5.** Let $H$ be a graph and $k$ be a positive integer. Then
$$X_H^k(x) = \sum \mathcal{X}_G(x)$$
where the sum runs over all digraphs $G$, such that $G$ is a $k$-balanced orientation of $H$ and all edges are solid.

### 6. Product and coproduct formulas for generalized chromatic functions

We now establish the product and coproduct formulas for generalized chromatic functions.

**Proposition 6.1.** Let $G_1$ and $G_2$ be edge-coloured digraphs. Then
$$\mathcal{X}_{G_1}(x,t) \mathcal{X}_{G_2}(x,t) = \mathcal{X}_{G_1 \oplus G_2}(x,t).$$
Proof. Note that by definition

$$X_{G_1}(x,t)X_{G_2}(x,t) = \sum t^{asc(\kappa_1)}t^{asc(\kappa_2)}x_{\kappa_1}x_{\kappa_2},$$

where the sum runs over all elements of the set

$$\{(\kappa_1, \kappa_2) : \kappa_1 \in C(G_1), \kappa_2 \in C(G_2)\}.$$ 

For each \(\kappa \in C(G_1 \cup G_2)\), there are \(\kappa_1 \in C(G_1)\) and \(\kappa_2 \in C(G_2)\) such that for each vertex \(a\) of \(G_1 \cup G_2\), \(\kappa_1(a) = \kappa(a)\) if \(a \in V(G_1)\), and \(\kappa_2(a) = \kappa(a)\) if \(a \in V(G_2)\). The map that sends \(\kappa\) to \((\kappa_1, \kappa_2)\) is a bijection between \(C(G_1 \cup G_2)\) and \(C(G_1) \times C(G_2)\). Moreover,

$$t^{asc(\kappa)}x_{\kappa} = t^{asc(\kappa_1)}t^{asc(\kappa_2)}x_{\kappa_1}x_{\kappa_2}.$$ 

Therefore,

$$X_{G_1}(x,t)X_{G_2}(x,t) = \sum_{\kappa \in C(G_1 \cup G_2)} t^{asc(\kappa)}x_{\kappa} = X_{G_1 \cup G_2}(x,t).$$

To give our coproduct formula, we first need to introduce some notation. Let \(G\) be an edge-coloured digraph. An induced subdigraph \(F = (V(F), E(F))\) of \(G\) is a subdigraph of \(G\) whose edge set \(E(F)\) is the set of all edges \((a, b)\) of \(G\) with \(a, b \in V(F)\).

For a subset \(A\) of \(V(G)\), let \(G|_A\) be the induced subdigraph of \(G\) with vertex set \(A\) where the colours of edges of \(G|_A\) are the same as their colours in \(G\). An edge-coloured subdigraph \(F\) of \(G\) is called a \(\rightarrow, \Rightarrow\)-induced subdigraph of \(G\) if \(F\) is an induced subdigraph of \(G\) and if \(a \in V(F)\) and either \(a \rightarrow b\) or \(a \Rightarrow b\) in \(G\), then \(b \in V(F)\).

Example 6.2. Let \(G\) be the following edge-coloured digraph

\begin{figure}
\centering
\begin{tikzpicture}
\node[vertex] (v1) at (0,0) {$v_1$};
\node[vertex] (v2) at (1,0) {$v_2$};
\node[vertex] (v3) at (0,-1) {$v_3$};
\node[vertex] (v4) at (1,-1) {$v_4$};
\draw[->] (v1) to (v2);
\draw[-] (v1) to (v3);
\draw[->] (v2) to (v4);
\draw[-] (v3) to (v4);
\end{tikzpicture}
\end{figure}

then the \(\rightarrow, \Rightarrow\)-induced subdigraphs of \(G\) are

\begin{align*}
&v_1 \rightarrow v_2, \quad v_1 \rightarrow v_2, \quad v_2 \rightarrow v_2, \\
&v_1 \rightarrow v_3, \quad v_3 \rightarrow v_3, \\
&v_2 \rightarrow v_3, \quad v_3 \rightarrow v_3, \\
&\emptyset.
\end{align*}

We now give the coproduct formula for \(\mathcal{X}_G(x)\).

Theorem 6.3. Let \(G\) be an edge-coloured digraph. Then

$$\Delta(\mathcal{X}_G(x)) = \sum \mathcal{X}_{G|_{V(G) \setminus V(F)}}(x) \otimes \mathcal{X}_F(x),$$

where the sum runs over all \(\rightarrow, \Rightarrow\)-induced subdigraphs \(F\) of \(G\).
**Proof.** Recall that the coproduct of $\text{QSym}(x)$ can be seen as the composite of the following functions

$$\text{QSym}(x) \cong \text{QSym}(x, y) \to \text{QSym}(x) \otimes \text{QSym}(y) \cong \text{QSym}(x) \otimes \text{QSym}(x),$$

where $\text{QSym}(x) \cong \text{QSym}(x, y)$ is defined by $f(x_1, x_2, \ldots) \mapsto f(x_1, x_2, \ldots, y_1, y_2, \ldots)$. We have

$$\Delta_G(x, y) = \sum_{\kappa} \prod_{a \in V(G)} x_{\kappa(a)}$$

where the sum runs over all functions $\kappa$ from $V(G)$ to the alphabet

$$x_1 < x_2 < \cdots < y_1 < y_2 < \cdots$$

such that when $(a, b)$ is in $E(G)$,

1. If $a \rightarrow b$, then $\kappa(a) \neq \kappa(b)$.
2. If $a \rightarrow b$, then $\kappa(a) < \kappa(b)$.
3. If $a \Rightarrow b$, then $\kappa(a) \leq \kappa(b)$.

For the induced subdigraph $F$ with the vertex set $\{a : \kappa(a) \in \{y_1, y_2, \ldots\}\}$, we see that if $a \in V(F)$ and either $a \rightarrow b$ or $a \Rightarrow b$ in $G$, then $\kappa(a) \leq \kappa(b)$, and so $b \in V(F)$. Therefore, $F$ is a $\{\rightarrow, \Rightarrow\}$-induced subdigraph of $G$. Also, note that the rest of vertices produce the edge coloured digraph $G_{|V(G) - V(F)}$. Applying the above composite, we have

$$\Delta_G(x, y) = \sum \Delta_{G_{|V(G) - V(F)}}(x) \otimes \Delta_F(y) = \sum \Delta_{G_{|V(G) - V(F)}}(x) \otimes \Delta_F(x),$$

where the sums run over all $\{\rightarrow, \Rightarrow\}$-induced subdigraphs $F$ of $G$. □

**Remark.** Theorem 6.3 is a generalization of [2, Proposition 3.4]. Moreover, when $G$ is an edge-coloured digraph whose edges are all dashed, then the generalized chromatic function $\Delta_G(x)$ is equal to Stanley’s chromatic symmetric function $\Delta_{\text{ch}}(x)$, so this coproduct formula will give a coproduct formula for Stanley’s chromatic symmetric functions.

**Example 6.4.**

\[
\Delta(\Delta_{\text{ch}}) = 1 \otimes \Delta_{\text{ch}} + \Delta_{\text{ch}}(x) \otimes \Delta_{\text{ch}}(x) + \Delta_{\text{ch}}(x) \otimes \Delta_{\text{ch}}(x) + \Delta_{\text{ch}}(x) \otimes \Delta_{\text{ch}}(x)
\]

\[
\Delta(\Delta_{\text{ch}}) = 1 \otimes \Delta_{\text{ch}} + \Delta_{\text{ch}}(x) \otimes \Delta_{\text{ch}}(x) + \Delta_{\text{ch}}(x) \otimes \Delta_{\text{ch}}(x) + \Delta_{\text{ch}}(x) \otimes \Delta_{\text{ch}}(x)
\]
7. Bases for \( \text{QSym}^r(x) \) using generalized chromatic functions

In this section, we realize different bases for the Hopf algebras in the following chain as special cases of the generalized chromatic functions of certain edge-coloured digraphs.

\[
\text{QSym}(x) = \text{QSym}^1(x) \supset \text{QSym}^2(x) \supset \cdots \supset \text{QSym}^\infty(x) = \text{Sym}(x)
\]

Given a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n \), the edge-coloured digraph \( G_\lambda \) is an edge-coloured digraph where

1. The vertex set of \( G \) is \( \{(i, j) : 1 \leq i \leq l, 1 \leq j \leq \lambda_i \} \).
2. \( (i, j) \rightarrow (i', j') \) in \( G \) if and only if \( i + 1 = i' \) and \( j = j' \).
3. \( (i, j) \Rightarrow (i', j') \) in \( G \) if and only if \( i = i' \) and \( j + 1 = j' \).

