ROW-STRONG DUAL IMMACULATE FUNCTIONS

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ABSTRACT. We define a new basis of quasisymmetric functions, the row-strict dual immaculate functions, as the generating function of a particular set of tableaux. We establish that this definition gives a function that can also be obtained by applying the $\psi$ involution to the dual immaculate functions of Berg, Bergeron, Saliola, Serrano, and Zabrocki (2014) and establish numerous combinatorial properties for our functions. We give an equivalent formulation of our functions via Bernstein-like operators, in a similar fashion to Berg et. al (2014). We conclude the paper by defining skew dual immaculate functions and hook dual immaculate functions and establishing combinatorial properties for them.

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Quasisymmetric functions were first defined formally by Gessel [9] in relation to the theory of $P$-partitions, and have since grown to be a vibrant area of research in their own right, including playing a crucial role in the resolution of the Shuffle Conjecture. As a natural nonsymmetric generalization of symmetric functions, one avenue of research has been to establish analogies of classical symmetric functions, for example monomial symmetric functions and chromatic symmetric functions. However an analogy to the ubiquitous Schur functions remained elusive until 2011, when [10] discovered quasisymmetric Schur functions that naturally arose from the combinatorics of nonsymmetric Macdonald polynomials. These functions became the genesis of the now flourishing area of Schur-like functions throughout algebraic combinatorics, for example [1, 6, 7, 11, 12, 14]. Remaining in the algebra of quasisymmetric functions, two further bases rose to attention: the dual immaculate functions [4], and the row-strict quasisymmetric Schur functions [14], the latter of which are quasisymmetric Schur functions under the $\psi$ involution. In this paper we will interpolate between these two bases to yield row-strict dual immaculate functions.

More precisely, quasisymmetric Schur functions, all forms, can be defined combinatorially as the generating function of composition fillings (resp. row-strict composition fillings) where there is a requirement that the first column strictly (resp. weakly) increase, each row increases weakly (resp. strictly), and a triple rule is satisfied. The dual immaculate functions were introduced by Berg et al. [4] as the dual basis of the noncommutative symmetric immaculate functions. Combinatorially the dual immaculate functions can be viewed as the generating functions of composition fillings that satisfy just the first column and row requirements of the quasisymmetric Schurs, omitting the triple rule.

The triple rules required to define all versions of quasisymmetric Schur functions allow those functions to retain many of the combinatorial properties of Schur functions, including an RSK-style insertion algorithm, a JDT algorithm, a Murnaghan-Nakayama rule, and Littlewood-Richardson rules. Without the triple rule, some combinatorial similarities to Schur functions are lost, but others are gained. For example, the immaculate functions satisfy a noncommutative analogue of the Jacobi-Trudi rule.

In this paper we define row-strict immaculate tableaux of a given composition shape, and study their generating function. By identifying the correct descent set, we show that our combinatorial definition of the row-strict dual immaculate functions is equivalent to applying the involution $\psi$ to the dual immaculate functions in Theorem 3.7, and can also be obtained from the Hopf algebra of noncommutative symmetric functions by suitably defined creation operators in Theorem 3.17.

We are able to quickly obtain many results from [4] by application of the $\psi$ involution in Theorem 3.19. We also carefully construct skew row-strict dual immaculate functions and define hook dual immaculate functions and obtain results for them in our final two sections.
In this work we focus primarily on combinatorial aspects of the row-strict dual immaculate functions. 0-Hecke modules for these new functions are defined in [16].

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2. Background

In this section we introduce much of the background on quasisymmetric and noncommutative symmetric functions needed for our results. We refer the reader to [12] for additional details.

A composition of a positive integer \( n \) is a sequence \( \alpha = (\alpha_1, \ldots, \alpha_k) \) such that \( \sum \alpha_i = n \). We write \( \alpha \vDash n \). We sometimes denote \( n \) by \( |\alpha| \) and \( k \) by \( \ell(\alpha) \), and \( \alpha_j = \cdots = \alpha_{j+m} = i \) as \( i^m \). The diagram of \( \alpha = (\alpha_1, \ldots, \alpha_k) \) is a collection of left-justified boxes with \( \alpha_i \) boxes in row \( i \), where row 1 is the bottom row.

Example 2.1. For \( \alpha = (3, 1, 4, 2, 5, 1) \), the diagram is as follows.

\[
\begin{array}{cccccc}
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{array}
\]

Compositions of \( n \) are in bijection with subsets of \( \{1, 2, \ldots, n-1\} \). Given a composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \) of \( n \), the corresponding set is \( \text{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\} \). For \( \alpha = (3, 1, 4, 2, 5, 1) \) that is a composition of 16, \( \text{set}(\alpha) = \{3, 4, 8, 10, 15\} \subseteq \{1, 2, \ldots, 15\} \). Given a subset \( S = \{s_1, s_2, \ldots, s_j\} \) of \( \{1, 2, \ldots, n-1\} \), the corresponding composition of \( n \) is \( \text{comp}(S) = (s_1, s_2 - s_1, \ldots, s_j - s_{j-1}, n - s_j) \). For \( S = \{2, 3, 5, 9, 10, 14\} \subseteq \{1, 2, \ldots, 15\} \), \( \text{comp}(S) = (2, 1, 2, 4, 1, 4, 2) \). The composition obtained by reversing the order of the parts of \( \alpha \), the reverse of \( \alpha \), is \( \text{rev}(\alpha) = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1) \). The complement of a composition \( \alpha \), denoted \( \alpha^c \) is the composition obtained from \( \alpha \) by taking the complement of the set corresponding to \( \alpha \). That is, \( \alpha^c = \text{comp}(\text{set}(\alpha)^c) \). The transpose of a composition \( \alpha \), denoted \( \alpha^t \) is the composition obtained from \( \alpha \) by taking the complement of the set corresponding to the reverse of \( \alpha \). That is, \( \alpha^t = \text{comp}(\text{set}(\text{rev}(\alpha))^c) \).

For example, if \( \alpha = (3, 1, 2, 4) \), \( \text{rev}(\alpha) = (4, 2, 1, 3) \), \( \text{set}(\text{rev}(\alpha)) = \{4, 6, 7\} \), \( \text{set}(\text{rev}(\alpha))^c = \{1, 2, 3, 5, 8, 9\} \), so \( \alpha^t = (1, 1, 2, 3, 1, 1) \).

We will use several different orders on compositions. The lexicographic order will be denoted \( \leq \). We say that a composition \( \beta = (\beta_1, \ldots, \beta_m) \) is a refinement of a composition \( \alpha =\)
(\alpha_1, \ldots, \alpha_k), denoted \( \beta \preceq \alpha \), if each part of \( \alpha \) can be obtained by adding consecutive parts of \( \beta \). Equivalently, we say that \( \alpha \) is a **coarsening** of \( \beta \). For example, \( \beta = (1, 2, 1, 1, 3, 2) \) is a refinement of \( \alpha = (3, 2, 5) \). Finally, we use an order, defined in [4], where \( \alpha \subset_s \beta \) if

1. \( |\beta| = |\alpha| + s \),
2. \( \alpha_j \leq \beta_j, \ \forall \ 1 \leq j \leq \ell(\alpha) \), and
3. \( \ell(\beta) \leq \ell(\alpha) + 1 \).

Note that the last two parts guarantee that \( \ell(\alpha) \leq \ell(\beta) \leq \ell(\alpha) + 1 \). If we have only the second condition then this is denoted \( \alpha \subseteq \beta \).

A function \( f \in \mathbb{Q}[x_1, x_2, \ldots] \) is **quasisymmetric** if the coefficient of \( x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_k^{\alpha_k} \) is the same as the coefficient of \( x_1^{i_1}x_2^{i_2} \cdots x_k^{i_k} \) for every \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \) and \( i_1 < i_2 < \cdots < i_k \). The set of all quasisymmetric functions forms a Hopf algebra graded by degree, \( \text{QSym} = \bigoplus_n \text{QSym}_n \), where each \( \text{QSym}_n \) is a vector space over \( \mathbb{Q} \) with bases indexed by compositions of \( n \).

The pertinent bases for our purposes include the **monomial**, **fundamental**, **dual immaculate**, and **quasisymmetric Schur** bases. We define the monomial and fundamental bases here and defer the remaining definitions until later.

Given a composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \) of \( n \), the **monomial quasisymmetric function** is

\[
M_{\alpha} = \sum_{(i_1, i_2, \ldots, i_k)} x_1^{i_1}x_2^{i_2} \cdots x_k^{i_k}.
\]

A second important quasisymmetric basis is the fundamental basis. Given a composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \) of \( n \), the **fundamental quasisymmetric function** indexed by \( \alpha \) is

\[
F_{\alpha}(x_1, x_2, \ldots) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1}x_{i_2} \cdots x_{i_n}.
\]

Note that

\[
1 \quad F_{\alpha} = \sum_{\beta \preceq \alpha} M_{\beta} \quad \text{and} \quad M_{\alpha} = \sum_{\beta \preceq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} F_{\beta}.
\]

In [8] the **noncommutative symmetric functions** are defined as the algebra \( \text{NSym} = \mathbb{Q}\langle e_1, e_2, \ldots \rangle \) generated by noncommuting indeterminates \( e_n \) of degree \( n \). The set of noncommutative symmetric functions forms a graded Hopf algebra \( \text{NSym} = \bigoplus_n \text{NSym}_n \) where the degree of functions in \( \text{NSym}_n \) is \( n \). Each \( \text{NSym}_n \) has bases indexed by compositions of \( n \).

The \( n \)th elementary noncommutative symmetric function is the indeterminate \( e_n \), where \( e_0 = 1 \). Given a composition \( \alpha = (\alpha_1, \ldots, \alpha_k) \), we define the **elementary noncommutative symmetric function** by

\[
e_{\alpha} = e_{\alpha_1} \cdots e_{\alpha_k}.
\]
The $n$th complete homogeneous noncommutative symmetric function is defined by
\[ h_n = \sum_{(\alpha_1, \ldots, \alpha_m) \vdash n} (-1)^{n-m} e_{\alpha} \]
with $h_0 = 1$. Then, for $\alpha = (\alpha_1, \ldots, \alpha_k)$, the complete homogeneous noncommutative symmetric function is defined by
\[ h_\alpha = h_{\alpha_1} \cdots h_{\alpha_k}. \]

We can write $h_\alpha$ in terms of the elementary noncommutative symmetric functions by
\[ h_\alpha = \sum_{\beta \preceq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} e_\beta \]
where the sum is over all $\beta$ that refine $\alpha$.

