PIERI RULES FOR SKEW DUAL IMMACULATE FUNCTIONS

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Abstract. In this paper we give Pieri rules for skew dual immaculate functions and their recently discovered row-strict counterparts. We establish our rules using a right-action analogue of the skew Littlewood-Richardson rule for Hopf algebras of Lam-Lauve-Sottile. We also obtain Pieri rules for row-strict (dual) immaculate functions.

1. Introduction

Schur-like functions are a new and flourishing area since the discovery of quasisymmetric Schur functions in 2011 [11], which led to numerous other similar functions being discovered, for example [1, 4, 6, 10, 14, 15, 16, 17]. In essence, Schur-like functions are functions that refine the ubiquitous Schur functions and reflect many of their properties, such as their combinatorics [2, 9], their representation theory [5, 7, 21, 22], and in the case of quasisymmetric Schur functions have already been applied to resolve conjectures [13]. Of the various Schur-like functions to arise after the quasisymmetric Schur functions, two were naturally related to them: the dual immaculate functions [6] and the row-strict quasisymmetric Schur functions [17]. Recently a fourth basis that interpolates between these latter two bases, the row-strict dual immaculate functions, was discovered [20], thus completing the picture. The representation theory of these functions was revealed in [19], in addition to the fundamental combinatorics in [20]. In this paper we extend the combinatorics to uncover skew Pieri rules in the spirit of [3, 12, 23] for both row-strict and classical dual immaculate functions.

More precisely, our paper is structured as follows. In Section 2 we establish a right-action analogue of [12, Theorem 2.1] in Theorem 2.6. We then recall required background for the Hopf algebras of quasisymmetric functions, QSym, and noncommutative symmetric functions, NSym, in Section 3. Finally, in Section 4 we give (left) Pieri rules for row-strict immaculate functions and row-strict dual immaculate functions in Corollaries 4.3 and 4.5 respectively. Our final theorem is Theorem 4.7, in which we establish Pieri rules for skew dual immaculate functions, and row-strict skew dual immaculate functions.

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2. The right-action skew Littlewood-Richardson rule for Hopf algebras

We begin by recalling and deducing general Hopf algebra results that will be useful later. Following Tewari and van Willigenburg [23], let $H$ and $H^*$ be a pair of dual Hopf algebras over a field $k$ with duality pairing $\langle \ , \ \rangle : H \otimes H^* \to k$ for which the structure of $H^*$ is dual to that of $H$ and vice versa. Let $h \in H, a \in H^*$. By Sweedler notation, we have coproduct denoted by $\Delta h = \sum h_1 \otimes h_2$, and similarly $h_1 h_2 = h_1 \cdot h_2$ denotes product. We define the action of one algebra on the other one by the following.

\begin{align*}
(1) \quad h \rightarrow a &= \sum \langle h, a \rangle a_1 \\
(2) \quad a \rightarrow h &= \sum \langle h_2, a \rangle h_1
\end{align*}

Let $S : H \to H$ denote the antipode map. Then for $\Delta h = \sum h_1 \otimes h_2$,

\begin{equation}
\sum (Sh_1)h_2 = \varepsilon(h)1_H = \sum h_1(Sh_2),
\end{equation}

where $\varepsilon$ and 1 denote counit and unit, respectively. Following Montgomery [18], we can define the convolution product $\ast$ for $f$ and $g$ in $H$ by

\begin{equation}
(f \ast g)(a) = \sum \langle f, a_1 \rangle \langle g, a_2 \rangle = \langle fg, a \rangle.
\end{equation}

Then it follows that

\begin{equation}
\langle g, f \rightarrow a \rangle = \langle gf, a \rangle.
\end{equation}

Similarly, $\langle a \rightarrow f, b \rangle = \langle f, ba \rangle$. Since $H^*$ is a left $H$-module algebra under $\rightarrow$, we have that $h \rightarrow (a \cdot b) = \sum (h_1 \rightarrow a) \cdot (h_2 \rightarrow b)$.

**Lemma 2.1.** ([12]) For $g, h \in H$ and $a \in H^*$,

\begin{equation}
(a \rightarrow g) \cdot h = \sum (S(h_2) \rightarrow a) \rightarrow (g \cdot h_1)
\end{equation}

where $S : H \to H$ is the antipode.