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models [n] \) and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) = (n^{m_1}, (n-1)^{m_{n-1}}, \ldots, 1^{m_1}) \vdash n \). In the following table, we see that many well-known bases of \( \text{Sym}(x) \) and \( \text{QSym}(x) \) are the generalized chromatic functions of some edge-coloured digraphs produced by the actions of the operators

\[ \mathbin{\biguplus}, \mathbin{\bigodot}, \mathbin{\bigtriangledown}, \mathbin{\bigsqcup} \]

on the edge-coloured digraphs in Table 1. For these well-known bases, the result follows immediately by definition, and hence readers unfamiliar with the classical definitions may take these to be definitions, or refer to [23] and for the upper-fundamental basis of \( \text{QSym}(x) \) take the commutative image of the noncommutative upper-fundamental basis in [9].

<table>
<thead>
<tr>
<th>Basis</th>
<th>( G )</th>
<th>( \mathcal{R}_G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monomial basis of ( \text{Sym}(x) )</td>
<td>( \bigodot_{j=1}^n \left( \bigoplus_{i=1}^m C_j \right) )</td>
<td>( m_\lambda )</td>
</tr>
<tr>
<td>Augmented monomial basis of ( \text{Sym}(x) )</td>
<td>( \bigoplus_{i=1}^l C_\lambda_i )</td>
<td>( \tilde{m}<em>\lambda = \lambda! m</em>\lambda )</td>
</tr>
<tr>
<td>Elementary basis of ( \text{Sym}(x) )</td>
<td>( \biguplus_{i=1}^l P_\lambda_i )</td>
<td>( e_\lambda )</td>
</tr>
<tr>
<td>Augmented elementary basis of ( \text{Sym}(x) )</td>
<td>( \biguplus_{i=1}^l K_\lambda_i )</td>
<td>( \lambda! e_\lambda )</td>
</tr>
<tr>
<td>Complete homogeneous basis of ( \text{Sym}(x) )</td>
<td>( \biguplus_{i=1}^l Q_\lambda_i )</td>
<td>( h_\lambda )</td>
</tr>
<tr>
<td>Power sum basis of ( \text{Sym}(x) )</td>
<td>( \biguplus_{i=1}^l C_\lambda_i )</td>
<td>( p_\lambda )</td>
</tr>
<tr>
<td>Schur basis of ( \text{Sym}(x) )</td>
<td>( G_\lambda )</td>
<td>( s_\lambda )</td>
</tr>
<tr>
<td>Monomial basis of ( \text{QSym}(x) )</td>
<td>( \bigodot_{i=1}^k C_\alpha_i )</td>
<td>( M_\alpha )</td>
</tr>
<tr>
<td>Fundamental basis of ( \text{QSym}(x) )</td>
<td>( \bigoplus_{i=1}^k Q_\alpha_i )</td>
<td>( F_\alpha )</td>
</tr>
<tr>
<td>Upper-fundamental basis of ( \text{QSym}(x) )</td>
<td>( \bigoplus_{i=1}^k C_\alpha_i )</td>
<td>( \overline{F}_\alpha )</td>
</tr>
</tbody>
</table>

Table 2. Bases for \( \text{Sym}(x) \) and \( \text{QSym}(x) \) reinterpreted
Remark. If we generalize the edge-coloured digraph $G_{\lambda}$ for a partition $\lambda$ to $G_{\alpha}$ for a composition $\alpha$ in the natural way, but restrict the second condition to only the first column, then by definition we have
\[(7.1) \quad \mathcal{X}_{G_{\alpha}}(x) = \mathcal{S}_{\alpha}^*\]
where $\mathcal{S}_{\alpha}^*$ is the dual immaculate function indexed by $\alpha$ [4], and switching the $\rightarrow$ and $\Rightarrow$ and vice versa in $G_{\alpha}$ we obtain the row-strict dual immaculate function $\mathcal{R}\mathcal{S}_{\alpha}^*$ [20].

Let $G$ be an edge-coloured digraph. Recall that $P(G, p)$ is the number of ways that one can properly colour the vertices of $G$ with $p$ colours. The following theorem shows that $P(G, p)$ is a polynomial in $p$.

**Theorem 7.1.** For any edge-coloured digraph $G$, $P(G, p)$ is a polynomial in $p$.

**Proof.** Note that when we set the first $p$ variables of $\mathcal{X}_{G}(x)$ equal to 1 and all others to 0, we have $P(G, p)$. Since $\mathcal{X}_{G}(x)$ is a quasisymmetric function by Proposition 4.2, it is a linear combination of the monomial basis elements of $\text{QSym}(x)$. Thus, we only need to show that for any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$,
\[\mathcal{X} \bigotimes_{i=1}^{k} \mathcal{C}_{\alpha_i} (1, 1, \ldots, 1, 0, 0, \ldots)\]
is a polynomial in $p$. Note that
\[\mathcal{X} \bigotimes_{i=1}^{k} \mathcal{C}_{\alpha_i} (1, 1, \ldots, 1, 0, 0, \ldots)\]
is equal to the number of strictly increasing sequences of length $k$ with elements in $[p]$. This number is equal to $\binom{p}{k}$, which is a polynomial in $p$. \qed

In [6], Cho and van Willigenburg constructed an infinite family of bases using chromatic symmetric functions of graphs. In the following theorem, we construct an infinite family of bases for $\text{QSym}(x)$ using generalized chromatic functions.

**Theorem 7.2.** Let $F_i$ be an edge-coloured digraph with $i$ vertices whose edges are of the form $\Rightarrow$. For each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$, define
\[F_{\alpha} = F_{\alpha_1} \bigoplus F_{\alpha_2} \bigoplus \cdots \bigoplus F_{\alpha_k}.\]
Then
\[\{ \mathcal{X}_{F_{\alpha}}(x) : \alpha \models n \}\]
is a basis for $\text{QSym}_n(x)$.

**Proof.** Writing $\mathcal{X}_{F_{\alpha}}(x)$ in terms of the monomial basis of $\text{QSym}(x)$, we have
\[\mathcal{X}_{F_{\alpha}}(x) = M_{\alpha} + \sum_{\beta \succ \alpha} c_{\alpha, \beta} M_{\beta}\]
for some coefficients $c_{\alpha, \beta}$. Since each $\mathcal{X}_{F_{\alpha}}(x)$ has a unique leading term $M_{\alpha}$ under the $<$ order, we can conclude that $\{ \mathcal{X}_{F_{\alpha}}(x) : \alpha \models n \}$ is a basis for $\text{QSym}_n(x)$. \qed

**Example 7.3.** If $F_i = C_i$, then $\{ \mathcal{X}_{F_{\alpha}}(x) \}$ is the monomial basis of $\text{QSym}(x)$. If $F_i = Q_i$, then $\{ \mathcal{X}_{F_{\alpha}}(x) \}$ is the fundamental basis of $\text{QSym}(x)$. 

We now establish several new bases for $QSym^r(x)$. Let $r$ be a positive integer or infinity. Let $(\beta, \mu) = ((\beta_1, \beta_2, \ldots, \beta_k), (\mu_1, \mu_2, \ldots, \mu_l))$ be an $r$-composition. Note that by definition we have that

$$M_{(\beta, \mu)} = \mathcal{X} \left( \bigotimes_{i=1}^{k} C_{\beta_i} \bigotimes \left( \bigotimes_{j=1}^{l} C_{\mu_j} \right) \right)(x),$$

and define

$$S_{(\beta, \mu)} = \mathcal{X}^* \left( \bigotimes_{i=1}^{k} C_{\beta_i} \bigotimes G_{\mu} \right)(x),$$

$$F_{(\beta, \mu)} = \mathcal{X}^* \left( \bigotimes_{i=1}^{k} C_{\beta_i} \bigotimes \left( \bigotimes_{j=1}^{l} C_{\mu_j} \right) \right)(x),$$

$$S_{(\beta, \mu)} = \mathcal{X}^* \left( \bigotimes_{i=1}^{k} C_{\beta_i} \bigotimes G_{\mu} \right).$$

**Theorem 7.4.** Each of the following is a basis for $QSym^r_n(x)$.

1. \( \{ M_{(\beta, \mu)} : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \).
2. \( \{ S_{(\beta, \mu)} : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \).
3. \( \{ F_{(\beta, \mu)} : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \).
4. \( \{ S_{(\beta, \mu)} : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \).

**Proof.** For the first part, note that this is the $r$-dominant monomial basis of $QSym^r_n(x)$.

For the second part, consider that $X G_{\mu}(x)$ is the Schur function $s_{\mu}$, thus

$$X G_{\mu} = s_{\mu} = \sum_{\nu \leq \mu} K_{\mu, \nu} m_{\nu},$$

where it is well known that $K_{\mu, \nu}$ is the Kostka number satisfying $K_{\mu, \mu} = 1$ and $K_{\mu, \nu} = 0$ if $\nu \succ \mu$. Thus,

$$S_{(\beta, \mu)} = \mathcal{X}^* \left( \bigotimes_{i=1}^{k} C_{\beta_i} \bigotimes G_{\mu} \right)(x) = \sum_{\nu \leq \mu} c_{(\beta, \nu)} M_{(\beta, \nu)}$$

for some coefficient $c_{(\beta, \nu)}$ such that $c_{(\beta, \mu)} \neq 0$.