The noncommutative ribbon Schur function is defined by
\[ r_\alpha = \sum_{\beta \succ \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} h_\beta \]
where the sum is over all $\beta$ that are coarsenings of $\alpha$.

As Hopf algebras, QSym and NSym are dual with the pairing
\[ \langle h_\alpha, M_\beta \rangle = \delta_{\alpha\beta} \]
and
\[ \langle r_\alpha, F_\beta \rangle = \delta_{\alpha\beta} \]
where $\delta_{\alpha\beta}$ is 1 if $\alpha = \beta$ and 0 otherwise.

Recall that in Sym there is an automorphism $\omega : \text{Sym} \to \text{Sym}$ such that $\omega(s_\lambda) = s_{\lambda'}$ where $\lambda'$ is the transpose of the partition $\lambda$ and $s_\lambda$ denotes the symmetric Schur function. In QSym we have three involutive automorphisms [12], $\psi, \rho,$ and $\omega$ defined on the fundamental basis by
\[ \psi(F_\alpha) = F_{\alpha^c} \]
\[ \rho(F_\alpha) = F_{\text{rev}(\alpha)} \]
\[ \omega(F_\alpha) = F_{\alpha^t}. \]

These maps all commute and $\omega = \rho \circ \psi = \psi \circ \rho$.

There are corresponding involutions in NSym, denoted by the same letters, and defined on the noncommutative ribbon basis by
\[ \psi(r_\alpha) = r_{\alpha^c} \]
\[ \rho(r_\alpha) = r_{\text{rev}(\alpha)} \]
\[ \omega(r_\alpha) = r_{\alpha^t} \]
In NSym, \(\rho\) and \(\omega\) are anti-automorphisms while \(\psi\) is an automorphism. We also have that \(\psi(h_\alpha) = e_\alpha, \rho(h_\alpha) = h_{\text{rev}(\alpha)}\) and \(\omega(h_\alpha) = e_{\text{rev}(\alpha)}\).

**Proposition 2.2.** The pairing between QSym and NSym is invariant under the map \(\psi\). That is, for \(F \in \text{QSym}\) and \(g \in \text{NSym}\), we have
\[
\langle g, F \rangle = \langle \psi(g), \psi(F) \rangle.
\]

**Proof.** It suffices to check that the equality holds for the noncommutative ribbon basis elements \(g = r_\alpha\) and the basis of fundamental quasisymmetric functions \(F = F_\beta\), where \(\alpha, \beta\) are compositions of \(n\). But this is clear from the preceding definitions. \(\square\)

Recall from [12, Section 3.4.2], the *forgetful* map
\[
\chi : \text{NSym} \longrightarrow \text{Sym}
\]
satisfying \(\chi(e_n) = e_n\). For a composition \(\alpha \vdash n\), as in [12, Section 2.2], let \(\tilde{\alpha}\) be the partition of \(n\) obtained by taking the parts of \(\alpha\) in weakly decreasing order. Then
\[
\chi(h_\alpha) = h_{\tilde{\alpha}}, \quad \chi(e_\alpha) = e_{\tilde{\alpha}}.
\]

**Proposition 2.3.** For \(g \in \text{NSym}\), \((\chi \circ \psi)(g) = (\omega \circ \chi)(g)\).

**Proof.** It suffices to verify the equality for the basis elements \(h_\alpha\). We have
\[
\chi(\psi(h_\alpha)) = \chi(e_\alpha) = e_{\tilde{\alpha}} = \omega(h_{\tilde{\alpha}}) = \omega(\chi(h_\alpha)),
\]
as claimed. \(\square\)

### 2.1. Dual immaculate functions.

The immaculate functions \(S_\alpha\) are a basis of NSym formed by iterated creation operators [4]. Their duals in QSym form the basis consisting of dual immaculate functions, \(S_\alpha^*\). These functions can be defined combinatorially as the generating function for immaculate tableaux.

**Definition 2.4.** Given a composition \(\alpha\), an *immaculate tableau* is a filling, \(D\), of the cells of the diagram of \(\alpha\) such that

1. The leftmost column entries strictly increase from bottom to top.
2. The row entries weakly increase from left to right.

An immaculate tableau of shape \(\alpha \vdash n\) is standard if it is filled with distinct entries taken from \(\{1, 2, \ldots, n\}\). Given an immaculate tableau \(D\), we form a content monomial, \(x^D\), by setting the exponent of \(x_i\) to be \(d_i\), the number of \(i\)'s in the tableau \(D\), namely, \(x^D = x_1^{d_1}x_2^{d_2} \cdots x_k^{d_k}\).
Definition 2.5. The dual immaculate function indexed by the composition $\alpha$ is

$$\mathcal{G}_\alpha^* = \sum_D x^D$$

where the sum is over all immaculate tableaux of shape $\alpha$.

We can rewrite the dual immaculate functions in terms of the fundamental basis as a sum over standard immaculate tableaux. To do this, we first standardize each immaculate tableau and define a descent set on the standard immaculate tableaux. The reading word of an immaculate tableau $D$ is obtained by reading the entries of $D$ from left to right starting with the top row. We can standardize a semi-standard tableau (repeated entries allowed) by replacing all the 1’s in the reading word by 1,2,..., in reading order, then the 2’s, etc.

Example 2.6. An immaculate tableau of shape $\alpha = (3,2,4,1,2)$ that has reading word 6,7,5,3,4,4,5,2,2,1,1,2 and its standardization.

$$T = \begin{array}{c}
6 & 7 \\
5 \\
3 & 4 & 4 & 5 \\
2 & 2 \\
1 & 1 & 2 \\
\end{array} \quad S = \begin{array}{c}
11 & 12 \\
9 \\
6 & 7 & 8 & 10 \\
3 & 4 \\
1 & 2 & 5 \\
\end{array}$$

For a composition $\alpha$, let $\text{SIT}(\alpha)$ denote the set of standard immaculate tableaux of shape $\alpha$.

Given a standard immaculate tableau $S$, the descent set of $S$, denoted $\text{Des}_{\mathcal{G}^*}(S)$, is

$$\text{Des}_{\mathcal{G}^*}(S) = \{ i : i + 1 \text{ appears strictly above } i \text{ in } S \}.$$ 

For the standard immaculate tableau in Example 2.6, $\text{Des}_{\mathcal{G}^*}(S) = \{ 2, 5, 8, 10 \}$.

Then

$$\mathcal{G}_\alpha^* = \sum_S F_{\text{comp}(\text{Des}_{\mathcal{G}^*}(S))}$$

where the sum is over all standard immaculate tableaux.

3. Row-strict dual immaculate functions

In this section we start with a combinatorial definition of a new quasisymmetric function we call the row-strict dual immaculate function.

Definition 3.1. Given a composition $\alpha$, a row-strict immaculate tableau is a filling $U$ such that

1. The leftmost column entries weakly increase from bottom to top.
2. The row entries strictly increase from left to right.
The row-strict dual immaculate function indexed by $\alpha$ is $\mathcal{RG}^*_\alpha = \sum_U x^U$ where the sum is over all row-strict immaculate tableaux of shape $\alpha$, and $x^U$ is the content monomial of the tableau $U$, as in Definition 2.5.

We say the row-strict tableau $U$ is standard if $x^U = x_1 \cdots x_n$. Thus standard row-strict immaculate tableaux coincide with standard immaculate tableaux.

As before, standardization provides us with a way to expand $\mathcal{RG}^*_\alpha$ in terms of the fundamental basis using only standard tableaux.

**Definition 3.2.** Given a row-strict immaculate tableau $T$, the row-strict immaculate reading word of $T$, denoted $\text{rw}_{\mathcal{RG}^*}(T)$, is the word obtained by reading the entries in the rows of $T$ from right to left starting with the bottom row and moving up.

To standardize a row-strict immaculate tableau $T$, replace the 1’s in $T$ with 1, 2, ..., in the order they appear in $\text{rw}_{\mathcal{RG}^*}(T)$, then the 2’s, etc.

**Definition 3.3.** The descent set of a standard row-strict immaculate tableau $T$ is the set

$$\text{Des}_{\mathcal{RG}^*}(T) = \{i : i + 1 \text{ is weakly below } i \text{ in } T\}.$$ 

**Example 3.4.** Consider the row-strict immaculate tableau

$$T = \begin{array}{cccc}
4 & 3 & 4 & 5 & 6 \\
2 & 5 & & \\
1 & 2 & 6 & \\
\end{array}$$

The row-strict immaculate reading word of $T$ is 6, 2, 1, 5, 2, 6, 5, 4, 3, 4 and the corresponding standardized row-strict immaculate tableau is

$$S = \begin{array}{cccc}
6 & 4 & 5 & 8 & 10 \\
3 & 7 & & \\
1 & 2 & 9 & \\
\end{array}$$

and $\text{Des}_{\mathcal{RG}^*}(T) = \{1, 4, 6, 8\}$.

The row-strict dual immaculate functions expand positively in the fundamental basis.

**Theorem 3.5.** Let $\alpha \models n$. Then

$$\mathcal{RG}^*_\alpha = \sum_S F_{\text{comp}((\text{Des}_{\mathcal{RG}^*}(S)))}$$

where the sum is over all standard row-strict immaculate tableaux of shape $\alpha$. 
Proof. Let $T$ be a row-strict immaculate tableau of shape $\alpha$. Then $T$ standardizes to some standard row-strict immaculate tableau $S$. Suppose $i \in \text{Des}_{R^*}(S)$. Then $i + 1$ is weakly below $i$ in $S$. If $i$ and $i + 1$ are in the same row of $S$, then the entry of $T$ replaced by $i$ is strictly less than the label replaced by $i + 1$ since rows of $T$ strictly increase. If $i + 1$ is in a lower row than $i$, then the entry of $T$ replaced by $i$ must be strictly less than the entry replaced by $i + 1$, else the standardization process was not followed. Thus $x^T$ has strict increases at each position in $\text{Des}_{R^*}(S)$ and $x^T$ is a monomial in $F_{\text{comp}}(\text{Des}_{R^*}(S))$. Thus every monomial in $R^*_\alpha$ appears in $\sum_S F_{\text{comp}}(\text{Des}_{R^*}(S))$.