As in Montgomery [18], define a right action by the following.

\begin{align*}
(4) \quad h \leftarrow a &= \sum \langle h, a \rangle a_2 \\
(5) \quad a \leftarrow h &= \sum \langle h_1, a \rangle h_2
\end{align*}

As before, it follows that $\langle g, f \leftarrow a \rangle = \langle fg, a \rangle$ and $\langle a \leftarrow f, b \rangle = \langle f, ab \rangle$. 

\[\text{\hspace{1cm}}\]
Lemma 2.2. Let $f \in H$ and $a, b \in H^*$. Then
\[ f \leftarrow a \cdot b = \sum (f_1 \leftarrow a) \cdot (f_2 \leftarrow b). \]

Proof. Let $f, g \in H$ and $a, b \in H^*$. Then
\[
\langle g, f \leftarrow (a \cdot b) \rangle = \langle fg, ab \rangle = \langle a \leftarrow (fg), b \rangle = \sum \langle f_1 g_1, a \rangle \langle f_2 g_2, b \rangle = \sum \langle g_1, f_1 \leftarrow a \rangle \langle g_2, f_2 \leftarrow b \rangle = \sum \langle g, (f_1 \leftarrow a) \cdot (f_2 \leftarrow b) \rangle.
\]
Thus $f \leftarrow a \cdot b = \sum (f_1 \leftarrow a) \cdot (f_2 \leftarrow b)$. \hfill \Box

Lemma 2.3. Let $a \in H^*$. Then
\[ \varepsilon(h) \cdot 1_H \leftarrow a = a \]
for any $h \in H$.

Proof. Let $a \in H^*$ and $h \in H$. Then
\[ \varepsilon(h) \cdot 1_H \leftarrow a = \sum \langle \varepsilon(h) \cdot 1_H, a_1 \rangle a_2. \]
This is only nonzero when $a_1 = 1_{H^*}$.

Lemma 2.4. Let $h \in H$ and $a, b \in H^*$. Then
\[ a \cdot (h \leftarrow b) = \sum h_1 \leftarrow ((S(h_2) \leftarrow a) \cdot b). \]

Proof. Expand the sum using Lemma 2.2 and coassociativity, $(\Delta \otimes 1) \circ \Delta(h) = (1 \otimes \Delta) \circ \Delta(h) = \sum h_1 \otimes h_2 \otimes h_3$, to get
\[
\sum h_1 \leftarrow ((S(h_2) \leftarrow a) \cdot b) = \sum (h_1 \leftarrow (S(h_2) \leftarrow a)) \cdot (h_3 \leftarrow b) = \sum (h_1 \cdot S(h_2) \leftarrow a) \cdot (h_3 \leftarrow b) \text{ since } H^* \text{ is an } H\text{-module} = ((\varepsilon(h) \cdot 1_H) \leftarrow a) \cdot (h \leftarrow b) \text{ by (3)} = a \cdot (h \leftarrow b) \text{ by Lemma 2.3} \hfill \Box
\]

Lemma 2.5. Let $g, h \in H$ and $a \in H^*$. Then
\[ h \cdot (a \leftarrow g) = \sum (S(h_2) \leftarrow a) \leftarrow h_1 \cdot g. \]
Proof. Let \( g, h \in H \) and \( a, b \in H^* \). Then
\[
\langle h \cdot (a \leftarrow g), b \rangle = \langle a \leftarrow g, h \leftarrow b \rangle = \langle g, a \cdot (h \leftarrow b) \rangle
\]
by Lemma 2.4
\[
= \sum \langle g, h_1 \leftarrow (S(h_2) \leftarrow a) \cdot b \rangle
\]
\[
= \sum \langle h_1 \cdot g, (S(h_2) \leftarrow a) \cdot b \rangle
\]
\[
= \sum \langle (S(h_2) \leftarrow a) \leftarrow h_1 \cdot g, b \rangle.
\]
\( \square \)

We can use the right action to obtain an algebraic Littlewood-Richardson formula analogous to [12, Theorem 2.1] for those bases whose skew elements appear as the right tensor factor in the coproduct.

Let \( \{L_\alpha\} \subset H \) and \( \{R_\beta\} \subset H^* \) be dual bases with indexing set \( \mathcal{P} \). Then

\[
L_\alpha \cdot L_\beta = \sum_\gamma b^\gamma_{\alpha,\beta} L_\gamma \quad \Delta(L_\gamma) = \sum_{\alpha,\beta} c^\gamma_{\alpha,\beta} L_\alpha \otimes L_\beta
\]
(6)
\[
R_\alpha \cdot R_\beta = \sum_\gamma c^\gamma_{\alpha,\beta} R_\gamma \quad \Delta(R_\gamma) = \sum_{\alpha,\beta} b^\gamma_{\alpha,\beta} R_\alpha \otimes R_\beta
\]
(7)

where \( b^\gamma_{\alpha,\beta} \) and \( c^\gamma_{\alpha,\beta} \) are structure constants. We can also write

\[
\Delta(L_\gamma) = \sum_\delta L_\delta \otimes L_\gamma/\delta \quad \Delta(R_\gamma) = \sum_\delta R_\delta \otimes R_\gamma/\delta.
\]
(8)

Note that \( L_\alpha \leftarrow R_\beta = R_\beta/\alpha \) and \( R_\beta \leftarrow L_\alpha = L_\alpha/\beta \). Further,

\[
\Delta(L_{\alpha/\beta}) = \sum_{\pi,\rho} c^\alpha_{\pi,\rho,\beta} L_\pi \otimes L_\rho \quad \Delta(R_{\alpha/\beta}) = \sum_{\pi,\rho} b^\alpha_{\pi,\rho,\beta} R_\pi \otimes R_\rho.
\]
(9)

The antipode acts on \( L_\rho \) by \( S(L_\rho) = (-1)^{\theta(\rho)} L_{\rho^*} \) where \( \theta : \mathcal{P} \to \mathbb{N} \) and \( * : \mathcal{P} \to \mathcal{P} \).