Consequently, \( \{ S_{(\beta, \mu)} : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \) is a basis for $QSym^r_n(x)$.

For the third part, consider that

$$\mathcal{X} \left( \bigotimes_{i=1}^{k} C_{\beta_i} \bigotimes G_{\mu} \right) = F_{\beta} = \sum_{\gamma \leq \beta} M_{\gamma},$$

Therefore,

$$F_{(\beta, \mu)} = \mathcal{X}^* \left( \bigotimes_{i=1}^{k} C_{\beta_i} \bigotimes \left( \bigotimes_{j=1}^{l} C_{\mu_j} \right) \right)(x) = \sum_{\gamma \leq \beta} M_{(\gamma, \mu)}.$$ 

Thus, \( \{ F_{(\beta, \mu)} : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \) is a basis for $QSym^r_n(x)$.

Lastly, note that

$$S_{(\beta, \mu)} = \mathcal{X}^* \left( \bigotimes_{i=1}^{k} C_{\beta_i} \bigotimes G_{\mu} \right)(x) = \sum_{\gamma \leq \beta, \nu \leq \mu} c_{(\gamma, \nu)} M_{(\gamma, \nu)},$$

where it is well known that $K_{\mu, \nu}$ is the Kostka number satisfying $K_{\mu, \mu} = 1$ and $K_{\mu, \nu} = 0$ if $\nu \succ \mu$.
and $c_{(\beta, \mu)} \neq 0$. Thus, $\{S_{(\beta, \mu)} : (\beta, \mu) \text{ is an } r\text{-composition of } n\}$ is a basis for $\text{QSym}_n^r(x)$.

\[\square\]

**Remark.** Note that the basis $S_{(\beta, \mu)}$ comes from the monomial basis of $\text{QSym}(x)$ and the Schur basis of $\text{Sym}(x)$, and $\overline{S}(\beta, \mu)$ comes from the upper-fundamental basis of $\text{QSym}(x)$ and the Schur basis of $\text{Sym}(x)$; however, we do not have a basis that comes from the fundamental basis of $\text{QSym}(x)$ and the Schur basis of $\text{Sym}(x)$ since if $M_\alpha$ appears in the linear expansion of the fundamental basis element $F_\beta$, then $\beta$ coarsens $\alpha$.

8. Symmetric functions and generalizations in noncommuting variables

This section introduces the Hopf algebras of symmetric, quasisymmetric, and $r$-quasisymmetric functions in noncommuting variables. The bases of these Hopf algebras are indexed by set partitions, set compositions, and $r$-set-compositions, respectively. We recall the definitions and notation related to these combinatorial objects.

8.1. Set partitions, set compositions, and $r$-set-compositions. Given a finite subset $A$ of $\mathbb{P}$, the standardization of $A$ is the unique order preserving bijection

$$\text{std}_A : A \to [|[A]|].$$

A set partition $\Pi$ of a set $A$ is a set consisting of mutually disjoint nonempty subsets $\Pi_1, \Pi_2, \ldots, \Pi_l$ of $A$ such that their union is $A$; this is denoted by $\Pi = \Pi_1/\Pi_2/\cdots/\Pi_l \vdash A$. Each $\Pi_i$ is called a block of the set partition $\Pi$, and the length of $\Pi$ is $l$. By convention, we denote by $\emptyset$ the unique empty set partition of $[0] = \emptyset$. Let $\lambda(\Pi) = ([|\Pi_1|], [|\Pi_2|], \ldots, [|\Pi_l|])$ where we assume that $|\Pi_1| \geq |\Pi_2| \geq \cdots \geq |\Pi_l|$. We say $\Pi$ is of shape $\lambda$ if $\lambda(\Pi) = \lambda$. The standardization of $\Pi$, $\text{std}(\Pi)$, is a set partition of $[|[A]|]$ such that $\text{std}_A(a)$ and $\text{std}_A(b)$ are in the same block of $\text{std}(\Pi)$ if and only if $a$ and $b$ are in the same block of $\Pi$. For example, $\Pi = 35/67/9 \vdash \{3, 5, 6, 7, 9\}$, with length $3$, $\lambda(\Pi) = (2, 2, 1)$, and $\text{std}(\Pi) = 12/34/5$.

A set composition $\Phi$ of a finite set $A$ is the list of mutually disjoint nonempty subsets $\Phi_1, \Phi_2, \ldots, \Phi_k$ of $A$ such that their union is $A$; this is denoted by $(\Phi_1|\Phi_2|\cdots|\Phi_k) \vdash A$. Each $\Phi_i$ is called a block of the set composition $\Phi$, and the length of $\Phi$ is $k$. By convention, we denote by $\emptyset$ the unique empty set composition of $[0] = \emptyset$. Let $\alpha(\Phi)$ be the composition $(|\Phi_1|, |\Phi_2|, \ldots, |\Phi_k|)$. We say $\Phi$ is of shape $\alpha$ if $\alpha(\Phi) = \alpha$. The standardization of $\Phi$, $\text{std}(\Phi)$, is a set composition of $[|[A]|]$ such that $\text{std}_A(a)$ and $\text{std}_A(b)$ are in the block $\text{std}(\Phi)$, if and only if $a$ and $b$ are in the block $\Phi_i$. For example, $\Phi = (35|9|67) \vdash \{3, 5, 6, 7, 9\}$, with length $3$, $\alpha(\Phi) = (2, 1, 2)$, and $\text{std}(\Phi) = (12|5|34)$. We say a set composition $\Psi$ corrupts $\Phi$ if $\Psi$ is $\Phi$ with some bars removed, and say that $\Phi$ reforms $\Psi$ if $\Psi$ is $\Phi$ with some bars added. In particular, the numbers of both must be written in the same order. For example, $(13|2|456)$ corrupts $(13|2|4|56)$, and $(13|2|4|56)$ reforms $(13|2|4|56)$ but $(3|1|2|4|56)$ does not reform $(13|2|4|56)$. For neatness, once our calculations are completed, we write the elements of each block in increasing order.

Let $r$ be a positive integer or infinity. Let $B$ be a finite subset of $\mathbb{P}$, and let $A$ be a subset of $B$. A pair $(\Phi, \Pi)$ where $\Phi$ is a set composition of $A$ and $\Pi$ is a set partition of $B - A$ is called an $r$-set-composition of $B$ if $(\alpha(\Phi), \lambda(\Pi))$ is an $r$-composition. The standardization of an $r$-set-composition $(\Phi, \Pi)$ of $B$ is an $r$-set-composition $\text{std}(\Phi, \Pi) = (\Psi, \Omega)$ such that
(1) std\(B(a)\) and std\(B(b)\) are in the block \(\Psi_i\) of \(\Psi\) if and only if \(a\) and \(b\) are in the block \(\Phi_i\) of \(\Phi\).

(2) std\(B(a)\) and std\(B(b)\) are in the same block of \(\Omega\) if and only if \(a\) and \(b\) are in the block of \(\Pi\).

For example, \(((36|29), 4/5/8)\) is a 2-set-composition of \(\{2, 3, 4, 5, 6, 8, 9\}\) and we have \(\text{std}((36|29), 4/5/8) = ((25|17), 3/4/6)\).

8.2. Quasisymmetric functions in noncommuting variables. The Hopf algebra of quasisymmetric functions in noncommuting variables appeared in [16]. They are realized as power series in noncommuting variables, and the bases of this Hopf algebra are indexed by set compositions.

Let \(\mathbb{Q}\langle\langle x_1, x_2, \ldots \rangle\rangle\) be the algebra of formal power series in infinitely many noncommuting variables \(x = x_1, x_2, \ldots\) over \(\mathbb{Q}\).

**Definition 8.1.** A quasisymmetric function in noncommuting variables is a formal power series \(f \in \mathbb{Q}\langle\langle x_1, x_2, \ldots \rangle\rangle\) such that

1. The degrees of the monomials in \(f\) are bounded.
2. For every set composition \(\Phi = (\Phi_1|\Phi_2| \cdots |\Phi_k) \subseteq [n]\), all monomials \(x_{i_1}x_{i_2}\cdots x_{i_n}\) in \(f\) satisfying
   a. \(i_j = i_\ell\) if \(j\) and \(\ell\) are in the same block of \(\Phi\) and
   b. \(i_j < i_\ell\) if \(j \in \Phi_p\) and \(\ell \in \Phi_q\) with \(p < q\) have the same coefficient.

The set of all quasisymmetric functions in noncommuting variables is denoted by \(\text{NCQSym}(x)\).