Now let $S$ be a standard row-strict immaculate tableau and let $x_{i_1} \cdots x_{i_n}$ with $i_1 \leq i_2 \leq \cdots \leq i_n$ be a monomial in $F_{\text{comp}}(\text{Des}_{R^*}(S))$. Create a new diagram $T$ from $S$ by replacing each entry $k$ in $S$ with $i_k$. If $i_k = i_{k+1}$ then $k \notin \text{Des}_{R^*}(S)$, so $k$ must appear strictly below $k + 1$ in $S$ and thus each entry in a row of $T$ is distinct and increases left to right. By construction, the first column will weakly increase from bottom to top. Thus $T$ is a semi-standard row-strict immaculate tableau with content $(i_1, \ldots, i_n)$, and $x_{i_1} \cdots x_{i_n}$ is a monomial in $R^*_\alpha$. □

Example 3.6. Let

$$S = \begin{array}{cccc}
6 & 5 & 8 & 10 \\
4 & 3 & 7 & \\
1 & 2 & \\
\end{array}$$

be a standard row-strict immaculate tableau. Then $\text{Des}_{R^*}(S) = \{1, 4, 6, 8\}$ and $x^{P} = x_1x_2^2x_3x_4^2x_5^2x_6^2$ is a monomial in $F_{\text{comp}}(\text{Des}_{R^*}(S))$. We can “destandardize” $S$ as described in the proof of Theorem 3.5 to obtain

$$T = \begin{array}{cccc}
4 & 5 & 6 \\
3 & 4 & \\
2 & 5 & \\
1 & \\
\end{array}$$

For any standard immaculate tableau $S$, note by definition that $\text{Des}(S) = \text{Des}(S)^c$.

It will be helpful to know how the involutions $\psi, \rho,$ and $\omega$ act on $\mathcal{S}^*_\alpha$.

Theorem 3.7. Let $\alpha$ be a composition. Then

\begin{align*}
(10) & \quad \psi(\mathcal{S}^*_\alpha) = R\mathcal{S}^*_\alpha \\
(11) & \quad \rho(\mathcal{S}^*_\alpha) = R\mathcal{S}^*_{\text{rev}(\alpha)} \\
(12) & \quad \omega(\mathcal{S}^*_\alpha(x_1, \ldots, x_n)) = R\mathcal{S}^*_\alpha(x_n, \ldots, x_1). 
\end{align*}
Proof. Let $\alpha$ be a composition. Recall from [4] that $\psi(F_\alpha) = F_{\alpha^c}$. Then

$$\psi(S^*_{\alpha}) = \psi\left(\sum_{S} F_{\text{comp}(\text{Des}_{\alpha^c}(S))}\right) = \sum_{S} \psi(F_{\text{comp}(\text{Des}_{\alpha^c}(S))}) = \sum_{S} F_{\text{comp}(\text{Des}_{\alpha^c}(S)^c)} = \sum_{S} F_{\text{comp}(\text{Des}_{R\alpha^c}(S))} = R_{\alpha^c} S^*_{\alpha}.$$ 

The other computations follow similarly. □

Corollary 3.8. We have that $\{R_{\alpha^c} S^*_{\alpha} \mid \alpha \models n\}$ is a basis for $\text{QSym}_n$.

Proof. Since $\{S^*_{\alpha} \mid \alpha \models n\}$ is a basis for $\text{QSym}_n$ and $\psi$ is an involution it follows by Theorem 3.7 that $\{R_{\alpha^c} S^*_{\alpha} \mid \alpha \models n\}$ is also a basis for $\text{QSym}_n$. □

Recall that the immaculate functions $S_{\alpha}$ satisfy, by definition,

$$\langle S_{\alpha}, S^*_{\beta} \rangle = \delta_{\alpha\beta}.$$ 

Similarly, by definition, we have row-strict immaculate functions $R S_{\alpha}$ satisfying

$$\langle R S_{\alpha}, R S^*_{\beta} \rangle = \delta_{\alpha\beta}.$$ 

An immediate consequence of these definitions is the effect of the map $\psi$ on $S_{\alpha}$. Using Proposition 2.2 we have, by duality,

$$\delta_{\alpha\beta} = \langle S_{\alpha}, S^*_{\beta} \rangle = \langle \psi(S_{\alpha}), \psi(S^*_{\beta}) \rangle = \langle \psi(S_{\alpha}), R S^*_{\beta} \rangle,$$

and hence $\psi(S_{\alpha}) = R S_{\alpha}$.

From [4 Proposition 3.36] we have that the dual immaculate functions are monomial positive:

$$S^*_{\alpha} = \sum_{\beta \leq \ell \alpha} K_{\alpha,\beta} M_{\beta}$$

and thus $K_{\alpha,\beta} = \langle h_{\beta}, S^*_{\alpha} \rangle = \langle e_{\beta}, R S^*_{\alpha} \rangle$. Similarly, for row-strict dual immaculate functions, we have by their definition and that of monomial quasisymmetric functions that

$$R S^*_{\alpha} = \sum_{\beta \leq \ell \alpha} K^*_{\alpha,\beta} M_{\beta}$$
where $K^*_{\alpha, \beta}$ is the number of row-strict immaculate tableaux of shape $\alpha$ and content $\beta$, and thus $K^*_{\alpha, \beta} = \langle h_\beta, R\mathcal{G}_\alpha^* \rangle = \langle e_\beta, \mathcal{S}_\alpha^* \rangle$. Note that $K_{\alpha, \beta} \neq K^*_{\alpha, \beta}$ in general.

Let $L_{\alpha, \beta}$ denote the number of standard immaculate tableaux of shape $\alpha$ with $\mathcal{S}^*$-descent composition $\beta$ and $L^*_{\alpha, \beta}$ denote the number of standard immaculate tableaux of shape $\alpha$ with $R\mathcal{G}^*$-descent composition $\beta$. Given a standard immaculate tableau $T$, we have $L^*_{\alpha, \beta} = L_{\alpha, \beta}^c$ since $\text{Des}_{\mathcal{S}^*}(T)^c = \text{Des}_{R\mathcal{G}^*}(T)$.

**Theorem 3.9.** For $\gamma \leq \ell \alpha$,

$$K^*_{\alpha, \gamma} = \sum_{\beta \succ \gamma} L_{\alpha, \beta}^c = \sum_{\beta \succ \gamma} L^*_{\alpha, \beta}.$$

**Proof.** We have

$$R\mathcal{G}^*_\alpha = \sum_{\gamma} K^*_{\alpha, \gamma} M_\gamma,$$

and

$$R\mathcal{G}^*_\alpha = \sum_{T \in \text{SIT}(\alpha)} F_{\text{comp}(\text{Des}_{R\mathcal{G}^*}(T))} = \sum_{\beta} F_\beta L^*_{\alpha, \beta} = \sum_{\beta} F_\beta L_{\alpha, \beta}^c.$$

Since the monomial expansion of $F_\beta$ is $F_\beta = \sum_{\gamma \leq \beta} M_\gamma$, equating coefficients of $M_\gamma$ gives

$$K^*_{\alpha, \gamma} = \sum_{\beta \succ \gamma} L_{\alpha, \beta}^c = \sum_{\beta \succ \gamma} L^*_{\alpha, \beta}.$$

\[\square\]

3.1. **Creation operators and row-strict immaculate functions.** In [4], the authors defined a family of operators on NSym, modelled after Bernstein’s operators that were used to define the ordinary Schur functions in the Hopf algebra of symmetric functions [13, p. 96 Exercise 29]. This new family of “creation operators” is then used to define the immaculate basis of NSym, and, via the pairing between NSym and its dual QSym, the dual immaculate quasisymmetric functions $\mathcal{S}^*_\alpha$.

In this section we define a variant of the creation operators of [4], and show how they in turn lead to a definition of the row-strict immaculate basis of NSym and our row-strict dual immaculate quasisymmetric functions $\mathcal{R}\mathcal{G}^*_\alpha$.

A pair of dual Hopf algebras $A$ and $B$ over a field $\mathbb{K}$ induces a pairing $\langle \cdot, \cdot \rangle : A \times B \to \mathbb{K}$. Hence for each element $F \in B$, one can define the adjoint operator $F^\perp : A \to A$ by

$$\langle F^\perp(a), b \rangle = \langle a, Fb \rangle.$$
Explicitly, if $\{a_\alpha\}$ and $\{b_\alpha\}$ are bases of $A$ and $B$ respectively so that $\langle a_\alpha, b_\beta \rangle = \delta_{\alpha\beta}$ as before, then the operator $F^\perp$ may be computed according to the formula

\[(13)\quad F^\perp(g) = \sum_\alpha \langle g, Fb_\alpha \rangle a_\alpha.\]

As in [4], we apply this to the graded dual Hopf algebras $A = \text{NSym}$ and $B = \text{QSym}$. Let $\{F_\alpha\}_{\alpha = n}$ be the basis of fundamental quasisymmetric functions in $\text{QSym}$, indexed by the compositions $\alpha$ of the nonnegative integer $n$. We will consider the linear transformation $F^\perp_\alpha$ of $\text{NSym}$ that is adjoint to multiplication by $F_\alpha$ in $\text{QSym}$.

First we record the following important effect of the involution $\psi$ on the adjoint transformation.

**Proposition 3.10.** Let $F \in \text{QSym}, H \in \text{NSym}$. Then

$$\psi[F^\perp(\psi(H))] = [\psi(F)]^\perp(H),$$

or equivalently,

$$\psi[F^\perp(H)] = [\psi(F)]^\perp(\psi(H)).$$

In particular, for the fundamental quasisymmetric function $F_\alpha$ indexed by the composition $\alpha$, we have $F^\perp_\alpha(\psi(H)) = \psi[F^\perp_{\alpha^e}(H)]$ and hence

$$F^\perp_{(1^i)}(\psi(H)) = \psi[F^\perp_{(i)}(H)], \quad F^\perp_{(i)}(\psi(H)) = \psi[F^\perp_{(1^i)}(H)].$$

**Proof.** Let $\{a_\alpha\}_{\alpha = n}$ and $\{b_\alpha\}_{\alpha = n}$ be dual bases of $\text{NSym}$ and $\text{QSym}$ respectively, so that $\langle a_\alpha, b_\beta \rangle = \delta_{\alpha\beta}$.