Theorem 2.6. For \( \alpha, \beta, \gamma, \delta \in \mathcal{P} \),

\[
L_{\alpha/\beta} \cdot L_{\gamma/\delta} = \sum_{\pi,\rho,\nu,\mu} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} b^\nu_{\pi,\gamma} b^\delta_{\mu,\rho^*} L_{\nu/\mu}.
\]
Proof. We use Lemma 2.5 and the preceding facts about the product, coproduct, and antipode maps on $H$ and $H^*$ to obtain

$$L_{\alpha/\beta} \cdot L_{\gamma/\delta} = L_{\alpha/\beta} \cdot (R_\delta \leftarrow L_{\gamma})$$

$$= \sum_{\pi, \rho} c^\alpha_{\pi, \rho, \beta} (S(L_\rho) \leftarrow R_\delta) \leftarrow (L_\pi \cdot L_{\gamma})$$

$$= \sum_{\pi, \rho} (-1)^{\theta(\rho)} c^\alpha_{\pi, \rho, \beta} (L_{\rho} \leftarrow \leftarrow \left(\sum_{\nu} b^\nu_{\pi, \gamma} L_\nu\right))$$

$$= \sum_{\pi, \rho, \nu} (-1)^{\theta(\rho)} c^\alpha_{\pi, \rho, \beta} b^\nu_{\pi, \gamma} (R_{\delta/\rho} \leftarrow L_{\nu})$$

$$= \sum_{\pi, \rho, \nu, \mu} (-1)^{\theta(\rho)} c^\alpha_{\pi, \rho, \beta} b^\nu_{\pi, \gamma} b^\delta_{\mu, \rho} (R_{\mu} \leftarrow L_{\nu})$$

$$= \sum_{\pi, \rho, \nu, \mu} (-1)^{\theta(\rho)} c^\alpha_{\pi, \rho, \beta} b^\nu_{\pi, \gamma} b^\delta_{\mu, \rho} L_{\nu/\mu}.$$ 

\[\square\]

3. The dual Hopf algebras \textsc{QSym} and \textsc{NSym}

We now focus our attention on the dual Hopf algebra pair of noncommutative symmetric functions and quasisymmetric functions, and introduce our main objects of study the (row-strict) dual immaculate functions.

A composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of $n$, denoted by $\alpha \vdash n$, is a list of positive integers such that $\sum_{i=1}^{k} \alpha_i = n$. We call $n$ the size of $\alpha$ and sometimes denote it by $|\alpha|$, and call $k$ the length of $\alpha$ and sometimes denote it by $\ell(\alpha)$. If $\alpha_j = \cdots = \alpha_m = i$ we sometimes abbreviate this to $i^m$, and denote the empty composition of 0 by $\emptyset$. There exists a natural correspondence between compositions $\alpha \vdash n$ and subsets $S \subseteq \{1, \ldots, n-1\} = [n-1]$. More precisely, $\alpha = (\alpha_1, \ldots, \alpha_k)$ corresponds to $\text{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$, and conversely $S = \{s_1, \ldots, s_{k-1}\}$ corresponds to $\text{comp}(S) = (s_1, s_2 - s_1, \ldots, n - s_{k-1})$. We also denote by $S^c$ the set complement of $S$ in $[n-1]$.

Given a composition $\alpha$, its diagram, also denoted by $\alpha$, is the array of left-justified boxes with $\alpha_i$ boxes in row $i$ from the bottom. Given two compositions $\alpha, \beta$ we say that $\beta \subseteq \alpha$ if $\beta_j \leq \alpha_j$ for all $1 \leq j \leq \ell(\beta) \leq \ell(\alpha)$, and given $\alpha, \beta$ such that $\beta \subseteq \alpha$, the skew diagram $\alpha/\beta$ is the array of boxes in $\alpha$ but not $\beta$ when $\beta$ is placed in the bottom-left corner of $\alpha$. If, furthermore, $\beta \subseteq \alpha$ and $\alpha_j - \beta_j \in \{0, 1\}$ for all $1 \leq j \leq \ell(\beta) \leq \ell(\alpha)$ then we call $\alpha/\beta$ a vertical strip.
Example 3.1. If $\alpha = (3, 4, 1)$, then $|\alpha| = 8$, $\ell(\alpha) = 3$, and $\text{set}(\alpha) = \{3, 7\}$. Its diagram is

\[
\alpha = \begin{array}{ccc}
\vline & \vline & \vline \\
\hline
& \vline & \vline \\
\hline
\end{array}
\]

and if $\beta = (2, 4)$, then

\[
\frac{\alpha}{\beta} = \begin{array}{c}
\vline \\
\hline
\vline \\
\hline
\end{array}
\]

is a vertical strip.