The vector space \(\text{NCQSym}(x)\) is a Hopf algebra where its product is the same as the product of the formal power series in noncommuting variables, and its coproduct \(\Delta\) is defined as follows (for more details see [5, Section 5]). Evaluate an element \(f(x) \in \text{NCQSym}(x) \cong \text{NCQSym}(x, y)\) as \(f(x, y)\) using the linearly ordered noncommuting variables \((x, y) = (x_1 < x_2 < \cdots < y_1 < y_2 < \cdots)\). Denote by \(\bar{f}(x, y)\) the image of \(f(x, y)\) after imposing the partial commutativity relations

\[ x_i y_j = y_j x_i \quad \text{for every pair } (x_i, y_j) \in x \times y. \]

We have that \(\bar{f}(x, y)\) lies in a subalgebra isomorphic to \(\text{NCQSym}(x) \otimes \text{NCQSym}(y)\). Let the image of \(\bar{f}(x, y)\) in \(\text{NCQSym}(x) \otimes \text{NCQSym}(y)\) be

\[ \sum \bar{f}_1(x) \otimes \bar{f}_2(y). \]

Applying the following isomorphism

\[ \text{NCQSym}(x) \otimes \text{NCQSym}(y) \cong \text{NCQSym}(x) \otimes \text{NCQSym}(x), \]

we have

\[ \sum \bar{f}_1(x) \otimes \bar{f}_2(y) \mapsto \sum \bar{f}_1(x) \otimes \bar{f}_2(x). \]

Now we define \(\Delta : \text{NCQSym}(x) \rightarrow \text{NCQSym}(x) \otimes \text{NCQSym}(x)\) such that

\[ \Delta(f(x)) = \sum \bar{f}_1(x) \otimes \bar{f}_2(x). \]
8.3. **Symmetric functions in noncommuting variables.** The Hopf algebra of symmetric functions in noncommuting variables is a Hopf subalgebra of NCQSym(\(x\)).

**Definition 8.2.** A symmetric function in noncommuting variables is a formal power series \(f \in \mathbb{Q}\langle\langle x_1, x_2, \ldots \rangle\rangle\) such that

1. The degrees of the monomials in \(f\) are bounded.
2. For any permutation \(\sigma \in \mathcal{S}_\infty\),

\[
\sigma.f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots).
\]

The set of all symmetric functions in noncommuting variables is denoted by \(NCSym(x)\).

8.4. **\(r\)-quasisymmetric functions in noncommuting variables.** We now introduce the new Hopf algebra of \(r\)-quasisymmetric functions.

For each \(r\)-set-composition \((\Phi, \Pi)\) of \([n]\), define the \(r\)-dominant monomial function in noncommuting variables to be

\[
M_{(\Phi, \Pi)} = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1}x_{i_2}\cdots x_{i_n}
\]

where the sum runs over all tuples \((i_1, i_2, \ldots, i_n)\) such that

1. \(i_j = i_k\) if and only if \(j\) and \(k\) are in the same block of either \(\Phi\) or \(\Pi\), and
2. \(i_j < i_k\) if \(j \in \Phi_l\) and \(k \in \Phi_m\) with \(l < m\).

Let

\[
NCQSym^r(x) = \bigoplus_{n \geq 0} NCQSym^n_\mathbb{Q}(x),
\]

where

\[
NCQSym^n_\mathbb{Q}(x) = \mathbb{Q}\text{-span}\{M_{(\Phi, \Pi)} : (\Phi, \Pi) \text{ is an } r\text{-set-composition of } [n]\}.
\]

We have that

\[
NCQSym(x) = NCQSym^1(x) \supset NCQSym^2(x) \supset \cdots \supset NCQSym^\infty(x) = NCSym(x).
\]

We will show later in Section 13 that \(NCQSym^r(x)\) is a Hopf algebra and give natural bases for the Hopf algebras in this chain realized as generalized chromatic functions in noncommuting variables, which are defined in the next section.

9. **Generalized chromatic functions in noncommuting variables**

A labelled edge-coloured digraph is an edge-coloured digraph where each vertex has a label; that is, a pair \((G, L)\) where \(G\) is an edge-coloured digraph and \(L\) is a bijection from \(V(G)\) to a subset \(A\) of \(\mathbb{P}\). The label of a vertex \(a\) is \(L(a)\), and \(A\) is called the label set of \((G, L)\). We usually depict a labelled edge-coloured digraph with the label of each vertex written inside it.

![Figure 7. A labelled edge-coloured digraph](image-url)
Consider the labelled edge-coloured digraphs \((G_1, L_1)\) and \((G_2, L_2)\) where the intersection of their label sets is empty. The disjoint union of \((G_1, L_1)\) and \((G_2, L_2)\), denoted \((G_1, L_1) \uplus (G_2, L_2)\), is the labelled edge-coloured digraph \((G, L)\) where

1. \(G = G_1 \uplus G_2\), the disjoint union of edge-coloured digraphs \(G_1\) and \(G_2\).
2. \(L(a) = L_1(a)\) if \(a \in V(G_1)\), and \(L(a) = L_2(a)\) if \(a \in V(G_2)\).

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7
\end{array}
\begin{array}{c}
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
\end{array}
\uplus
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7
\end{array}
\begin{array}{c}
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
\end{array}
\]

**Figure 8.** The disjoint union of two labelled edge-coloured digraphs

Also, the dashed (solid and double, respectively) sum of the labelled edge-coloured digraphs \((G_1, L_1)\) and \((G_2, L_2)\) is the labelled edge-coloured digraph \((G, L)\) where

1. \(G\) is the dashed (solid and double, respectively) sum of edge-coloured digraphs \(G_1\) and \(G_2\).
2. \(L(a) = L_1(a)\) if \(a \in V(G_1)\), and \(L(a) = L_2(a)\) if \(a \in V(G_2)\).

The dashed, solid, and double sums of the labelled edge-coloured digraphs \((G_1, L_1)\) and \((G_2, L_2)\) are denoted by

\[(G_1, L_1) \ominus (G_2, L_2), \ (G_1, L_1) \ominus (G_2, L_2), \text{ and } (G_1, L_1) \ominus (G_2, L_2),\]

respectively.

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7
\end{array}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7
\end{array}
\ominus
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7
\end{array}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7
\end{array}
\]

**Figure 9.** The dashed sum of two labelled edge-coloured digraphs

We frequently use the labelled edge-coloured digraphs in the following table.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_A)</td>
<td>An arbitrary ((C_n, L)) where (L : V(C_n) \to A)</td>
</tr>
<tr>
<td>(P_A)</td>
<td>((P_n, L)) where (L : V(P_n) \to A) such that if (a \to b), then (L(a) &lt; L(b))</td>
</tr>
<tr>
<td>(Q_A)</td>
<td>((Q_n, L)) where (L : V(Q_n) \to A) such that if (a \Rightarrow b), then (L(a) &lt; L(b))</td>
</tr>
<tr>
<td>(K_A)</td>
<td>An arbitrary ((K_n, L)) where (L : V(K_n) \to A)</td>
</tr>
</tbody>
</table>

**Table 3.** Useful labelled edge-coloured digraphs

\[
\begin{array}{c}
2 \\
4 \\
7 \\
\end{array}
\begin{array}{c}
2 \\
4 \\
7 \\
\end{array}
\begin{array}{c}
2 \\
4 \\
7 \\
\end{array}
\begin{array}{c}
2 \\
4 \\
7 \\
\end{array}
\]

\[
C_{\{2,4,7\}} \quad P_{\{2,4,7\}} \quad Q_{\{2,4,7\}} \quad K_{\{2,4,7\}}
\]
Figure 10. The labelled edge-coloured digraphs $C_{\{2,4,7\}}, P_{\{2,4,7\}}, Q_{\{2,4,7\}}, K_{\{2,4,7\}}$

Definition 9.1. (The generalized chromatic function of a labelled edge-coloured digraph in noncommuting variables) Let $(G, L)$ be a labelled edge-coloured digraph where $L : V(G) \to |V(G)|$. Define
\[
Y_{(G,L)}(x, t) = \sum_{\kappa \in C(G)} t^{\text{asc}(\kappa)} x_{\kappa,L}
\]
where
\[
x_{\kappa,L} = \prod_{i=1}^{|V(G)|} x_{\kappa(L^{-1}(i))} \quad \text{and} \quad \text{asc}(\kappa) = |\{(a, b) \in E(G) : \kappa(a) < \kappa(b)\}|.
\]
The power series
\[
Y_{(G,L)}(x) = Y_{(G,L)}(x, 1)
\]
is called the generalized chromatic function of $(G, L)$.

Consider the following commutation map,
\[
\rho : Q\langle\langle x_1, x_2, \ldots \rangle\rangle \to Q[[x_1, x_2, \ldots]]
\]
\[
x_i \mapsto x_i.
\]
The following proposition follows by definition.

Proposition 9.2. For any labelled edge-coloured digraph $(G, L)$, we have
\[
\rho(Y_{(G,L)}(x, t)) = Y_G(x, t).
\]

Gebhard-Sagan’s chromatic symmetric function in noncommuting variables [11]: For any graph $H$ with vertices labelled by $a_1, a_2, \ldots, a_n$ in a fixed order, define
\[
Y_H(x) = \sum_{\kappa \in C(H)} x_{\kappa(a_1)} x_{\kappa(a_2)} \cdots x_{\kappa(a_n)}.
\]
By definition we have the following.