From Equation (13) we have

$$F^\perp(\psi(H)) = \sum_\alpha \langle \psi(H), Fb_\alpha \rangle a_\alpha = \sum_\alpha \langle H, \psi(F)\psi(b_\alpha) \rangle a_\alpha$$

by Proposition 2.2 and hence

$$\psi[F^\perp(\psi(H))] = \sum_\alpha \langle H, \psi(F)\psi(b_\alpha) \rangle \psi(a_\alpha) = [\psi(F)]^\perp(H),$$

since again Proposition 2.2 implies that duality of bases is preserved under $\psi$. \qed

**Lemma 3.11.** [4] Lemma 2.6] For $i, j > 0$ and $f \in \text{NSym},$

$$F^\perp_{(1^i)}(fh_j) = F^\perp_{(1^i)}(f)h_j + F^\perp_{(1^{i-1})}(f)h_{j-1}; \quad F^\perp_{(i)}(fh_j) = \sum_{k=0}^{\min(i,j)} F^\perp_{(i-k)}(f)h_{j-k}.$$  

In particular we have
The next two definitions are made in [4].

**Definition 3.12.** [4, Definition 3.1] The noncommutative Bernstein operator $B_m$ is defined by
\[ B_m = \sum_{i \geq 0} (-1)^i h_{m+i} F_{\alpha}^\perp(1), \]
and for $\alpha \in \mathbb{Z}^m$,
\[ B_\alpha = B_{\alpha_1} \cdots B_{\alpha_m}. \]
Note that when $i = 0$, $(1^0)$ is the empty composition and thus $F_{(1^0)}^\perp(f) = f = F_{(1)}^\perp(f)$ for all $f \in \text{NSym}$, since $F_{(0)} = 1$ in $\text{QSym}$. Also $F_{(1^1)}^\perp(1) = F_{(1)}^\perp(1) = \begin{cases} 0, & i > 0; \\ 1, & i = 0. \end{cases}$

While we chose duality to define immaculate functions, the following is the original definition, which was proven to be equivalent in [4].

**Definition 3.13.** [4, Definition 3.2] For any $\alpha \in \mathbb{Z}^m$, the immaculate function $S_\alpha \in \text{NSym}$ is given by
\[ S_\alpha = B_\alpha(1) = B_{\alpha_1} \cdots B_{\alpha_m}(1). \]
This definition was inspired by Bernstein’s original definition in the Hopf algebra of symmetric functions for a Schur function $s_\alpha$ indexed by any $m$-tuple $\alpha \in \mathbb{Z}^m$.

As observed in [4, Example 3.3], we have
\[ S_{(m)} = B_m(1) = h_m, \quad S_{(a,b)} = B_a(h_b) = h_a h_b - h_{a+1} h_{b-1}. \]
Applying $\psi$ to Lemma 3.11 and using Proposition 3.10 and the fact that $\psi(F_\alpha) = F_{\alpha e}$, so that $\psi(F_{(1^1)}) = F_{(1)}$ in $\text{NSym}_i$, we obtain

**Lemma 3.14.** For $i, j > 0$ and $f \in \text{NSym}$,
\[ F_{(i)}^\perp(1)(f) e_j = F_{(i)}^\perp(f) e_j + F_{(i-1)}^\perp(f) e_{j-1}; \quad F_{(i^1)}^\perp(f) e_j = \sum_{k=0}^{\min(i, j)} F_{(i-k)}^\perp(f) e_{j-k}. \]
In particular we have
\[
F_{(i)}^\perp(e_j) = \begin{cases} 
0, & i > j \\
e_{j-i}, & 1 \leq i \leq j \\
e_j, & i = 0;
\end{cases} 
F_{(i)}(e_j) = \begin{cases} 
0, & i > 1 \\
e_{j-1}, & i = 1 \\
e_j, & i = 0.
\end{cases}
\]

Now we define new operators as follows.

**Definition 3.15.** Define the noncommutative Bernstein operator \( B_{rs}^m \) by
\[
B_{rs}^m = \sum_{i \geq 0} (-1)^i e_{m+i} F_{(i)}^\perp,
\]
and for \( \alpha \in \mathbb{Z}^m \),
\[
B_{\alpha}^{rs} = B_{\alpha_1}^{rs} \cdots B_{\alpha_m}^{rs}.
\]

Note that when \( i = 0 \), this is the empty composition and \( F_\emptyset = 1 \) in \( \text{QSym} \), and thus \( F_{(0)}^\perp(f) = f = F_{(0)}^\perp(f) \) for all \( f \in \text{NSym} \).

Furthermore we have the following.

**Lemma 3.16.** For \( \alpha \in \mathbb{Z}^m \), \( \psi(\mathcal{S}_\alpha) = B_{\alpha}^{rs}(1) \).

*Proof.* From the above properties, it is clear that
\[
B_{m}^{rs}(1) = e_m, \quad \psi(\mathcal{S}_{(a,b)}) = B_{\alpha}^{rs}(e_b) = e_a e_b - e_{a+1} e_{b-1}.
\]

Hence the result is true for \( m \leq 2 \). Let \( f \in \text{NSym} \). We claim that
\[(14) \quad \psi(B_{m}(f)) = B_{m}^{rs}(\psi(f)).\]

We have
\[
\psi(B_{m}(f)) = \psi\left( \sum_{i \geq 0} h_{m+i} F_{(i')}^\perp(f) \right) = \sum_{i \geq 0} e_{m+i} \psi[F_{(i')}^\perp(f)]
= \sum_{i \geq 0} e_{m+i} F_{(i)}^\perp(\psi(f)) = B_{m}^{rs}(\psi(f)),
\]
where the penultimate equality is thanks to Proposition 3.10.

Since for \( \alpha \in \mathbb{Z}^m \),
\[
B_{\alpha}(1) = B_{\alpha_1}(f), \quad f = B_{\alpha_2} \cdots B_{\alpha_m}(1),
\]
the result now follows by induction. \( \square \)

**Theorem 3.17.** The row-strict immaculate function \( R\mathcal{S}_\alpha \) can be defined as the result of applying a creation operator as follows:
\[
R\mathcal{S}_\alpha = B_{\alpha}^{rs}(1).
\]
Proof. Immediate from the preceding lemma, since we already know that $\mathcal{R}\mathcal{S}_\alpha = \psi(\mathcal{S}_\alpha)$. □

Finally, just as left multiplication by $h_m$ can be expressed in terms of creation operators [4, Remark 3.6], we have the following.

Lemma 3.18. Left multiplication by $h_m$ in $\text{NSym}$ can be expressed as applying the operator

$$h_m = \sum_{i \geq 0} \mathcal{B}_{m+1} F_{(i)}^\dagger,$$

and left multiplication by $e_m$ in $\text{NSym}$ can be expressed as applying the operator

$$e_m = \sum_{i \geq 0} \mathcal{B}_{m+1} F_{(i)}^\dagger.$$

Proof. Immediate from Equation (14). □

3.2. Results obtained by using $\psi$. We can immediately obtain the row-strict analogue of many results in [4] by using the $\psi$ involution. We list here the most pertinent for the remainder of the paper. We leave results for skew row-strict dual immaculate functions to the next section, as the combinatorial definition is not obviously equivalent.

Theorem 3.19. (1) [4, Lemma 3.4] For $s \geq 0, m \in \mathbb{Z}$ and $f \in \text{NSym}$,

$$\mathcal{B}_m(f)h_s = \mathcal{B}_{m+1}(f)h_{s-1} + \mathcal{B}_m(fh_s)$$

$$\iff \mathcal{B}_m^r(f)e_s = \mathcal{B}_{m+1}^r(f)e_{s-1} + \mathcal{B}_m^r(fe_s).$$

(2) [4, Theorem 3.5] (Multiplicity-free right Pieri rule)

$$\mathcal{S}_\alpha h_s = \sum_{\alpha \leq \beta} \mathcal{S}_\beta \iff \mathcal{R}\mathcal{S}_\alpha e_s = \sum_{\alpha \leq \beta} \mathcal{R}\mathcal{S}_\beta.$$

(3) [4, Proposition 3.32] (Multiplicity-free right Pieri rule) For a composition $\alpha$ and $s \geq 0$,

$$\mathcal{S}_\alpha \mathcal{S}_{(1^s)} = \mathcal{S}_\alpha e_s = \sum_{\beta} \mathcal{S}_\beta \iff \mathcal{R}\mathcal{S}_\alpha \mathcal{R}\mathcal{S}_{(1^s)} = \mathcal{R}\mathcal{S}_\alpha h_s = \sum_{\beta} \mathcal{R}\mathcal{S}_\beta,$$

where the summation ranges over compositions of $\beta$ of $|\alpha| + s$ such that $\alpha_i \leq \beta_i \leq \alpha_i + 1$ and $\alpha_i = 0$ for $i > \ell(\alpha)$.

(4) [4, Corollary 3.31]

$$\mathcal{S}_{(1^n)} = \sum_{\alpha \vdash n} (-1)^{n-\ell(\alpha)} h_\alpha = e_n \iff \mathcal{R}\mathcal{S}_{(1^n)} = \sum_{\alpha \vdash n} (-1)^{n-\ell(\alpha)} e_\alpha = h_n.$$
Theorem 3.27 \((Jacobi-Trudi)\) For \(\ell(\alpha) = m\),

\[
\mathcal{S}_\alpha = \sum_{\sigma \in S_m} (-1)^{\text{sgn}(\sigma)}h_{(\alpha_1 + \sigma_1 - 1, \alpha_2 + \sigma_2 - 2, \ldots, \alpha_m + \sigma_m - m)}
\]

\(\Leftrightarrow \psi \quad \mathcal{R}\mathcal{S}_\alpha = \sum_{\sigma \in S_m} (-1)^{\text{sgn}(\sigma)}e_{(\alpha_1 + \sigma_1 - 1, \alpha_2 + \sigma_2 - 2, \ldots, \alpha_m + \sigma_m - m)}\)

where \(S_m\) is the symmetric group on \(m\) elements and \((-1)^{\text{sgn}(\sigma)}\) is the sign of \(\sigma\).