Definition 3.2. Given a composition $\alpha$, a standard immaculate tableau $T$ of shape $\alpha$ is a bijective filling of its diagram with $1, \ldots, |\alpha|$ such that

1. The entries in the leftmost column increase from bottom to top;
2. The entries in each row increase from left to right.

We obtain a standard skew immaculate tableau of shape $\alpha/\beta$ by extending the definition to skew diagrams $\alpha/\beta$ in the natural way.

Given a standard (skew) immaculate tableau, $T$, its descent set is

\[\text{Des}(T) = \{i : i + 1 \text{ appears strictly above } i \text{ in } T\} .\]

Example 3.3. A standard skew immaculate tableau of shape $(3, 4, 1)/(1)$ is

\[
T = \begin{array}{cccc}
7 \\
2 & 3 & 4 & 6 \\
1 & 5 \\
\end{array}
\]

with $\text{Des}(T) = \{1, 5, 6\}$.

We are now ready to define our Hopf algebras and functions of central interest.

Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vdash n$ and commuting variables $\{x_1, x_2, \ldots\}$ we define the monomial quasisymmetric function $M_\alpha$ to be

\[M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}\]

the fundamental quasisymmetric function $F_\alpha$ to be

\[F_\alpha = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}\]
the dual immaculate function $\mathcal{S}_\alpha^*$ to be

$$\mathcal{S}_\alpha^* = \sum_T F_{\text{comp}(\text{Des}(T))}$$

and the row-strict dual immaculate function $\mathcal{R}\mathcal{S}_\alpha^*$ to be

$$\mathcal{R}\mathcal{S}_\alpha^* = \sum_T F_{\text{comp}(\text{Des}(T)^c)}$$

where the latter two sums are over all standard immaculate tableaux $T$ of shape $\alpha$. These extend naturally to give skew dual immaculate and row-strict dual immaculate functions $\mathcal{S}_{\alpha/\beta}^*$ $[6]$ $\mathcal{R}\mathcal{S}_{\alpha/\beta}^*$ $[20]$, where $\alpha/\beta$ is a skew diagram.

Example 3.4. We have that $M_2 = x_1x_2 + x_1x_3 + x_2x_3 + \cdots$ and $F_2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots = \mathcal{S}_2^* = \mathcal{R}\mathcal{S}_{(1^2)}^*$ from the following standard immaculate tableau $T$ with Des($T$) = $\emptyset$.

$$T = \begin{array}{c} 1 \\ 2 \end{array}$$

The set of all monomial or fundamental quasisymmetric functions forms a basis for the Hopf algebra of quasisymmetric functions $\text{QSym}$, as do the set of all (row-strict) dual immaculate functions. There exists an involutory automorphism $\psi$ defined on fundamental quasisymmetric functions by

$$\psi(F_\alpha) = F_{\text{comp}(\text{set}(\alpha^c))}$$

such that $[20]$

$$\psi(\mathcal{S}_\alpha^*) = \mathcal{R}\mathcal{S}_\alpha^*$$

for a composition $\alpha$. This extends naturally to skew diagrams $\alpha/\beta$ to give

$$\psi(\mathcal{S}_{\alpha/\beta}^*) = \mathcal{R}\mathcal{S}_{\alpha/\beta}^*.$$ Dual to the Hopf algebra of quasisymmetric functions is the Hopf algebra of noncommutative symmetric functions $\text{NSym}$. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ and noncommuting variables $\{y_1, y_2, \ldots\}$ we define the $n$th elementary noncommutative symmetric function $e_n$ to be

$$e_n = \sum_{i_1 < \cdots < i_n} y_{i_1} \cdots y_{i_n}$$

and the elementary noncommutative symmetric function $e_\alpha$ to be

$$e_\alpha = e_{\alpha_1} \cdots e_{\alpha_k}.$$ Meanwhile, we define the $n$th complete homogeneous noncommutative symmetric function $h_n$ to be

$$h_n = \sum_{i_1 \leq \cdots \leq i_n} y_{i_1} \cdots y_{i_n}.$$
and the complete homogeneous noncommutative symmetric function $h_{\alpha}$ to be

$$h_{\alpha} = h_{\alpha_1} \cdots h_{\alpha_k}.$$ 

The set of all elementary or complete homogeneous noncommutative symmetric functions forms a basis for NSym. The duality between QSym and NSym is given by

$$\langle M_{\alpha}, h_{\alpha} \rangle = \delta_{\alpha\beta}$$

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise. This induces the bases dual to the (row-strict) dual immaculate functions via

$$\langle S^*_\alpha, S_\alpha \rangle = \delta_{\alpha\beta} \quad \langle R S^*_\alpha, R S_\alpha \rangle = \delta_{\alpha\beta}$$

and implicitly defines the bases of immaculate and row-strict immaculate functions. While concrete combinatorial definitions of these functions have been established [6, 20], we will not need them here. However, what we will need is the involutory automorphism in NSym corresponding to $\psi$ in QSym, defined by $\psi(e_{\alpha}) = h_{\alpha}$ that gives [20] $\psi(S_\alpha) = R S_\alpha$.