Proposition 9.3. Let $H$ be a graph with labelled vertices $a_1, a_2, \ldots, a_n$. Then
\[
Y_H(x) = Y_{(G,L)}(x)
\]
where the underlying graph of $G$, $\overline{G}$, is isomorphic to $H$, $G$ only has dashed edges, and $L(a_i) = i$.

Let $A$ be a nonempty subset of $\mathbb{P}$. Define $\mathfrak{S}_A$ to be the set of all bijections from $A$ to itself. Let $(G, L)$ be a labelled edge-coloured digraph with label set $A$. Then for $\sigma \in \mathfrak{S}_n$, define
\[
\sigma \circ (G, L) = (G, \sigma \circ L).
\]
The symmetrized generalized chromatic function of $(G, L)$ is
\[
\mathfrak{S}Y_{(G,L)}(x) = \sum_{\sigma \in \mathfrak{S}_A} Y_{\sigma(G,L)}(x).
\]
10. Product and coproduct formulas for generalized chromatic functions in noncommuting variables

We now establish the product and coproduct formulas for generalized chromatic functions in noncommuting variables.

Let \((G, L)\) be a labelled edge-coloured digraph, and let \(n\) be a positive integer. Then define \(L + n\) to be a labelling for \(G\) where \((L + n)(a) = L(a) + n\); that is, we add \(n\) to the label of each vertex. By a proof analogous to that of Proposition 6.1 we have the following product formula for generalized chromatic functions of labelled edge-coloured digraphs in noncommuting variables.

**Proposition 10.1.** Let \((G_1, L_1)\) and \((G_2, L_2)\) be labelled edge-coloured digraphs. Then

\[
\mathcal{Y}(G_1, L_1)(x, t)\mathcal{Y}(G_2, L_2)(x, t) = \mathcal{Y}(G_1, L_1)_\oplus(G_2, L_2 + |V(G_1)|)(x, t).
\]

Let \((G, L)\) be a labelled edge-coloured digraph. Let \(F\) be an induced subdigraph of \(G\). Let \(B = \{L(a) : a \in V(F)\}\). Define \(L_V(F) : V(F) \to |V(F)|\) by

\[
L_V(F)(a) = \text{std}_B(L(a)).
\]

For example, let \(G\) be a digraph with vertex set \(V(G) = \{r, s, t, u, v, w\}\), and the labelling \(L : V(G) \to [6]\) is defined by \(L(r) = 2, L(s) = 5, L(t) = 1, L(u) = 6, L(v) = 3,\) and \(L(w) = 4\). If \(V(F) = \{r, s, u, w\}\), then \(B = \{L(a) : a \in V(F)\} = \{2, 4, 5, 6\}\). Thus, \(L_V(F) : V(F) \to [4]\) is defined by \(L_V(F)(r) = \text{std}_B(L(r)) = 1, L_V(F)(s) = \text{std}_B(L(s)) = 3, L_V(F)(u) = \text{std}_B(L(u)) = 4,\) and \(L_V(F)(w) = \text{std}_B(L(w)) = 2\).

By a proof analogous to that of Theorem 6.3, we have the following coproduct formula for generalized chromatic functions of labelled edge-coloured digraphs.

**Proposition 10.2.** Let \((G, L)\) be a labelled edge-coloured digraph. Then

\[
\Delta(\mathcal{Y}(G, L)(x)) = \sum \mathcal{Y}(G|V(G)\setminus V(F), L|V(G)\setminus V(F))(x) \otimes \mathcal{Y}(F, L_V(F))(x),
\]

where the sum runs over all \(\{\to, \Rightarrow\}\)-induced subdigraphs \(F\) of \(G\).

11. Bases for \(\text{NCSym}(x)\) and \(\text{NCQSym}(x)\) using generalized chromatic functions in noncommuting variables

In this section, we realize different bases for the Hopf algebras \(\text{NCSym}(x)\) and \(\text{NCQSym}(x)\) as generalized chromatic functions in noncommuting variables of certain labelled edge-coloured digraphs. Since these bases are less well known than their (quasi)symmetric counterparts, we provide their classical definitions taken from [21] for \(\text{NCSym}(x)\) and [5, 9] for \(\text{NCQSym}(x)\).

**Monomial basis \(\{m_\Pi\}\) of \(\text{NCSym}(x)\).** Given \(\Pi = \Pi_1/\Pi_2/\cdots/\Pi_l \vdash [n]\), define the *monomial symmetric function* in noncommuting variables to be

\[
m_\Pi = \sum_{i_1, i_2, \ldots, i_n} x_{i_1}x_{i_2}\cdots x_{i_n},
\]
where the sum is over all \(n\)-tuples \((i_1, i_2, \ldots, i_n)\) with \(i_j = i_k\) if and only if \(j\) and \(k\) are in the same block in \(\Pi\). For example,
\[
m_{13/24} = x_1 x_2 x_1 x_2 + x_2 x_1 x_2 x_1 + x_1 x_3 x_1 x_3 + x_3 x_1 x_3 x_1 + x_2 x_3 x_2 x_3 + x_3 x_2 x_3 x_2 + \cdots.
\]

By definition we have that \(m_{\Pi} = \mathcal{Y}(G,L)(x)\) where
\[
(G, L) = C_{\Pi_1} \bigoplus C_{\Pi_2} \bigoplus \cdots \bigoplus C_{\Pi_l}.
\]
For example, if \((G, L)\) is the following labelled edge-coloured digraph

\[
\begin{array}{c}
1 \\
\downarrow \\
3 \\
\downarrow \\
2 \\
\downarrow \\
4
\end{array}
\]

then \(\mathcal{Y}(G,L)(x) = m_{13/24}\).

**Power sum basis** \(\{p_{\Pi}\}\) of \(\text{NCSym}(x)\). Given \(\Pi = \Pi_1/\Pi_2/\cdots/\Pi_l \vdash [n]\), define the **power sum symmetric function** in noncommuting variables to be
\[
p_{\Pi} = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n},
\]
where \(i_j = i_k\) if \(j, k\) are in the same block in \(\Pi\). For example,
\[
p_{13/24} = x_1 x_2 x_1 x_2 + x_2 x_1 x_2 x_1 + x_1^4 + x_2^4 + \cdots.
\]
By definition we have that \(p_{\Pi} = \mathcal{Y}(G,L)(x)\) where
\[
(G, L) = K_{\Pi_1} \bigcup K_{\Pi_2} \bigcup \cdots \bigcup K_{\Pi_l}.
\]
For example, if \((G, L)\) is the following labelled edge-coloured digraph

\[
\begin{array}{c}
1 \\
\downarrow \\
3 \\
\downarrow \\
2 \\
\downarrow \\
4
\end{array}
\]

then \(\mathcal{Y}(G,L)(x) = p_{13/24}\).

**Remark 11.1.** Note that the underlying edge-coloured digraphs for the augmented monomial basis elements of \(\text{Sym}(x)\) and the monomial basis elements of \(\text{NCSym}(x)\) are the same. Moreover, the underlying edge-coloured digraphs for power sum basis elements of \(\text{Sym}(x)\) and power sum basis elements in \(\text{NCSym}(x)\) are the same.

**Elementary basis** \(\{e_{\Pi}\}\) of \(\text{NCSym}(x)\). Given \(\Pi = \Pi_1/\Pi_2/\cdots/\Pi_l \vdash [n]\), define the **elementary symmetric function** in noncommuting variables to be
\[
e_{\Pi} = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n},
\]
where \(i_j \neq i_k\) if \(j, k\) are in the same block in \(\Pi\). For example,
\[
e_{13/24} = x_1 x_1 x_2 x_2 + x_2 x_2 x_1 x_1 + x_1 x_2 x_2 x_1 + x_1 x_2 x_3 x_4 + \cdots.
\]

It follows by definition that the elementary symmetric functions in noncommuting variables can be written in two ways using labelled edge-coloured digraphs. The first is that \(e_{\Pi} = \mathcal{Y}(G,L)(x)\) where
\[
(G, L) = K_{\Pi_1} \bigcup K_{\Pi_2} \bigcup \cdots \bigcup K_{\Pi_l}.
\]
For example, if \((G, L)\) is the following labelled edge-coloured digraph
\[
\begin{array}{c}
1 \rightarrow 3 \\
2 \rightarrow 4
\end{array}
\]
then \(\mathcal{Y}_{(G,L)}(x) = e_{13/24}\).