Corollary 3.31

\[
\mathcal{S}_{(1^n)} = \sum_{\alpha \models n} (-1)^{n-\ell(\alpha)}h_\alpha = \mathcal{R}\mathcal{S}_{(1^n)} = \sum_{\alpha \models n} (-1)^{n-\ell(\alpha)}e_\alpha = h_n.
\]

Also from Lemma 2.5 and Equation (14),

\[
F_{(1^r)}(\mathcal{S}_{(1^n)}) = \mathcal{S}_{(n-r)}, \quad \text{and for } s > 1, F_{(s)}(\mathcal{S}_{(1^n)}) = 0;
\]

\(\Leftrightarrow \quad F_{(r)}(\mathcal{R}\mathcal{S}_{(1^n)}) = \mathcal{R}\mathcal{S}_{(n-r)}, \quad \text{and for } s > 1, F_{(1^s)}(\mathcal{R}\mathcal{S}_{(1^n)}) = 0.\)

Proposition 3.16 and Corollary 3.18

\[
h_\beta = \sum_{\alpha \geq \ell} K_{\alpha, \beta} \mathcal{S}_\alpha \Leftrightarrow e_\beta = \sum_{\alpha \geq \ell} K_{\alpha, \beta} \mathcal{R}\mathcal{S}_\alpha
\]

and by Theorem 3.9

\[
h_\beta = \sum_{\alpha \geq \ell} K^*_{\alpha, \beta} \mathcal{R}\mathcal{S}_\alpha \Leftrightarrow e_\beta = \sum_{\alpha \geq \ell} K^*_{\alpha, \beta} \mathcal{S}_\alpha.
\]

Theorem 3.25 \(\quad\text{The ribbon function } r_\beta \text{ expands positively in both immaculate bases:}\)

\[
r_\beta = \sum_{\alpha \geq \ell} L_{\alpha, \beta} \mathcal{S}_\alpha \Leftrightarrow r_\beta = \sum_{\alpha \geq \ell} L_{\alpha, \beta} \mathcal{R}\mathcal{S}_\alpha.
\]

Theorem 3.38 \(\quad\text{The Schur function } s_\lambda \text{ with } \ell(\lambda) = k \text{ expands into the dual immaculate and row-strict dual immaculate bases as follows:}\)

\[
s_\lambda = \sum_{\sigma \in S_k} (-1)^{\text{sgn}(\sigma)} \mathcal{S}_\sigma^* \Leftrightarrow s_\lambda = \sum_{\sigma \in S_k} (-1)^{\text{sgn}(\sigma)} \mathcal{R}\mathcal{S}_\sigma^* \]

\(\text{taking } \mathcal{S}_\sigma^*(\lambda) = 0 = \mathcal{R}\mathcal{S}_\sigma^*(\lambda) = 0 \text{ if } \sigma \text{ and } \lambda \text{ do not satisfy the condition below: for } \lambda \text{ a partition and } \sigma \in S_{\ell(\lambda)}, \text{ we define } \sigma(\lambda) = (\lambda_{\sigma_1} + 1 - \sigma_1, \ldots, \lambda_{\sigma_k} + k - \sigma_k) \text{ provided } \lambda_{\sigma_i} + i - \sigma_i > 0 \text{ for each } i.\)
(10) [2] Theorem 1.1] For $\alpha$ a composition and $c_{\alpha\beta} \geq 0$,
\[ S_\alpha^* = \sum_{\beta} c_{\alpha\beta} \hat{S}_\beta \iff R S_\alpha^* = \sum_{\beta} c_{\alpha\beta} \hat{R} S_\beta, \]
where $\hat{S}$ and $\hat{R} S$ are the Young quasisymmetric Schur and row-strict quasisymmetric Schur functions.

4. Skew row-strict dual immaculate functions

Following the work of Berg et. al. [4], we define the poset $\mathcal{P}$ of immaculate tableaux. The labelled poset $\mathcal{P}$ is on the set of all compositions. Place an arrow from $\alpha$ to $\beta$ if $\alpha$ and $\beta$ differ by a single box, denoted $\beta \subset_1 \alpha$. The label of $m$ on each cover $\alpha \xrightarrow{m} \beta$ denotes the row containing the single additional box. Denote the path from $\alpha$ to $\beta$ in $\mathcal{P}$ by $P = [\alpha, \beta]$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The start of the poset $\mathcal{P}$ with edge labels. A horizontal 3-strip is shown in red and a vertical 3-strip is shown in blue.}
\end{figure}

To obtain a standard skew immaculate tableau from a path $P = [\alpha, \beta]$, for each $m_i$, $1 \leq i \leq k$, label the rightmost unlabeled cell in row $m_i$ of $\alpha$ with $k - i + 1$, see Example 4.2. In order to understand the combinatorial models for skew dual immaculate and skew row-strict dual immaculate functions we define two special types of paths.

Definition 4.1. A path $P = \{ \alpha = \beta^{(0)} \xrightarrow{m_1} \beta^{(1)} \xrightarrow{m_2} \cdots \xrightarrow{m_k} \beta^{(k)} = \beta \}$ in the poset $\mathcal{P}$ is a
\begin{itemize}
\item \textit{horizontal $k$-strip} if $m_1 \leq m_2 \leq \cdots \leq m_k$, and a
\item \textit{vertical $k$-strip} if $m_1 > m_2 > \cdots > m_k$.
\end{itemize}
The horizontal 3-strip (red path) and vertical 3-strip (blue path) in Figure 1 give rise to the following tableaux.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 2 & 1 \\
\end{array}
\]

\begin{itemize}
\item horizontal strip
\item vertical strip
\end{itemize}

We can directly define a standard skew immaculate tableau of shape $\alpha/\beta$ as a standard filling of the shape $\alpha/\beta$ such that rows strictly increase from left to right and the labels in $\alpha/\beta$ in cells that are in the first column of $\alpha$ must increase from bottom to top. For a path $P = [\alpha, \beta]$ of length $k$, define the descent set of $P$ to be $D(P) = \{k - i : m_i > m_{i+1}\}$ and the weak ascent set of $P$ to be $A(P) = \{k - i : m_i \leq m_{i+1}\}$. Each such path $P = [\alpha, \beta]$ corresponds to a unique standard skew immaculate tableau $T$ of shape $\alpha/\beta$, and conversely. Furthermore, the descent set $D(P)$ coincides with the descent set $\text{Des}_{S^*}(T) = \{i : i + 1 \text{ appears strictly above } i \text{ in } T\}$, and similarly the ascent set $A(P)$ coincides with the descent set $\text{Des}_{R^*}(T) = \{i : i + 1 \text{ appears weakly below } i \text{ in } T\}$.

**Example 4.2.** For $\alpha/\beta = (3, 2, 3)/(1, 1, 2)$,

\[
T = \begin{array}{ccc}
& 1 \\
2 & \\
3 & 4 \\
\end{array}
\]

is a valid standard skew immaculate tableau. It corresponds to the path $P = (3, 2, 3) \rightarrow (2, 2, 3) \rightarrow (1, 2, 3) \rightarrow (1, 1, 3) \rightarrow (1, 1, 2)$. Further, $\text{Des}_{R^*}(T) = \{1, 2, 3\}$, $D(P)$ is empty, and $A(P) = \{1, 2, 3\}$.

Given a path $P = [\alpha, \emptyset]$ corresponding to a standard immaculate tableau $T$, we have that $\text{Des}_{S^*}(T) = D(P)$ and $\text{Des}_{R^*}(T) = A(P)$, by comparing the definitions, and is illustrated in Figure 2

\[
T = \begin{array}{ccc}
4 & 7 \\
2 & 3 & 5 \\
1 & 6 \\
\end{array}
\]

$P = (2, 3, 2) \rightarrow (2, 3, 1) \rightarrow (1, 3, 1) \rightarrow (1, 2, 1) \rightarrow (2, 1) \rightarrow (1, 1) \rightarrow (1) \rightarrow \emptyset$

**Figure 2.** The path $P$ has $D(P) = \{1, 3, 6\}$ and $A(P) = \{2, 4, 5\}$, while $\text{Des}_{S^*}(T) = \{1, 3, 6\}$ and $\text{Des}_{R^*}(T) = \{2, 4, 5\}$. 
Note that given a skew immaculate tableau, it can be decomposed into horizontal or vertical strips in several ways. An example of decomposing a tableau into either horizontal or vertical strips is given in Figure 3.

\[
T = \begin{array}{ccc}
2 & 4 & 5 \\
3 \\
1
\end{array}
\]

\[
P = (3, 2, 3) \rightarrow (3, 2, 2) \rightarrow (3, 2, 1) \rightarrow (3, 1, 1) \rightarrow (3, 1) \rightarrow (2, 1)
\]

**Figure 3.** The standard skew immaculate tableau \(T\) and its corresponding path can be decomposed into maximal horizontal strips \((3, 2, 3) \rightarrow (3, 2, 2) \rightarrow (3, 1, 1), (3, 1) \rightarrow (2, 1)\). Alternatively, decompose \(P\) into maximal vertical strips \((3, 2, 3) \rightarrow (3, 2, 2), (3, 2, 1) \rightarrow (3, 1, 1), (3, 1, 1) \rightarrow (3, 1) \rightarrow (2, 1)\).

In [4] the poset \(\mathcal{P}\) and horizontal strips are used to define the skew dual immaculate functions as follows.

**Definition 4.3.** For \(\{\gamma : \beta \subseteq \gamma \subseteq \alpha\}\) an interval in \(\mathcal{P}\), define the *skew dual immaculate function* to be

\[
\mathcal{G}^*_{\alpha/\beta} = \sum_\gamma \langle \mathcal{G}_\beta h_\gamma, \mathcal{G}^*_\alpha \rangle M_\gamma.
\]

This can be rewritten in terms of both the fundamental basis and the dual immaculate basis.

**Proposition 4.4.** [4, Propositions 3.47 and 3.48] For \(\{\gamma : \beta \subseteq \gamma \subseteq \alpha\}\) an interval in \(\mathcal{P}\),

\[
\mathcal{G}^*_{\alpha/\beta} = \sum_\gamma \langle \mathcal{G}_\beta r_\gamma, \mathcal{G}^*_\alpha \rangle F_\gamma \tag{15}
\]

\[
= \sum_\gamma \langle \mathcal{G}_\beta \mathcal{G}_\gamma, \mathcal{G}^*_\alpha \mathcal{G}^*_\gamma \rangle \tag{16}
\]

\[
= \sum_{P=[\beta, \alpha] \in \mathcal{P}} F_{\text{comp}(D(P))} = \sum_{T \text{ a standard skew immaculate tableau of shape } \alpha/\beta} F_{\text{comp}(\text{Des}_{\mathcal{G}^*}(T))}; \tag{17}
\]

in the last line, each path \(P\) from \(\beta\) to \(\alpha\) corresponds to a unique standard skew immaculate tableau \(T\) of shape \(\alpha/\beta\).