4. The Pieri rules for skew dual immaculate functions

A left Pieri rule for immaculate functions was conjectured in [6, Conjecture 3.7] and proved in [8]. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ we say that $\text{tail}(\alpha) = (\alpha_2, \ldots, \alpha_k)$. If $\beta \in \mathbb{Z}^k$, then $\text{neg}(\alpha - \beta) = |\{i : \alpha_i - \beta_i < 0\}|$. Let $\text{sgn}(\beta) = (-1)^{\text{neg}(\beta)}$ with $\text{neg}(\beta) = |\{i : \beta_i < 0\}|$.

Following [8], we define $Z_{s,\alpha}$ to be a set of all $\beta \in \mathbb{Z}^k$ such that

1. $\beta_1 + \cdots + \beta_k = s$ and $\beta_1 + \cdots + \beta_i \leq s$ for all $i \leq k$;

2. $\alpha_i - \beta_i \geq 0$ for all $1 \leq i \leq k$ and $|i : \alpha_i - \beta_i = 0| \leq 1$;

3. For all $1 \leq i \leq k$,
   - if $\alpha_i > s - (\beta_1 + \cdots + \beta_{i-1})$, then $0 \leq \beta_i \leq s - (\beta_1 + \cdots + \beta_{i-1})$,
   - if $\alpha_i < s - (\beta_1 + \cdots + \beta_{i-1})$, then $\beta_i < 0$, and
   - if $\alpha_i = s - (\beta_1 + \cdots + \beta_{i-1})$, then either $\beta_i < 0$ or $\beta_i = \alpha_i$ and $\beta_{i+1} = \cdots = \beta_k = 0$.

Now we are ready to define the coefficients of the immaculate basis appearing in the left Pieri rule.

**Definition 4.1** ([8]). For a positive integer $s$ and compositions $\alpha, \gamma$ with $|\alpha| - |\gamma| = s$, let $1 \leq j \leq k$ be the smallest integer such that $\alpha_i = \gamma_{i-1}$ for all $j < i \leq k$ where $j = k$ when
\[ \alpha_k \neq \gamma_{k-1}. \] Let \( j \leq r \leq k \) be the largest integer such that \( \alpha_j < \alpha_{j+1} < \cdots < \alpha_r \). Let \( \alpha^{(i)} = (\alpha_1, \ldots, \alpha_i) \) Then define

\[
c^j_{s, \alpha} = \begin{cases} 
\text{sgn}(\alpha - \gamma), & \text{if } \ell(\gamma) = \ell(\alpha) \text{ and } \alpha - \gamma \in \mathbb{Z}_{s, \alpha}; \\
\text{sgn}(\alpha^{(j-1)} - \gamma^{(j-1)}), & \text{if } \ell(\gamma) = \ell(\alpha) - 1, \\
0, & \text{if } r - j \text{ is even, and }
\end{cases}
\]

\[
(\alpha^{(j-1)} - \gamma^{(j-1)}, \alpha_j, 0, \ldots, 0) \in \mathbb{Z}_{s, \alpha}; \\
0, & \text{otherwise.}
\]

**Theorem 4.2** ([6, 8]). Let \( m > 0 \) and \( \alpha \) be a composition. Then

\[
h_{m} \mathcal{S}_\alpha = \sum_{\beta \models |\alpha| + m \atop \beta_1 \geq m \atop 0 \leq \ell(\beta) - \ell(\alpha) \leq 1} c_{\beta_1 - m, \alpha}^{\text{tail}(\beta)} \mathcal{S}_\beta.
\]

Applying \( \psi \) to both sides of the left Pieri rule in Theorem 4.2 immediately yields a left Pieri rule for row-strict immaculate functions.

**Corollary 4.3.** Let \( m > 0 \) and \( \alpha \) be a composition. Then

\[
e_{m} \mathcal{R}\mathcal{S}_\alpha = \sum_{\beta \models |\alpha| + m \atop \beta_1 \geq m \atop 0 \leq \ell(\beta) - \ell(\alpha) \leq 1} c_{\beta_1 - m, \alpha}^{\text{tail}(\beta)} \mathcal{R}\mathcal{S}_\beta.
\]

Lemma 3.1 of [8] shows that for \( s \geq 0, r > 0 \) and compositions \( \alpha, \beta \) with \( |\alpha| = |\beta| + s \),

\[
\langle \mathcal{S}_\alpha, F_s \mathcal{S}_{\beta}^* \rangle = \langle h_s \mathcal{S}_\alpha, \mathcal{S}^*_{(s+r, \beta)} \rangle.
\]

This leads to the following Pieri rule for dual immaculate functions.

