SOLUTIONS: ONE-SIDED AND TWO-SIDED LIMIT PROBLEMS

1. Evaluate the one-sided limits below.

a) i) $\lim\limits_{x \to 2^-} |x-2|$ ii) $\lim\limits_{x \to 2^+} |x-2|$

i) As x approaches 2 from the left, it must be true that $x < 2$. We further obtain $x - 2 < 0$ by subtracting 2 from both sides of the inequality. The absolute value $|x-2|$ is therefore equal to $-(x-2)$ for $x < 2$. Evaluating the limit yields $\lim\limits_{x \to 2^-} |x-2| = \lim\limits_{x \to 2^-} (2-x) = 2-2 = 0$.

ii) Approaching x from the right of 2, we know that $x > 2$. This gives $x - 2 > 0$ so $|x-2| = x-2$. Hence $\lim\limits_{x \to 2^+} |x-2| = \lim\limits_{x \to 2^+} (x-2) = 2-2 = 0$

b) i) $\lim\limits_{x \to -1^-} \sqrt{x^2-1}$ ii) Why do we not evaluate $\lim\limits_{x \to -1^+} \sqrt{x^2-1}$?

i) Check: if $x < -1$, then $x^2 > 1$. It follows that $x^2 - 1 > 0$ and so the square root is defined. Thus $\lim\limits_{x \to -1^-} \sqrt{x^2-1} = \sqrt{(-1)^2-1} = 0$.

ii) When $x > -1$ and x is sufficiently close to -1 (e.g. $x < 1$), $x+1 > 0$ and $x - 1 < 0$. It follows that $x^2 - 1 = (x+1)(x-1) < 0$. Taking this limit would require us to consider the square root of negative numbers, an operation that we have not defined. (Search "sqrt(x^2-1)" in Google to generate a graph of the function.)

c) i) $\lim\limits_{x \to 1^-} \sqrt[3]{\dfrac{x^3-4x^2+3x}{x^2-2x+2}}$ ii) $\lim\limits_{x \to 1^+} \sqrt[3]{\dfrac{x^3-4x^2+3x}{x^2-2x+2}}$

The cube root of any real number is defined, so there is no need to consider whether or not the term inside the root is negative. Thus

$$\lim_{x \to 1^-} \sqrt[3]{\frac{x^3-4x^2+3x}{x^2-2x+2}} = \lim_{x \to 1^+} \sqrt[3]{\frac{x^3-4x^2+3x}{x^2-2x+2}} = \sqrt[3]{\frac{1^3-4\cdot 1^2+3\cdot 1}{1^2-2\cdot 1+2}} = 0.$$

2. Compute the following limits:

a) $\lim_{x\to 2}(|x-2|+x)^5$

$= (0+2)^5$ ($\lim_{x\to 2}|x-2| = 0$ from 1a)

$= 32$

b) $\lim_{x\to 3}\dfrac{x^2-9}{x^2-8x+15}$

$$= \lim_{x\to 3}\frac{(x+3)(x-3)}{(x-5)(x-3)}$$
$$= \lim_{x\to 3}\frac{x+3}{x-5}$$
$$= \frac{3+3}{3-5}$$
$$= -3$$

c) $\lim_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}$

$$= \lim_{x\to 0}\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}$$
$$= \lim_{x\to 0}\frac{x+1-1}{x(\sqrt{x+1}+1)}$$
$$= \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}$$
$$= \frac{1}{\sqrt{0+1}+1}$$
$$= \frac{1}{2}$$

d) $\lim_{x\to 1}\dfrac{x-1}{\sqrt{2x-1}-1}$

$$= \lim_{x\to 1}\frac{(x-1)(\sqrt{2x-1}+1)}{(\sqrt{2x-1}-1)(\sqrt{2x-1}+1)}$$
$$= \lim_{x\to 1}\frac{(x-1)(\sqrt{2x-1}+1)}{2x-1-1}$$

$$
\begin{aligned}
&= \lim_{x\to 1} \frac{\sqrt{2x-1}+1}{2} \\
&= \lim_{x\to 1} \frac{\sqrt{2\cdot 1-1}+1}{2} \\
&= 1
\end{aligned}
$$

3. Find the values of the parameters a and b such that the function

$$
f(x) = \begin{cases} (2x+a)^3, & \text{if } x < 0 \\ 5bx+8, & \text{if } 0 \le x < 1 \\ x^2+12, & \text{if } x \ge 1 \end{cases}
$$

is continuous at all the points in its domain.

We know polynomial functions are continuous. Hence, the only points at which discontinuities can occur are the points where pieces of $f(x)$ join. That is, the pieces of $f(x)$ must connect at $x = 0$ and $x = 1$. This requires the left-hand and right-hand limits of $f(x)$ to be equal. We begin by evaluating these limits:

$$
\begin{aligned}
&\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x+a)^3 = (2\cdot 0+a)^3 = a^3 \\
&\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (5bx+8) = 5b\cdot 0+8 = 8 \\
&\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (5bx+8) = 5b\cdot 1+8 = 5b+8 \\
&\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2+12) = 1^2+12 = 13
\end{aligned}
$$

Letting $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$ gives $a^3 = 8$ and $5b+8 = 13$, which solves to $a = 2$ and $b = 1$.