Note that the number of standard skew immaculate tableaux \(T\) of shape \(\alpha/\beta\) with \(\text{comp}(\text{Des}_{\mathcal{G}^*}(T)) = \gamma\) is \(\langle \mathcal{G}_\beta r_\gamma, \mathcal{G}^*_\alpha \rangle\).
**Definition 4.5.** For \( \{ \gamma : \beta \subseteq \gamma \subseteq \alpha \} \) an interval in \( \mathfrak{P} \), define the *skew row-strict dual immaculate function* to be

\[
\mathcal{R} \mathcal{S}_{\alpha/\beta}^* = \sum_{\gamma} \langle \mathcal{R} \mathcal{S}_{\beta} h_{\gamma}, \mathcal{R} \mathcal{S}_{\alpha}^* \rangle M_{\gamma}.
\]

We now quickly obtain the following.

**Theorem 4.6.** For \( \{ \gamma : \beta \subseteq \gamma \subseteq \alpha \} \) an interval in \( \mathfrak{P} \),

\[
\mathcal{R} \mathcal{S}_{\alpha/\beta}^* = \sum_{\gamma} \langle \mathcal{R} \mathcal{S}_{\beta} r_{\gamma}, \mathcal{R} \mathcal{S}_{\alpha}^* \rangle F_{\gamma} = \psi(\mathcal{S}_{\alpha/\beta}^*) = \sum_{\gamma} \langle \mathcal{R} \mathcal{S}_{\beta} \mathcal{R} \mathcal{S}_{\gamma}, \mathcal{R} \mathcal{S}_{\alpha}^* \mathcal{R} \mathcal{S}_{\gamma}^* \rangle
\]

\[
= \sum_{P=\beta,\alpha} F_{\text{comp}(\mathcal{A}(P))} = \sum_{T \text{ a standard skew immaculate tableau of shape } \alpha/\beta} F_{\text{comp}(\text{Des}_{\mathcal{R} \mathcal{S}_{\alpha}^*}(T))}.
\]

**Proof.** The first equality is immediate from Definition 4.5 by using (3) to expand \( h_{\gamma} \) in terms of the ribbon basis, interchanging the order of summation, and finally using (1):

\[
\mathcal{R} \mathcal{S}_{\alpha/\beta}^* = \sum_{\gamma} \langle \mathcal{R} \mathcal{S}_{\beta} r_{\tau}, \mathcal{R} \mathcal{S}_{\alpha}^* \rangle F_{\tau} = \psi(\mathcal{S}_{\alpha/\beta}^*) = \sum_{\gamma} \langle \mathcal{R} \mathcal{S}_{\beta} \mathcal{R} \mathcal{S}_{\gamma}, \mathcal{R} \mathcal{S}_{\alpha}^* \mathcal{R} \mathcal{S}_{\gamma}^* \rangle
\]

The second line now follows by applying \( \psi \) to the first equality in Proposition 4.4, and using the invariance of the pairing under \( \psi \), which gives

\[
\psi(\mathcal{S}_{\alpha/\beta}^*) = \sum_{\gamma} \langle \psi(\mathcal{S}_{\beta} \psi(\mathcal{S}_{\alpha}) \rangle \psi(F_{\gamma}) = \sum_{\gamma} \langle \mathcal{R} \mathcal{S}_{\beta} r_{\gamma}, \mathcal{R} \mathcal{S}_{\alpha}^* \rangle F_{\gamma},
\]

where we have used (10), (7) and (4). The last two lines are now immediate by applying \( \psi \) to the last two equations in Proposition 4.4, and since \( \mathcal{A}(P) \) and \( \mathcal{D}(P) \) are complementary by definition, and each path \( P \) from \( \beta \) to \( \alpha \) corresponds to a unique standard skew immaculate tableau \( T \) of shape \( \alpha/\beta \). \( \square \)

**Definition 4.7.** Let \( \alpha \) and \( \beta \) be compositions with \( \beta \subseteq \alpha \). Then a filling \( T \) of the diagram of \( \alpha/\beta \) is a *skew immaculate tableau* provided

1. the entries in the first column of \( \alpha \) (if any remain in \( \alpha/\beta \)) are strictly increasing from bottom to top, and
2. rows weakly increase from left to right.

Similarly, \( T \) is a *skew row-strict immaculate tableau* if
(1) the entries in the first column of \( \alpha \) (if any remain in \( \alpha/\beta \)) are weakly increasing from bottom to top, and

(2) rows strictly increase from left to right.

We now have the needed interpretation of the coefficients in Definitions 4.3 and 4.5 to rewrite \( \mathcal{G}^*_{\alpha/\beta} \) and \( \mathcal{R}\mathcal{G}^*_{\alpha/\beta} \) as generating functions of skew immaculate tableaux.

**Theorem 4.8.** Let \( \alpha \) and \( \beta \) be compositions with \( \beta \subseteq \alpha \). Then

\[
\mathcal{G}^*_{\alpha/\beta} = \sum_T x^T
\]

where the sum is over all skew immaculate tableaux of shape \( \alpha/\beta \), and

\[
\mathcal{R}\mathcal{G}^*_{\alpha/\beta} = \sum_T x^T
\]

where the sum is over all skew row-strict immaculate tableaux of shape \( \alpha/\beta \).

**Proof.** By Point (3) in Theorem 3.19, we know that for \( \gamma = \gamma_1 \gamma_2 \cdots \gamma_k \), \( \alpha \) can be obtained from \( \beta \) by a series of vertical strips of lengths \( \gamma_1, \gamma_2, \ldots, \gamma_k \). Thus the coefficient \( \langle \mathcal{R}\mathcal{G}_\beta h_\gamma, \mathcal{G}^*_{\alpha} \rangle \) represents the number of ways to add a sequence of vertical strips of lengths \( \gamma_1, \gamma_2, \ldots, \gamma_k \) from \( \beta \) to \( \alpha \), which counts the number of skew immaculate tableaux \( T \) of shape \( \alpha/\beta \) such that the descent composition of \( T \) is coarser than \( \gamma \), since adding a vertical strip after another one may or may not create a descent. Thus

\[
\langle \mathcal{G}_\beta h_\gamma, \mathcal{G}^*_{\alpha} \rangle
\]

is the number of skew immaculate tableaux of shape \( \alpha/\beta \) of content \( \gamma \) and

\[
\langle \mathcal{R}\mathcal{G}_\beta h_\gamma, \mathcal{R}\mathcal{G}^*_{\alpha} \rangle
\]

is the number of skew row-strict immaculate tableaux of shape \( \alpha/\beta \) of content \( \gamma \). The result now follows immediately from the definitions. \( \square \)

**Example 4.9.** Consider

\[
T = \begin{bmatrix}
1 & 4 \\
3 & \\
2 &
\end{bmatrix}
\]

and corresponding path

\[
P = (2, 2, 2) \xrightarrow{3} (2, 2, 1) \xrightarrow{2} (2, 1, 1) \xrightarrow{1} (1, 1, 1) \xrightarrow{3} (1, 1).
\]

Note that \( T \) can be considered to be formed from vertical strips corresponding to \( \gamma = (1, 3) \) or \((1, 1, 2), \) or \((1, 2, 1) \) or \((1, 1, 1, 1) \) since \( \text{comp}(\text{Des}_{\mathcal{R}\mathcal{G}^*}(T)) = (1, 3) \) and is coarser than the listed options for \( \gamma \).
4.1. **Hopf algebra approach.** We consider the Hopf algebra approach to defining skew dual immaculate functions and establish that it is equivalent to the previous definition. To start, we provide a brief introduction to the necessary Hopf algebra background.

We have that NSym and QSym form dual Hopf algebras using the pairing $\langle \cdot, \cdot \rangle : \text{NSym} \otimes \text{QSym} \to \mathbb{Q}$ defined by $\langle h_\alpha, M_\beta \rangle = \delta_{\alpha\beta}$ where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise.

Given dual bases $\{B_i\}_{i \in I}$ and $\{D_i\}_{i \in I}$, $B_i \cdot B_j = \sum_k b_{i,j}^k B_k \iff \Delta D_k = \sum_{i,j} b_{i,j}^k D_i \otimes D_j$

$D_i \cdot D_j = \sum_k d_{i,j}^k D_k \iff \Delta B_k = \sum_{i,j} d_{i,j}^k B_i \otimes B_j$

where $\cdot$ is the product and $\Delta$ is the coproduct.

For the fundamental quasisymmetric functions, we have that

$\Delta F_\alpha = \sum_{(\beta, \gamma) \text{ with } \beta \cdot \gamma = \alpha \text{ or } \beta \circ \gamma = \alpha} F_\beta \otimes F_\gamma$ (18)

where for $\beta = (\beta_1, \ldots, \beta_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\beta \cdot \gamma = (\beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_n)$ is the concatenation of $\beta$ and $\gamma$, and $\beta \circ \gamma = (\beta_1, \ldots, \beta_{k-1}, \beta_k + 1, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is the near-concatenation of $\beta$ and $\gamma$.

Following [5], we can define the coproduct $\Delta \mathcal{S}_\alpha^*$ in terms of skew elements $\widetilde{\mathcal{S}}_{\alpha/\gamma}^*$.

**Definition 4.10.** Let $\alpha \vdash n$ and define

$\Delta \mathcal{S}_\alpha^* = \sum_{\gamma} \mathcal{S}_\gamma^* \otimes \widetilde{\mathcal{S}}_{\alpha/\gamma}^*.$

We show that $\widetilde{\mathcal{S}}_{\alpha/\gamma}^* = \mathcal{S}_{\alpha/\gamma}^*$ as described in Proposition 4.4.

**Lemma 4.11.**

$\widetilde{\mathcal{S}}_{\alpha/\gamma}^* = \mathcal{S}_{\alpha/\gamma}^* = \sum_T F_{\text{comp}(\text{Des}_{\mathcal{S}_\alpha^*}(T))}$

where the sum is over all standard skew immaculate tableaux $T$ of shape $\alpha/\gamma$.