**Theorem 4.4** ([8]). Let \( s > 0 \) and \( \alpha \) be a composition. Then

\[
F_s \mathcal{S}_\alpha^* = \sum_{\beta \models |\alpha| + s \atop 0 \leq \ell(\beta) - \ell(\alpha) \leq 1} c_{s, \beta}^\alpha \mathcal{S}_\beta^*.
\]

Again, applying \( \psi \) to both sides gives a Pieri rule for row-strict dual immaculate functions.

**Corollary 4.5.** Let \( s > 0 \) and \( \alpha \) be a composition. Then

\[
F_{(1^s)} \mathcal{R}\mathcal{S}_\alpha^* = \sum_{\beta \models |\alpha| + s \atop 0 \leq \ell(\beta) - \ell(\alpha) \leq 1} c_{s, \beta}^\alpha \mathcal{R}\mathcal{S}_\beta^*.
\]
We use these results together with Hopf algebra computations to construct a Pieri rule for skew dual immaculate functions. Using the map $\psi$, this also gives a Pieri rule for row-strict skew dual immaculate functions. But first we have a small, yet crucial, lemma.

**Lemma 4.6.** Let $\alpha$ and $\gamma$ be compositions. Then $\mathcal{G}_\gamma \prec \mathcal{G}_\alpha^* = \mathcal{G}_{\alpha/\gamma}^*$.

**Proof.** Recall that if $H = \text{QSym}$ and $H^* = \text{NSym}$ are our pair of dual Hopf algebras, then we know $\Delta \mathcal{G}_\alpha^* = \sum_\beta \mathcal{G}_\beta^* \otimes \mathcal{G}_{\alpha/\beta}^*$ and we have

$$\mathcal{G}_\gamma \prec \mathcal{G}_\alpha^* = \sum_\beta \langle \mathcal{G}_\gamma, \mathcal{G}_\beta^* \rangle \mathcal{G}_{\alpha/\beta}^* = \mathcal{G}_{\alpha/\gamma}^*$$

since $\langle \mathcal{G}_\gamma, \mathcal{G}_\beta^* \rangle = \delta_{\gamma\beta}$, where $\delta_{\gamma\beta} = 1$ if $\gamma = \beta$ and 0 otherwise. \hfill $\Box$

We can now give our Pieri rule for (row-strict) skew dual immaculate functions.

**Theorem 4.7.** Let $\gamma \subseteq \alpha$. Then

$$\mathcal{G}_{(s)}^* \mathcal{G}_{\alpha/\gamma}^* = \sum_{\beta/\tau} (-1)^{|\gamma|-|\tau|} \cdot c_{|\beta|-|\alpha|, |\beta|}^{|\gamma|, |\tau|} \mathcal{G}_{\beta/\tau}^*$$

and hence by applying $\psi$ to both sides

$$\mathcal{R} \mathcal{G}_{(s)}^* \mathcal{G}_{\alpha/\gamma}^* = \sum_{\beta/\tau} (-1)^{|\gamma|-|\tau|} \cdot c_{|\beta|-|\alpha|, |\beta|}^{|\gamma|, |\tau|} \mathcal{R} \mathcal{G}_{\beta/\tau}^*$$

where $|\beta/\tau| = |\alpha/\gamma| + s$, $\gamma/\tau$ is a vertical strip of length at most $s$, $\ell(\beta) - \ell(\alpha) \in \{0, 1\}$ and $c_{|\beta|-|\alpha|, |\beta|}^{|\gamma|, |\tau|}$ is the coefficient of Definition 4.1. These decompositions are multiplicity-free up to sign.

**Proof.** Note that $\mathcal{G}_{(1^s)}^* = F_{(1^s)}$ and $\mathcal{G}_{(s)}^* = F_{(s)}$. Recall that

$$\Delta F_{\alpha} = \sum_{\beta \gamma = \alpha \text{ or } \beta \otimes \gamma = \alpha} F_{\gamma} \otimes F_{\gamma}$$

where for $\beta = (\beta_1, \ldots, \beta_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_l)$, $\beta \cdot \gamma = (\beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_l)$ is the concatenation of $\beta$ and $\gamma$, and $\beta \otimes \gamma = (\beta_1, \ldots, \beta_k - 1, \beta_k + \gamma_1, \gamma_2, \ldots, \gamma_l)$ is the near-concatenation of $\beta$ and $\gamma$.