The second is that
\[
e_{\Pi} = \sum_{(\sigma_1, \ldots, \sigma_i) \in \mathcal{S}_{\Pi_1} \times \cdots \times \mathcal{S}_{\Pi_l}} \mathcal{Y}_{(\sigma_{i+1} \circ P_{\Pi_{i+1}})}(x).
\]

For example, \(e_{13/24}\) is equal to
\[
\mathcal{Y}_{1 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 4}(x) + \mathcal{Y}_{3 \rightarrow 1 \rightarrow 2 \rightarrow 4}(x).
\]

**Complete homogeneous basis** \(\{h_\Pi\}\) of \(NCSym(x)\). For set partitions \(\Pi\) and \(\Omega\) of \([n]\), let \(\Omega \leq \Pi\) if each block of \(\Omega\) is contained in some block of \(\Pi\). The set of all set partitions of \([n]\) with this partial ordering gives a lattice; the meet (greatest lower bound) and join (least upper bound) operations of this lattice are denoted by \(\land\) and \(\lor\), respectively. The **complete homogeneous symmetric function** in noncommuting variables is
\[
h_\Pi = \sum_{\Omega \leq \Pi} \lambda(\Omega \land \Pi)!m_\Omega.
\]

For example,
\[
h_{13/24} = m_{1/2}/3/4 + m_{12}/3/4 + 2m_{13}/2/4 + m_{14}/2/3 + m_{1}/23/4 + 2m_{1}/24/3 + m_{1}/2/34 + m_{12}/34 + 4m_{13}/24 + m_{14}/23 + 2m_{123}/4 + 2m_{124}/3 + 2m_{134}/2 + 2m_{1}/234 + 4m_{1234}.
\]

Using labelled edge-coloured digraphs we now present the complete homogeneous basis. Let \(\Pi = \Pi_1/\Pi_2/\cdots/\Pi_l\) be a set partition of \([n]\). By [1, Lemma 2.14], we have that
\[
h_{\Pi} = \sum_{(\sigma_1, \ldots, \sigma_i) \in \mathcal{S}_{\Pi_1} \times \cdots \times \mathcal{S}_{\Pi_l}} \mathcal{Y}_{(\sigma_{i+1} \circ Q_{\Pi_{i+1}})}(x).
\]

For example, \(h_{13/24}\) is equal to
\[
\mathcal{Y}_{1 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 4}(x) + \mathcal{Y}_{3 \rightarrow 1 \rightarrow 2 \rightarrow 4}(x) + \mathcal{Y}_{1 \rightarrow 3 \rightarrow 4 \rightarrow 2}(x) + \mathcal{Y}_{3 \rightarrow 1 \rightarrow 4 \rightarrow 2}(x) + \mathcal{Y}_{1 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 2}(x) + \mathcal{Y}_{1 \rightarrow 3 \rightarrow 4 \rightarrow 2 \rightarrow 1}(x).
\]

**Remark 11.2.** The elementary symmetric functions in \(NCSym(x)\) are produced by the symmetrized generalized chromatic functions of the connected components of labelled edge-coloured digraphs where the generalized chromatic functions of their edge-coloured digraphs give the elementary symmetric functions in \(Sym(x)\). Also, the complete homogeneous symmetric functions in \(NCSym(x)\) are the symmetrized generalized chromatic functions of the connected components of labelled edge-coloured digraphs where the generalized chromatic functions of their edge-coloured digraphs give the complete homogeneous symmetric functions in \(Sym(x)\).
Rosas-Sagan Schur functions of NCSym($x$). Rosas and Sagan in [21], as an analogy for the monomial, power sum, elementary and homogeneous bases of Sym($x$), introduced the above bases for NCSym($x$), recalling the elementary basis from the work of Wolf [27]. Their proposed analogy for Schur functions did not produce enough distinct elements to make a basis for NCSym($x$). However, their functions have a natural realization in terms of generalized chromatic functions, so we include them here.

Let $\Pi = \Pi_1/\Pi_2/\cdots/\Pi_l$ be a set partition of $[n]$ such that $|\Pi_1| \geq |\Pi_2| \geq \cdots \geq |\Pi_l|$. Let $\lambda = \lambda(\Pi)$. Consider the labelled edge-coloured digraph $(G, L)$ such that
\begin{enumerate}
  \item $G = G_\lambda$.
  \item $L(i, j) = \lambda_1 + \cdots + \lambda_i - 1 + j$.
\end{enumerate}
Now let
$$S_\Pi = \sum_{\sigma \in S_n} \mathcal{Y}_{\sigma\circ (G, L)}(x).$$
Then $S_\Pi \in \text{NCSym}(x)$, and $S_\Pi = S_{\Omega}$ if and only if $\lambda(\Pi) = \lambda(\Omega) = \lambda$. Note that $S_\Pi$ is the same as $S_\lambda$ introduced in [21]. Moreover, $\rho(S_\Pi) = n!s_\lambda$. However, the set
$$\{S_\Pi : \Pi \vdash [n]\}$$
is not a basis for NCSym$_n(x)$ since the dimension of the space spanned by this set is equal to the number of partitions, which is less than the number of set partitions for $n > 2$.

Monomial basis $\{M_\Phi\}$ of NCQSym($x$). Given $\Phi = (\Phi_1 | \Phi_2 | \cdots | \Phi_k) \vdash [n]$, define the monomial quasisymmetric function in noncommuting variables to be
$$M_\Phi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1}x_{i_2}\cdots x_{i_n}$$
where the sum runs over all tuples $(i_1, i_2, \ldots, i_n)$ such that
\begin{enumerate}
  \item $i_j = i_\ell$ if $j$ and $\ell$ are in the same block of $\Phi$ and
  \item $i_j < i_\ell$ if $j \in \Phi_p$ and $\ell \in \Phi_q$ with $p < q$.
\end{enumerate}
For example,
$$M_{(13|24)} = x_1x_2x_1x_2 + x_1x_3x_1x_3 + x_2x_3x_2x_3 + \cdots.$$  
By definition, we have that $M_\Phi = \mathcal{Y}_{(G, L)}(x)$, where
$$(G, L) = C_{\Phi_1} \bigoplus C_{\Phi_2} \bigoplus \cdots \bigoplus C_{\Phi_k}.$$  
For example, if $(G, L)$ is the following labelled edge-coloured digraph
\[
\begin{array}{cccccccc}
2 & \rightarrow & 4 & \bigoplus & 3 & \rightarrow & 7 & \bigoplus & 6 & \rightarrow & 1 & \rightarrow & 5
\end{array}
\]
then $\mathcal{Y}_{(G, L)}(x) = M_{(24|378|6|15)}$.

To the best of our knowledge the following basis is not in the literature, however, similar bases exist [5, 9]. Therefore we will define the basis elements in terms of generalized chromatic functions in noncommuting variables first, before deriving an explicit formula for them, and establishing that they are a basis for NCQSym($x$). Since they can be
defined using generalized chromatic functions in noncommuting variables their product and coproduct formulas follow from Propositions 10.1 and 10.2.

**Fundamental basis** \{F_\Phi\} of NCQSym(x). Given \Phi = (\Phi_1|\Phi_2|\ldots|\Phi_\ell) \models [n], define the **fundamental quasisymmetric function** in noncommuting variables to be

\[ F_\Phi = \mathcal{Y}_{(G,L)}(x), \]

where

\[ (G, L) = Q_{\Phi_1} \bigoplus Q_{\Phi_2} \bigoplus \cdots \bigoplus Q_{\Phi_\ell}. \]

For example, if \Phi = (24|378|6|15) then \((G, L)\) is the following labelled edge-coloured digraph

\[ \begin{array}{cccccccc}
2 & \rightarrow & 4 & \bigoplus & 3 & \rightarrow & 7 & \rightarrow & 8 & \bigoplus & 6 & \bigoplus & 1 & \rightarrow & 5
\end{array} \]

and \( F_{(24|378|6|15)} = \mathcal{Y}_{(G,L)}(x). \)

By comparing the labelled edge-coloured digraphs in the realizations of \( F_\Phi \) and \( M_\Phi \) as generalized chromatic functions in noncommuting variables, we immediately get the following.

**Proposition 11.3.** Let \( \Phi \) be a set composition. Then

\[ F_\Phi = \sum_{\Psi \text{ reforms } \Phi} M_\Psi \]

and hence \( \{F_\Phi\} \) is a basis for NCQSym(x).