**Proof.** We use the technique of [5] Proposition 3.1. Let $T$ be a standard skew immaculate tableaux such that $|T| = n$. For any $k$ with $0 \leq k \leq n$, let $\mathcal{U}_k(T)$ be the standardization of the skew tableaux consisting of cells of $T$ with entries $\{n-k+1, \ldots, n\}$. Also let $\Omega_k(T)$ be the skew tableaux consisting of the cells of $T$ after removing the entries $\{k+1, \ldots, n\}$ as in Figure 4.
$T = \begin{array}{ccc}
4 & 5 & 8 \\
* & * & 6 \\
* & 2 & 3 \\
* & 1 & 9 
\end{array}$ \quad \Omega_4(T) = \begin{array}{ccc}
4 & * & * \\
* & * & 2 \\
* & 1 & 3 
\end{array} \quad \mathcal{U}_5(T) = \begin{array}{ccc}
* & 1 & 4 \\
* & 2 & 3 \\
* & * & * \\
* & * & 5 
\end{array}$

\textbf{Figure 4.} An example of $\Omega_{n-k}(T)$ and $\mathcal{U}_k(T)$.

Note that if $T$ is a standard immaculate tableau of shape $\alpha$, then $T = \Omega_{n-k}(T) \cup (\mathcal{U}_k(T) + (n-k))$ where $\mathcal{U}_k(T) + (n-k)$ is $\mathcal{U}_k(T)$ with $n-k$ added to each entry. Suppose $\text{Des}_{\mathcal{S}^*}(T) = \alpha$ with $|\alpha| = n$. Then we can rewrite (18) as

$$\Delta F_\alpha = \sum_{i=0}^{n} F_{\beta_i} \otimes F_{\gamma_i}$$

where $|\beta_i| = n-i$, $|\gamma_i| = i$, and either $\beta_i \cdot \gamma_i = \alpha$ or $\beta_i \odot \gamma_i = \alpha$. Observe that $\beta_i = \text{comp}(\text{Des}_{\mathcal{S}^*}(\Omega_{n-i}(T)))$ and $\gamma_i = \text{comp}(\text{Des}_{\mathcal{S}^*}(\mathcal{U}_i(T)))$.

Then

$$\Delta \mathcal{G}^*_\alpha = \Delta \left( \sum_T F_{\text{comp}(\text{Des}_{\mathcal{S}^*}(T))} \right)$$

$$= \sum_T \Delta F_{\text{comp}(\text{Des}_{\mathcal{S}^*}(T))}$$

$$= \sum_T \sum_{i=0}^{n} F_{\beta_i} \otimes F_{\gamma_i}$$

where $T$ is a standard immaculate tableau of shape $\alpha$.

Further, by Definition 4.10 we have

$$\Delta \mathcal{G}^*_\alpha = \sum_{\delta} \mathcal{G}^*_\delta \otimes \mathcal{G}^*_{\alpha/\delta}$$

$$= \sum_{\delta} \sum_{S} F_{\text{comp}(\text{Des}_{\mathcal{S}^*}(S))} \otimes \mathcal{G}^*_{\alpha/\delta}$$

where $S$ is a standard immaculate tableau of shape $\delta$.

For a fixed $S$ of shape $\delta$ with $|\delta| = n-k$ for some $k$, there exists a standard immaculate tableau $T$ of shape $\alpha$ such that $S = \Omega_{n-k}(T)$. Then $\mathcal{U}_k(T)$ has shape $\alpha/\delta$. Similarly, given a standard immaculate tableau $T$ of shape $\alpha$, $T = \Omega_{n-k}(T) \cup (\mathcal{U}_k(T) + (n-k))$ where $\Omega_{n-k}(T)$ has shape $\delta$ with $|\delta| = n-k$ and $\mathcal{U}_k(T)$ has shape $\alpha/\delta$. Thus

$$\mathcal{G}^*_{\alpha/\delta} = \sum_T F_{\text{comp}(\text{Des}_{\mathcal{S}^*}(T))} = \mathcal{G}^*_{\alpha/\delta}$$
where $T$ is a standard skew immaculate tableau of shape $\alpha/\delta$.

It follows by Theorem 4.6 that we have

$$\Delta R^*_{\alpha} = \sum_{\beta} R^*_{\beta} \otimes R^*_{\alpha/\beta}.$$

4.2. Expansions of skew Schur functions. We can also use a Hopf algebra approach to establish skew versions of Point (9) in Theorem 3.19, from where we recall that for a partition $\lambda$ and $\sigma \in S(\lambda)$, define $\sigma(\lambda) = (\lambda_{\sigma_j} + 1 - \sigma_1, \ldots, \lambda_{\sigma_k} + k - \sigma_k)$ provided $\lambda_{\sigma_i} + i - \sigma_i > 0$ for each $i$.

Also recall that $s_{\lambda/\mu} = \det(h_{\lambda_j - \mu_{j+i-j}})$. If we consider compositions $\alpha \subseteq \lambda$, we can define $s_{\lambda/\alpha} = \det(h_{\lambda_j - \alpha_j - i+j})$. Note that if there exists some $\alpha_j - j = \alpha_k - k$ for some $j \neq k$, $s_{\lambda/\alpha} = 0$ since two columns of the matrix will be equal. If no such pair $j, k$ exists, then there exists a unique permutation $\tau$ such that $\tau(\alpha) = (\alpha_{\tau_1} + 1 - \tau_1, \ldots, \alpha_{\tau_k} + k - \tau_k) = \mu$ where $\mu$ is a partition. In this case,

$$s_{\lambda/\mu} = (-1)^{\text{sgn}(\tau)} s_{\lambda/\alpha}.$$ (19)

**Theorem 4.12.** Let $\lambda$ and $\mu$ be partitions with $\mu \subseteq \lambda$. Then

$$s_{\lambda/\mu} = \sum_{\sigma \in S(\lambda)} (-1)^{\text{sgn}(\sigma) + \text{sgn}(\tau)} \mathcal{S}^*_{\sigma(\lambda)/\tau(\mu)}$$

for any choice of $\tau$ such that $\tau(\mu)$ is a composition.

**Proof.** Recall that $\Delta(s_\lambda) = \sum_\mu s_{\lambda/\mu} \otimes s_\mu = \sum_\mu s_\mu \otimes s_{\lambda/\mu}$ because the Hopf algebra of symmetric functions is cocommutative. We can rewrite $\Delta(s_\lambda)$ using Theorem 3.19 Point (9). Then

$$\Delta(s_\lambda) = \Delta \left( \sum_{\sigma \in S(\lambda)} (-1)^{\text{sgn}(\sigma)} \mathcal{S}^*_{\sigma(\lambda)} \right)$$

$$= \sum_{\sigma \in S(\lambda)} (-1)^{\text{sgn}(\sigma)} \Delta \mathcal{S}^*_{\sigma(\lambda)}$$

$$= \sum_{\sigma \in S(\lambda)} (-1)^{\text{sgn}(\sigma)} \left( \sum_\beta \mathcal{S}^*_{\beta} \otimes \mathcal{S}^*_{\sigma(\lambda)/\beta} \right)$$

$$= \sum_\beta \mathcal{S}^*_{\beta} \otimes \left( \sum_{\sigma \in S(\lambda)} (-1)^{\text{sgn}(\sigma)} \mathcal{S}^*_{\sigma(\lambda)/\beta} \right).$$
On the other hand,

\[
\sum_{\mu} s_{\mu} \otimes s_{\lambda/\mu} = \sum_{\mu} \left( \sum_{\tau \in S_{\ell(\mu)}} (-1)^{\text{sgn}(\tau)} G_{\tau(\mu)}^* \right) \otimes s_{\lambda/\mu}
\]

\[
= \sum_{\mu} \sum_{\tau \in S_{\ell(\mu)}} (-1)^{\text{sgn}(\tau)} \left( G_{\tau(\mu)}^* \otimes s_{\lambda/\mu} \right)
\]

\[
= \sum_{\beta} G_{\beta}^* \otimes \left( \sum_{\tau \in S_{\ell(\beta)}} (-1)^{\text{sgn}(\tau)} s_{\lambda/\tau^{-1}(\beta)} \right)
\]

where \(\beta\) is a composition and \(\tau^{-1}(\beta)\) is a partition. Thus for a fixed choice of \(\beta\),

\[
\sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma)} G_{\sigma(\lambda)/\beta}^* = \sum_{\tau \in S_{\ell(\beta)}} (-1)^{\text{sgn}(\tau)} s_{\lambda/\tau^{-1}(\beta)}.
\]

Note that for each \(\beta\), there is at most one \(\tau \in S_{\ell(\mu)}\) such that \(s_{\lambda/\tau^{-1}(\beta)} = s_{\lambda/\mu} \neq 0\) for a partition \(\mu\). Thus

\[
s_{\lambda/\mu} = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma)+\text{sgn}(\tau)} G_{\sigma(\lambda)/\tau(\mu)}^*
\]

for any valid choice of \(\tau\). \(\square\)

Choosing \(\tau\) as the identity gives the following corollary.

**Corollary 4.13.** For partitions \(\lambda\) and \(\mu\) with \(\mu \subseteq \lambda\),

\[
s_{\lambda/\mu} = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma)} G_{\sigma(\lambda)/\mu}^*.
\]

Applying \(\psi\) to both sides of Theorem 4.12 gives us an expansion in terms of the row-strict dual immaculate functions.

**Corollary 4.14.** For partitions \(\lambda\) and \(\mu\) with \(\mu \subseteq \lambda\) and \(\tau \in S_{\ell(\mu)}\) such that \(\tau(\mu)\) is a composition,

\[
s_{\lambda/\mu'} = \sum_{\sigma \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\sigma)+\text{sgn}(\tau)} \mathcal{R} G_{\sigma(\lambda)/\tau(\mu)}^*.
\]
5. Hook dual immaculate functions

Now that we have skew row-strict dual immaculate functions, we can define hook dual immaculate functions in a combinatorial manner analogous to the hook Schur functions [17] and hook quasisymmetric Schur functions [15].

**Definition 5.1.** Let $A = \{1, 2, \ldots, \ell\}$ and $A' = \{1', 2', \ldots, k'\}$ be two alphabets with $1 < 2 < \cdots < \ell < 1' < 2' < \cdots < k'$. Then a semistandard hook immaculate tableau of shape $\alpha$ is a filling of the diagram of $\alpha$ such that

1. the first column increases from bottom to top with the increase strict in $A$ and weak in $A'$, and
2. each row increases from left to right, weakly in $A$ and strictly in $A'$.

Denote the set of all semistandard hook immaculate tableaux of shape $\alpha$ by $HI_{\alpha}$.