Then we have

$$\Delta (F_{(s)}) = \sum_{i=0}^{s} F_{(i)} \otimes F_{(s-i)}.$$
Thus,

\[ \mathcal{S}_\alpha^* \mathcal{S}_\alpha^* = \mathcal{S}_\alpha^* (\mathcal{S}_\gamma \leftarrow \mathcal{S}_\alpha^*) \quad \text{by Lemma 4.6} \]

\[ = F_\alpha (\mathcal{S}_\gamma \leftarrow \mathcal{S}_\alpha^*) \]

\[ = \sum_{i=0}^s (S(F_{(s-i)}) \leftarrow \mathcal{S}_\gamma) \leftarrow (F_{(i)} \mathcal{S}_\alpha^*) \quad \text{by Lemma 2.5} \]

We first compute \( S(F_{(s-i)}) \leftarrow \mathcal{S}_\gamma \). Since it is well known that \( S(F_{\alpha}) = (-1)^{|\alpha|} F_{\text{comp(set}(\alpha)^c)} \) we have \( S(F_{(s-i)}) = (-1)^{s-i} F_{(1^{s-i})} \). Furthermore, we can write the coproduct as

\[ \Delta(\mathcal{S}_\gamma) = \sum_{\delta,\tau} b_{\delta,\tau} \mathcal{S}_\delta \otimes \mathcal{S}_\tau. \]

Thus,

\[ S(F_{(s-i)}) \leftarrow \mathcal{S}_\gamma = (-1)^{s-i} F_{(1^{s-i})} \leftarrow \mathcal{S}_\gamma \]

\[ = \sum_{\delta,\tau} (-1)^{s-i} b_{\delta,\tau}^\gamma (F_{(1^{s-i})}, \mathcal{S}_\delta) \mathcal{S}_\tau \]

\[ = \sum_{\delta,\tau} (-1)^{s-i} b_{\delta,\tau}^\gamma (\mathcal{S}_{(1^{s-i})}, \mathcal{S}_\delta) \mathcal{S}_\tau \]

\[ = \sum_{\tau} (-1)^{s-i} b_{(1^{s-i}),\tau}^\gamma \mathcal{S}_\tau. \]

By the definition of product and coproduct on NSym, we have

\[ b_{\delta,\tau}^\gamma = \langle \Delta \mathcal{S}_\gamma, \mathcal{S}_\delta^* \otimes \mathcal{S}_\tau^* \rangle = \langle \mathcal{S}_\gamma, \mathcal{S}_\delta^* \cdot \mathcal{S}_\tau^* \rangle. \]

To compute this for \( \delta = (1^{s-i}) \) we use Proposition 3.34 from [6] which states that \( F_{(1^r)} \mathcal{S}_\alpha = \sum_{\beta} \mathcal{S}_\beta \) where \( \beta \in \mathbb{Z}^{l(\alpha)}, \alpha_k - \beta_k \in \{0, 1\} \) for all \( k \) and \( |\beta| = |\alpha| - r \). The operator \( F_{(r)} \) is used throughout [6], and has the property that \( \langle F_{(r)} \mathcal{S}_\alpha, \mathcal{S}_\beta^* \rangle = \langle \mathcal{S}_\alpha, F \mathcal{S}_\beta \rangle \).

Thus,

\[ b_{(1^{s-i}),\tau}^\gamma = \langle \mathcal{S}_\gamma, \mathcal{S}_{(1^{s-i})} \mathcal{S}_\tau^* \rangle \]

\[ = \langle \mathcal{S}_\gamma, F_{(1^{s-i})} \mathcal{S}_\tau^* \rangle \]

\[ = \langle F_{(1^{s-i})} \mathcal{S}_\gamma, \mathcal{S}_\tau^* \rangle \]

\[ = \langle \sum_{\beta} \mathcal{S}_\beta, \mathcal{S}_\tau^* \rangle \]

\[ = \delta_{\beta,\tau} \]

where the sum is over all \( \beta \) such that \( \beta \in \mathbb{Z}^{l(\gamma)}, \gamma_k - \beta_k \in \{0, 1\} \) for all \( k \), and \( |\beta| = |\gamma| - (s - i) \).
Then using the above calculations, Theorem 4.4 and Lemma 4.6 we have

\[
\mathfrak{S}^*_\alpha \mathfrak{S}^*_{\alpha/\gamma} = \mathfrak{S}^*_\alpha (\mathfrak{S}_\gamma \leftarrow \mathfrak{S}^*_\alpha) \\
= \sum_{i=0}^{s} \left( (S(F_{s-i}) \leftarrow \mathfrak{S}_\gamma) \leftarrow (F_{(i)} \mathfrak{S}^*_\alpha) \right) \\
= \sum_{i=0}^{s} \left( (-1)^{(s-i)} \sum_{\tau \in \mathbb{Z}^{(\gamma)}} \mathfrak{S}_\tau \right) \leftarrow \left( \sum_{\beta=|\alpha|+i}^{c^\alpha_{i,\beta} \mathfrak{S}^*_\beta} \right) \\
= \sum_{i=0}^{s} \sum_{\tau,\beta \in \mathbb{Z}^{(\gamma)}} \left( (-1)^{(s-i)} \cdot c^\alpha_{i,\beta} \mathfrak{S}^*_{\beta/\tau} \right) \\
= \sum_{\beta/\tau} (-1)^{|\gamma| - |\tau|} \cdot c^\alpha_{|\beta|-|\alpha|,\beta} \mathfrak{S}^*_{\beta/\tau}
\]

where \(|\beta/\tau| = |\alpha/\gamma| + s\), \(\gamma/\tau\) is a vertical strip of length at most \(s\), and \(|\beta| - |\alpha| \in \{0, 1\}\).