For example,

\[ F_{(13|24)} = M_{(13|24)} + M_{(1|324)} + M_{(13|24)} + M_{(13|24)}. \]

**Upper-fundamental basis** \{F_\Phi\} of NCQSym(x). This basis also appears in [9] as the \( L \) basis, but we reinterpret it here as generalized chromatic functions. Given \( \Phi = (\Phi_1|\Phi_2|\ldots|\Phi_\ell) \models [n] \), define the **upper-fundamental quasisymmetric function** in noncommuting variables to be

\[ F_\Phi = \sum_{\Psi \text{ corrupts } \Phi} M_\Psi. \]

For example,

\[ F_{(13|24)} = M_{(13|24)} + M_{(1324)} = M_{(13|24)} + M_{(1234)}. \]

By definition, we have that \( F_\Phi = \mathcal{Y}_{(G,L)}(x), \) where

\[ (G, L) = C_{\Phi_1} \bigoplus C_{\Phi_2} \bigoplus \cdots \bigoplus C_{\Phi_\ell}. \]

For example, if \((G, L)\) is the following labelled edge-coloured digraph

\[ \begin{array}{cccccccc}
2 & \rightarrow & 4 & \bigoplus & 3 & \rightarrow & 7 & \rightarrow & 8 & \bigoplus & 6 & \bigoplus & 1 & \rightarrow & 5
\end{array} \]

then \( \mathcal{Y}_{(G,L)}(x) = F_{(24|378|6|15)}. \)
Remark 11.4. Note that the underlying edge-coloured digraphs of monomial (fundamental and upper-fundamental, respectively) bases of $QSym(x)$ and $NCQSym(x)$ are the same. Moreover, for any set composition $\Phi$, 
\[
\rho(M_\Phi) = M_{\alpha(\Phi)}, \quad \rho(F_\Phi) = F_{\alpha(\Phi)}, \quad \text{and} \quad \rho(F_\Phi) = F_{\alpha(\Phi)}.
\]

Let $\Phi = (\Phi_1|\Phi_2|\cdots|\Phi_k)$ be a set composition, and $\Pi = \Pi_1/\Pi_2/\cdots/\Pi_l$ be a set partition. In the following table, we summarize our results for this section using the operators $\sqcup$, $\otimes$, $\ominus$, $\odot$ on the edge-coloured digraphs in Table 3.

<table>
<thead>
<tr>
<th>Basis</th>
<th>Expression</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monomial basis of $NCSym(x)$</td>
<td>$\mathcal{Y} \left( \bigotimes_{i=1}^{l} C_{\Pi_i} \right)(x)$</td>
<td>$m_\Pi$</td>
</tr>
<tr>
<td>Elementary basis of $NCSym(x)$</td>
<td>$\sum_{(\sigma_1,\ldots,\sigma_l)\in S_{\Pi_1} \times \cdots \times S_{\Pi_l}} \mathcal{Y} \left( (w_1^{i_1} \cdots w_{i_l})<em>{\sigma</em>{i_1}} \cdots \sigma_{i_l} \right)(x)$</td>
<td>$e_\Pi$</td>
</tr>
<tr>
<td>Elementary basis of $NCSym(x)$</td>
<td>$\mathcal{Y} \left( (w_1^{i_1} \cdots w_{i_l})<em>{\sigma</em>{i_1}} \cdots \sigma_{i_l} \right)(x)$</td>
<td>$e_\Pi$</td>
</tr>
<tr>
<td>Complete homogeneous basis of $NCSym(x)$</td>
<td>$\sum_{(\sigma_1,\ldots,\sigma_l)\in S_{\Pi_1} \times \cdots \times S_{\Pi_l}} \mathcal{Y} \left( (w_1^{i_1} \cdots w_{i_l})<em>{\sigma</em>{i_1}} \cdots \sigma_{i_l} \right)(x)$</td>
<td>$h_\Pi$</td>
</tr>
<tr>
<td>Power sum basis of $NCSym(x)$</td>
<td>$\mathcal{Y} \left( (w_1^{i_1} \cdots w_{i_l})<em>{\sigma</em>{i_1}} \cdots \sigma_{i_l} \right)(x)$</td>
<td>$p_\Pi$</td>
</tr>
<tr>
<td>Monomial basis of $NCQSym(x)$</td>
<td>$\mathcal{Y} \left( \bigotimes_{i=1}^{k} C_{\Phi_i} \right)(x)$</td>
<td>$M_\Phi$</td>
</tr>
<tr>
<td>Fundamental basis of $NCQSym(x)$</td>
<td>$\mathcal{Y} \left( \bigotimes_{i=1}^{k} Q_{\Phi_i} \right)(x)$</td>
<td>$F_\Phi$</td>
</tr>
<tr>
<td>Upper-fundamental basis of $NCQSym(x)$</td>
<td>$\mathcal{Y} \left( \bigotimes_{i=1}^{k} C_{\Pi_i} \right)(x)$</td>
<td>$F_\Phi$</td>
</tr>
</tbody>
</table>

Table 4. Bases for $NCSym(x)$ and $NCQSym(x)$ reinterpreted.
The Malvenuto-Reutenauer Hopf algebra is the graded Hopf algebra

$$\mathcal{G} \text{Sym}(x) = \bigoplus_{n \geq 0} \mathcal{G} \text{Sym}_n(x),$$

where

$$\mathcal{G} \text{Sym}_n(x) = \mathbb{Q}\text{-span}\{F_\sigma : \sigma \in \mathcal{S}_n\}.$$  

For more details about the Malvenuto-Reutenauer Hopf algebra, see [14, Section 8].

Given any permutation $\sigma$, let $\text{Des}(\sigma) = \{i : \sigma(i) > \sigma(i + 1)\}$. Let $i_1 < i_2 < \cdots < i_t$ be all elements of $\text{Des}(\sigma)$. By definition we have that $F_\sigma = \mathcal{Y}_{(G,L)}(x)$ where

$$(G,L) = Q_{\{\sigma(1),\ldots,\sigma(i_1)\}} \bigoplus Q_{\{\sigma(i_1+1),\ldots,\sigma(i_2)\}} \bigoplus \cdots \bigoplus Q_{\{\sigma(i_t+1),\ldots,\sigma(n)\}}.$$  

For the permutation $\sigma$, define the set composition

$$\Phi_\sigma = (\sigma(1) \cdots \sigma(i_1))|\sigma(i_1+1) \cdots \sigma(i_2)| \cdots |\sigma(i_t-1+1) \cdots \sigma(i_t)).$$

For example, if $\sigma = 836791524$, then

$$\text{Des}(\sigma) = \{1, 5, 7\},$$

and $F_\sigma = \mathcal{Y}_{(G,L)}$ where

$$(G,L) = Q_{\{8\}} \bigoplus Q_{\{3,6,7,9\}} \bigoplus Q_{\{1,5\}} \bigoplus Q_{\{2,4\}}.$$  

Moreover,

$$\Phi_\sigma = (8\,|3679\,|15\,|24).$$

Considering the product and coproduct formulas for the fundamental bases of $\mathcal{G} \text{Sym}(x)$ and $\text{NCQSym}(x)$, we have the following injection,

$$\mathcal{G} \text{Sym}(x) \rightarrow \text{NCQSym}(x)$$

$$F_\sigma \mapsto F_{\Phi_\sigma}.$$  

13. $\text{NCQSym}^r(x)$ and its bases

To conclude we prove that $\text{NCQSym}^r(x)$ is a Hopf algebra and then establish two natural bases for this Hopf algebra using generalized chromatic functions, where

$$\text{NCQSym}(x) = \text{NCQSym}^1(x) \supset \text{NCQSym}^2(x) \supset \cdots \supset \text{NCQSym}^\infty(x) = \text{NCSym}(x).$$

The embedding of $\text{NCQSym}^r(x)$ to $\text{NCQSym}(x)$ is given by

$$M_{(\Phi, \Pi)} = \sum_{\Psi \in \Phi \boxtimes \Pi} M_\Psi$$

where $\boxtimes$ is defined as follows. If $\Phi = (\Phi_1|\Phi_2|\cdots|\Phi_k)$ and $\Pi = \Pi_1/\Pi_2/\cdots/\Pi_t$, then $\Phi \boxtimes \Pi$ is the set of set compositions $(\Psi_1|\Psi_2|\cdots|\Psi_{k+l})$ such that

$$\{\Psi_1, \Psi_2, \ldots, \Psi_{k+l}\} = \{\Phi_1, \Phi_2, \ldots, \Phi_k, \Pi_1, \Pi_2, \ldots, \Pi_t\}$$

and if $\Phi_i = \Psi_l$ and $\Phi_{i+1} = \Psi_m$, then we always have $l < m$.

In the following theorem, we show that $\text{NCQSym}^r(x)$ is a Hopf algebra by finding the product and coproduct formulas for the $r$-dominant monomial basis of $\text{NCQSym}^r(x)$. 
Theorem 13.1. For any positive integer \( r \), NCQSym\(^r\)(\( x \)) is a Hopf algebra.