The content monomial of a hook tableau $T$ is a monomial in two alphabets, $x_1, \ldots, x_\ell$ and $y_1, \ldots, y_k$, where

$$z^T = \prod_{i \in A \cup A'} z_i^{\# \text{ of } i's \text{ in } T}$$

where $z_i = x_i$ if $i \in A$ and $z_i = y_i$ if $i \in A'$.

**Example 5.2.** Let $\alpha = (3, 1, 2, 4, 3)$. Then $T$, as shown below, is a hook immaculate tableau with content monomial $z^T = x_1^2x_2^2x_3^2y_1^2y_2y_3y_4^2y_5$.

$$T = \begin{array}{cccc}
1' & 2' & 4' \\
1' & 3' & 4' & 5' \\
3 & 1' \\
2 \\
1 & 1 & 3
\end{array}$$

**Definition 5.3.** The hook dual immaculate function indexed by $\alpha$ is

$$\mathcal{H}\mathcal{S}_\alpha^*(X, Y) = \mathcal{H}\mathcal{S}_\alpha^*(x_1, \ldots, x_\ell, y_1, \ldots, y_k) = \sum_{T \in HI_{\alpha}} z^T.$$

It follows immediately from the definition that

$$\mathcal{H}\mathcal{S}_\alpha^*(X, Y) = \sum_{\gamma \subseteq \alpha} \mathcal{S}_\gamma^*(X) \mathcal{R}\mathcal{S}_{\alpha/\gamma}^*(Y).$$

We can also expand $\mathcal{H}\mathcal{S}_\alpha^*(X, Y)$ in terms of the super fundamental quasisymmetric functions. We use the definition in [15].
Definition 5.4. For $\alpha \vdash n$,

$$\bar{Q}_\alpha(X,Y) = \sum_{a_1 \leq a_2 \leq \cdots \leq a_n\atop a_i = a_{i+1} \in A \Rightarrow i \notin \text{set}(\alpha)} z_{a_1} z_{a_2} \cdots z_{a_n},$$

where $z_a = x_a$ if $a \in A$ and $z_{a'} = y_a$ for $a' \in A'$.

Theorem 5.5. [15, Theorem 4.1] For $\alpha \vdash n$,

$$\bar{Q}_\alpha(X,Y) = \sum_{i=0}^{n} F_\beta(X) F_\gamma(Y)$$

where $\beta \cdot \gamma = \alpha$ if $i \in \text{set}(\alpha)$ and $\beta \odot \gamma = \alpha$ if $i \notin \text{set}(\alpha)$.

As usual, we must have a standardization procedure for hook dual immaculate tableaux and an appropriate descent set to index the super fundamental quasisymmetric functions. To standardize a hook dual immaculate tableau $H$, first replace the entries of $H$ from $A$ by scanning unprimed entries from left to right, starting with the top row, replacing 1s as they are encountered in this reading order, followed by 2s, etc. Next continue with the entries of $A'$ by scanning from right to left starting with the bottom row.

Example 5.6. The reading word of $T$, as shown below, is $3, 2, 1, 1, 3, 1', 5', 4', 3', 1', 4', 2'$, giving rise to stdz($T$) below.

$$T = \begin{array}{ccccc}
1' & 2' & 4' \\
1' & 3' & 4' & 5' \\
3 & 1' \\
2 \\
1 & 1 & 1 & 3
\end{array} \quad \text{stdz}(T) = \begin{array}{ccccc}
8 & 9 & 12 \\
7 & 10 & 11 & 13 \\
4 & 6 \\
3 \\
1 & 2 & 5
\end{array}$$

Note that the standardization of a hook dual immaculate tableau is a standard dual immaculate tableau. Recall that the descent set of a standard dual immaculate tableau $S$ is $\text{Des}_S(S) = \{i : i + 1 \text{ is strictly above } i \text{ in } S\}$. The descent set for stdz($T$) in Example 5.6 is $\text{Des}_S(\text{stdz}(T)) = \{2, 3, 5, 6, 7, 11\}$. From the definition of standardization, we note that if $T$ is a hook immaculate tableau of shape $\alpha$ with $T = S \cup U$ where $S$ is an immaculate tableau of shape $\beta$ and $U$ is a skew row-strict immaculate tableau of shape $\alpha/\beta$, then

$$\text{Des}_S(\text{stdz}(T)) = \text{Des}_S(\text{stdz}(S)) \cup (\text{Des}_{R\text{G}}(\text{stdz}(U))^c + |\beta|)$$

if $|\beta| + 1$ is weakly lower than $|\beta|$ in stdz($T$) and

$$\text{Des}_S(\text{stdz}(T)) = \text{Des}_S(\text{stdz}(S)) \cup (\text{Des}_{R\text{G}}(\text{stdz}(U))^c + |\beta|) \cup \{|\beta|\}$$

if $|\beta| + 1$ appears strictly above $|\beta|$ in stdz($T$).
Theorem 5.7. Let $\alpha \vdash n$. Then

$$H_{S^\alpha}(X,Y) = \sum_{\tau} Q_{\text{comp}(\text{Des}_{S^\alpha}(\tau))}(X,Y)$$

where the sum is over all standard dual immaculate tableaux of shape $\alpha$.

Proof. We show that each polynomial consists of the same monomials. Suppose

$$x_{a_1} \cdots x_{a_k} y_{b_1} \cdots y_{b_m}$$

is the content monomial associated with a hook immaculate tableau $T$ of shape $\alpha$ with $a_1 \leq a_2 \leq \cdots \leq a_k$ and $b_1 \leq b_2 \leq \cdots \leq b_m$. Note that if $a_i = a_{i+1}$, then $i \notin \text{Des}_{S^\alpha}(\text{stdz}(T))$ by the standardization procedure. Similarly, if $b_i = b_{i+1}$, $i + k \in \text{Des}_{S^\alpha}(\text{stdz}(T))$, since $b'_i$ must occur in a lower row of $T$ than $b'_{i+1}$. Thus $x_{a_1} \cdots x_{a_k} y_{b_1} \cdots y_{b_m}$ is a monomial in $Q_{\text{comp}(\text{Des}_{S^\alpha}(\text{stdz}(T)))}(X,Y)$.

Now suppose $x_{a_1} \cdots x_{a_k} y_{b_1} \cdots y_{b_m}$ is a monomial in $Q_{\text{comp}(\text{Des}_{S^\alpha}(\tau))}(X,Y)$ for some standard immaculate tableau $S$ of shape $\alpha$. We must show that there exists a hook immaculate tableau with content $a_1, \ldots, a_k, b'_1, \ldots, b'_m$. Do this by replacing $n$ in $S$ with $b'_m$, $n-1$ in $S$ with $b'_{m-1}$ and so on. Since $b_i = b_{i+1}$ implies that $i + k \in \text{Des}_{S^\alpha}(\tau)$, we have that each primed entry in a row is distinct and increasing from left to right. Similarly, if $a_i = a_{i+1}$, then $i \notin \text{Des}_{S^\alpha}(\tau)$, guaranteeing that the first column is increasing bottom to top and has distinct unprimed entries. Thus the result is a hook immaculate tableau of content $x_{a_1} \cdots x_{a_k} y_{b_1} \cdots y_{b_m}$. $\square$

Berele and Regev \[3\] defined hook Schur functions indexed by a partition $\lambda$ as

$$H_{S^\lambda}(X,Y) = \sum_{\mu \subseteq \lambda} s_{\mu}(X)s_{\lambda'/\mu'}(Y).$$

We have the following analogue of Theorem 3.19 Point (9).

Theorem 5.8. Let $\lambda$ be a partition. Then

$$H_{S^\lambda}(X,Y) = \sum_{\tau \in S_{\ell(\lambda)}} (-1)^{\text{sgn}(\tau)} H_{S_{\tau(\lambda)}}(X,Y).$$
Proof. Let $\lambda$ be a partition. Then
\[
\mathcal{H}s_\lambda(X, Y) = \sum_{\mu \subseteq \lambda} s_\mu(X)s_{\lambda'/\mu'}(Y)
\]
\[
= \sum_{\mu \subseteq \lambda} \left( \sum_{\sigma \in S_{t(\mu)}} (-1)^{\text{sgn}(\sigma)} \mathcal{G}^*_\sigma(X)s_{\lambda'/\mu'}(Y) \right)
\]
\[
= \sum_{\mu \subseteq \lambda} \left( \sum_{\sigma \in S_{t(\mu)}} (-1)^{\text{sgn}(\sigma)} \mathcal{G}^*_\sigma(X)(-1)^{\text{sgn}(\sigma)}s_{\lambda'/\sigma(\mu)}(Y) \right) \quad \text{by (19)}
\]
\[
= \sum_{\mu \subseteq \lambda} \mathcal{G}^*_{\sigma(\mu)}(X) \sum_{\tau \in S_{t(\lambda)}} (-1)^{\text{sgn}(\tau)} \mathcal{R}\mathcal{G}^*_{\tau(\lambda)/\sigma(\mu)}(Y)
\]
\[
(21) = \sum_{\tau \in S_{t(\lambda)}} (-1)^{\text{sgn}(\tau)} \left( \sum_{\mu \subseteq \lambda} \sum_{\sigma \in S_{t(\mu)}} \mathcal{G}^*_\sigma(X)\mathcal{R}\mathcal{G}^*_{\tau(\lambda)/\sigma(\mu)}(Y) \right).
\]

Note that the only terms $\sigma(\mu)$ that appear in (21) are those such that $\sigma(\mu) = \beta$ for a composition $\beta$. We rewrite (21) as
\[
\mathcal{H}s_\lambda(X, Y) = \sum_{\tau \in S_{t(\lambda)}} (-1)^{\text{sgn}(\tau)} \left( \sum_{\mu \subseteq \lambda} \sum_{\sigma \in S_{t(\mu)}} \mathcal{G}^*_\sigma(X)\mathcal{R}\mathcal{G}^*_{\tau(\lambda)/\beta(\mu)}(Y) \right)
\]
\[
= \sum_{\tau \in S_{t(\lambda)}} (-1)^{\text{sgn}(\tau)} \mathcal{G}^*_{\beta(\lambda)}(X)\mathcal{R}\mathcal{G}^*_{\tau(\lambda)/\beta}(Y)
\]
\[
= \sum_{\tau \in S_{t(\lambda)}} (-1)^{\text{sgn}(\tau)} \mathcal{H}\mathcal{G}^*_{\tau(\lambda)}(X, Y).
\]

\[\square\]

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