**Example 4.8.** Let us compute \(\mathfrak{S}^*_\alpha \cdot \mathfrak{S}^*_{(2,1,1)/(1,1)}\).

First, we need to compute all compositions \(\beta \vdash 4 + i\) for \(i \in \{0, 1, 2\}\) and \(|\beta| = 3\) or \(4\). We list all possible choices for \(\beta\) as the set

\[
A = \{(1, 1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 1, 1, 2), (1, 1, 2, 1), (1, 2, 1, 1), \\
(2, 1, 1, 1), (1, 1, 3), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1), (3, 1, 1), (1, 1, 1, 3), \\
(1, 1, 2, 2), (1, 1, 3, 1), (1, 2, 1, 2), (1, 2, 2, 1), (1, 3, 1, 1), (2, 1, 1, 2), \\
(2, 1, 2, 1), (2, 2, 1, 1), (3, 1, 1, 1), (1, 1, 4), (1, 2, 3), (1, 3, 2), (1, 4, 1), \\
(2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 1, 2), (3, 2, 1), (4, 1, 1)\}.
\]

Next we need to find \(\tau\) by removing a vertical strip of length at most \(s = 2\) from \(\gamma = (1, 1)\). We list all options for \(\tau\) as the set \(B = \{\emptyset, (1, 1)\}\).
By Theorem 4.7, now we expand $\mathfrak{S}^*_{(2)} \cdot \mathfrak{S}^*_{(1,2,1)/(1,1)}$ by finding all valid pairs $(\beta, \tau)$ such that $|\beta/\tau| = 4$. Thus

\[
\mathfrak{S}^*_{(2)} \cdot \mathfrak{S}^*_{(1,2,1)/(1,1)} = c_{(1,2,1)}^{(1,2,1)} \mathfrak{S}^*_{(1,1,1,1)/(1,1)} + c_{(1,2,1)}^{(1,2,1)} \mathfrak{S}^*_{(1,1,2)/1} + c_{0,(1,2,1)}^{(1,2,1)} \mathfrak{S}^*_{(2,1,1)}
\]

\[
- c_{1,1,1,1}^{(1,2,1)} \mathfrak{S}^*_{(1,1,1,2)/(1,1)} - c_{1,1,2,1}^{(1,2,1)} \mathfrak{S}^*_{(1,1,2,1)/(1,1)}
\]

\[
- c_{1,2,1,1}^{(1,2,1)} \mathfrak{S}^*_{(1,2,1,1)/(1,1)} - c_{1,2,2,1}^{(1,2,1)} \mathfrak{S}^*_{(1,2,2,1)/(1,1)}
\]

\[
- c_{1,3,1,1}^{(1,2,1)} \mathfrak{S}^*_{(1,3,1,1)/(1,1)} - c_{1,3,2,1}^{(1,2,1)} \mathfrak{S}^*_{(1,3,2,1)/(1,1)}
\]

\[
+ c_{2,1,1,3}^{(1,2,1)} \mathfrak{S}^*_{(1,1,1,3)/(1,1)} + c_{2,1,2,2}^{(1,2,1)} \mathfrak{S}^*_{(1,1,2,2)/(1,1)}
\]

\[
+ c_{2,1,3,1}^{(1,2,1)} \mathfrak{S}^*_{(1,1,3,1)/(1,1)} + c_{2,1,4,1}^{(1,2,1)} \mathfrak{S}^*_{(1,1,4,1)/(1,1)}
\]

\[
+ c_{2,2,1,1}^{(1,2,1)} \mathfrak{S}^*_{(1,2,1,1)/(1,1)} + c_{2,2,2,1}^{(1,2,1)} \mathfrak{S}^*_{(1,2,2,1)/(1,1)}
\]

\[
+ c_{2,2,3,1}^{(1,2,1)} \mathfrak{S}^*_{(1,2,3,1)/(1,1)} + c_{2,2,4,1}^{(1,2,1)} \mathfrak{S}^*_{(1,2,4,1)/(1,1)}
\]

We can compute all the coefficients $c_{|\beta|-|\alpha|, \beta}$ and most of them turn out to be zero. Hence we have the following expansion after simplification.

\[
\mathfrak{S}^*_{(2)} \cdot \mathfrak{S}^*_{(1,2,1)/(1,1)} = \mathfrak{S}^*_{(1,2,1)/(1,1)} - \mathfrak{S}^*_{(1,1,2,1)/(1,1)} - \mathfrak{S}^*_{(2,2,1)/(1,1)} + \mathfrak{S}^*_{(2,1,2,1)/(1,1)}
\]

\[
+ \mathfrak{S}^*_{(3,2,1)/(1,1)}
\]

REFERENCES


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