Proof. It is enough to show that NCQSym\(^r\)(\( x \)) is closed under the product and coproduct of NCQSym(\( x \)). In NCSym(\( x \)), the coproduct on the monomial basis is taking subsets of blocks, followed by standardization. That is, given a set partition \( \Pi = \Pi_1/\Pi_2/\cdots/\Pi_l \), then

\[
\Delta(m_\Pi) = \sum_{\{i_1,i_2,\ldots,i_l\} \subseteq [n]} m_{\text{std}(\Pi_1/\Pi_2/\cdots/\Pi_l)} \otimes m_{\text{std}(\Pi_{i_1+1}/\Pi_{i_2+2}/\cdots/\Pi_{i_l})}.
\]

Recall that for a set composition \( \Phi = (\Phi_1|\Phi_2|\cdots|\Phi_k) \), in NCQSym(\( x \)), we have

\[
\Delta(M_\Phi) = \sum_{t=0}^{k} M_{\text{std}(\Phi_1|\Phi_2|\cdots|\Phi_t)} \otimes M_{\text{std}(\Phi_{t+1}|\Phi_{t+2}|\cdots|\Phi_k)}.
\]

Therefore, in NCQSym\(^r\)(\( x \)), if \( (\Phi, \Pi) = (\Phi_1|\Phi_2|\cdots|\Phi_k, \Pi_1/\Pi_2/\cdots/\Pi_l) \), we have

\[
\Delta(M_{(\Phi, \Pi)}) = \sum_{t=1}^{k} \sum_{j=0}^{l} M_{\text{std}(\Phi_1|\Phi_2|\cdots|\Phi_t, \Pi_{i_1}/\Pi_{i_2}/\cdots/\Pi_{i_j})} \otimes M_{\text{std}(\Phi_{t+1}|\Phi_{t+2}|\cdots|\Phi_k, \Pi_{i_{j+1}}/\cdots/\Pi_{i_l})}.
\]

For example, when \( r = 2 \),

\[
\Delta(M_{((24),1/3)}) = 1 \otimes M_{((24),1/3)} + M_{(0,1)} \otimes M_{(13,2)} + M_{(0,1)} \otimes M_{(23,1)} + M_{(23,1)} \otimes M_{(0,1)} + M_{((13,2)} \otimes M_{(0,1)} + M_{(0,1/2)} \otimes M_{(23,1)} + M_{(23,1/2)} \otimes M_{(0,1)} + M_{((23,1/2)} \otimes M_{(0,1/2)} + M_{(0,1/2)} \otimes M_{((23,1/2)} + M_{((24,1/3)} \otimes 1.
\]

The product formula is a little more complicated. Before we continue we need to recall the product for the monomial basis of NCQSym(\( x \)). Let \( \Phi = (\Phi_1|\Phi_2|\cdots|\Phi_k) \) be a set composition of \([n]\), and let \( A \) be a subset of \([n]\). The restriction of \( \Phi \) to \( A \), \( \Phi|_A \), is the set composition \( (\Phi_1 \cap A|\Phi_2 \cap A|\cdots|\Phi_k \cap A)^* \) where the bullet means we drop the empty parts. For example,

\[
(357|26|14)|_{\{1,3,4\}} = (3|14).
\]

Let \( \Phi \in [n] \) and \( \Psi \in [m] \). The shifted quasi-shuffle of \( \Phi \) and \( \Psi \), denoted \( \Phi \circlearrowleft \Psi \), is the set of set compositions \( \Gamma \) such that \( \Gamma|_{\{1,2,\ldots,n\}} = \Phi \) and \( \text{std}(\Gamma|_{\{n+1,n+2,\ldots,n+m\}}) = \Psi \). The product formula for the monomial basis of NCQSym(\( x \)) is

\[
M_\Phi \cdot M_\Psi = \sum_{\Gamma \in \Phi \circlearrowleft \Psi} M_\Gamma.
\]

Now for an \( r \)-set-composition \( (\Phi, \Pi) \), we have

\[
M_{(\Phi, \Pi)} = \sum_{\Psi \in \Phi \circlearrowleft \Pi} M_\Psi.
\]

Conversely, given a set composition \( \Psi \), there is a unique \( r \)-set-composition \( (\Phi, \Pi) \) such that \( \Psi \in \Phi \circlearrowleft \Pi \). We denote \( \Phi \) by \( \Psi|_{r\text{-comp}} \) and \( \Pi \) by \( \Psi|_{r\text{-par}} \).

Also, for set compositions \( \Gamma \) and \( \Theta \) we have

\[
M_\Gamma \cdot M_\Theta = \sum_{\Upsilon \in \Gamma \circlearrowleft \Theta} M_\Upsilon.
\]

Conversely, given a set composition \( \Upsilon \) with \( |\Upsilon| = n + m \), there is a unique pair of set compositions \( \Gamma, \Theta \) such that \( |\Gamma| = n \), \( |\Theta| = m \) and \( \Upsilon \in \Gamma \circlearrowleft \Theta \). We denote \( \Gamma \) by \( \Upsilon|_{\{1,\ldots,n\}} \) and \( \Theta \) by \( \Upsilon|_{\{n+1,\ldots,n+m\}} \).
Therefore,
\[ M_{(\Phi,\Pi)} \cdot M_{(\Psi,\Omega)} = \sum_{\Gamma \in \mathcal{F}_{\Phi \Pi}, \Theta \in \mathcal{F}_{\Psi \Omega}} \left( \sum_{\Upsilon \in \mathcal{F}_{\Gamma \Theta}} M_\Upsilon \right) = \sum_\Upsilon C_{(\Phi,\Pi),(\Psi,\Omega)}^\Upsilon M_\Upsilon. \]

Let \(|(\Phi,\Pi)| = n\) and \(|(\Psi,\Omega)| = m\). We first note that all coefficients \(C_{(\Phi,\Pi),(\Psi,\Omega)}^\Upsilon\) are 0 or 1. Indeed, \(C_{(\Phi,\Pi),(\Psi,\Omega)}^\Upsilon = 1\) if and only if

1. \(\Phi = (\Upsilon_{\{1,\ldots,n\}})_{r \text{-comp}},\)
2. \(\Pi = (\Upsilon_{\{1,\ldots,n\}})_{r \text{-par}},\)
3. \(\Psi = (\Upsilon_{\{n+1,\ldots,n+m\}})_{r \text{-comp}},\)
4. \(\Omega = (\Upsilon_{\{n+1,\ldots,n+m\}})_{r \text{-par}}.\)

Let \(C_{(\Phi,\Pi),(\Psi,\Omega)}^\Upsilon = 1\) for some \(\Upsilon',\) and let \(\Phi' = \Upsilon'_{r \text{-comp}}\) and \(\Pi' = \Upsilon'_{r \text{-par}}.\) We want to show that for any \(\Upsilon'' \in \Phi'\Pi',\) we have \(C_{(\Phi,\Pi),(\Psi,\Omega)}^{\Upsilon''} = 1.\) Then the product \(M_{(\Phi,\Pi)} \cdot M_{(\Psi,\Omega)}\) is in \(\text{NCQSym}^r(x).\)

Let \(\Gamma' = \Upsilon'_{\{1,\ldots,n\}},\) \(\Theta' = \Upsilon'_{\{n+1,\ldots,n+m\}},\) \(\Gamma'' = \Upsilon''_{\{1,\ldots,n\}}\) and \(\Theta'' = \Upsilon''_{\{n+1,\ldots,n+m\}}.\) Note the fact that in a quasi-shuffle, the blocks of size less than \(r\) can only be obtained from blocks of size less than \(r.\) Since \(\Upsilon'_{r \text{-comp}} = \Upsilon''_{r \text{-comp}} = \Phi',\) we must have \(\Gamma'_{r \text{-comp}} = \Gamma''_{r \text{-comp}} = \Phi\) and \(\Theta'_{r \text{-comp}} = \Theta''_{r \text{-comp}} = \Psi.\) And therefore, we must also have \(\Gamma'_{r \text{-par}} = \Gamma''_{r \text{-par}} = \Pi\) and \(\Theta'_{r \text{-par}} = \Theta''_{r \text{-par}} = \Omega.\) Hence, \(\Gamma'' \in \Phi'\Pi\) and \(\Theta'' \in \Psi'\Omega,\) that is, \(C_{(\Phi,\Pi),(\Psi,\Omega)}^{\Upsilon''} = 1.\)

Finally we establish two bases for \(\text{NCQSym}^r_n(x).\) Given an \(r\)-set-composition \((\Phi,\Pi) = (\Phi_1|\Phi_2|\cdots|\Phi_k),\Pi_1/\Pi_2/\cdots/\Pi_l)\) of \([n],\) note that by definition we have that
\[ M_{(\Phi,\Pi)} = \Upsilon \left( \bigotimes_{i=1}^k C_{\Phi_i} \otimes \bigotimes_{j=1}^l C_{\Pi_j} \right) (x) \]
and define
\[ \overline{F}_{(\Phi,\Pi)} = \Upsilon \left( \bigotimes_{i=1}^k C_{\Phi_i} \otimes \bigotimes_{j=1}^l C_{\Pi_j} \right) (x). \]

**Proposition 13.2.** Each of the following is a basis for \(\text{NCQSym}^r_n(x).\)

1. \(\{M_{(\Phi,\Pi)} : (\Phi,\Pi) \text{ is an } r\text{-set-composition of } [n]\}\)
2. \(\{\overline{F}_{(\Phi,\Pi)} : (\Phi,\Pi) \text{ is an } r\text{-set-composition of } [n]\}\)

**Proof.** The first set is the \(r\)-dominant monomial basis for \(\text{NCQSym}^r_n(x).\) Now, consider that
\[ \overline{F}_{(\Phi,\Pi)} = \sum_{\Psi \text{ corrupts } \Phi} M_{(\Psi,\Omega)}. \]
Therefore, the second set also is a basis for \(\text{NCQSym}^r_n(x).\)

\[ \square \]

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References


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